

Vibrations of Structures
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Modal Analysis-1

So, in the previous lectures, we had been looking at ways of deriving the equation of motion of continuous systems. So, we derived equation of motion, the boundary conditions; and we also looked at initial conditions which close the system, that means, we will then have unique solutions of our system. Now, so this part, we derived these equations by two ways; one was the Newtonian approach, the other was the variational approach. Now, in the next few lectures, what we are going to do is analyze these equations or solve these equations.

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EOM + BCs + ICs

Characteristic solutions

- Natural frequencies
- Modes of vibration

} Modal analysis

$$\rho A w_{,tt} - T w_{,xx} = 0 \quad w(0,t) = 0 \quad w(l,t) = 0$$

$$w(x,t) = W(x) e^{i\omega t} \quad i = \sqrt{-1} \quad \omega: \text{circular frequency}$$

$W(x)$: amplitude function

$w(x,t) = (C \cos \omega t + S \sin \omega t) W(x)$

- Separable
- System passes through equilibrium point at same time instant
- Nodes ($W(x^*) = 0$)
- Phase difference 0 or π
 $W(x_1) W(x_2) > 0$ < 0

$\frac{w(x_1,t)}{w(x_2,t)} = \frac{W(x_1)}{W(x_2)}$

So, the first thing that we note is that, we have the equation of motion, then we have the boundary conditions and we have the initial conditions. So, this is what we have in hand. Now, what can we do with these? So, in the general situation, what we can do is given these three things, we can solve the system. We can find out the solution; we can find out the behavior of the system. But, then with change in initial conditions for example, the

solution will change. Now, if the solution changes, then we are not able to get a feel of the system. So, the question that we put is whether we can find out some characteristic solution of the system, so what we are interested in is finding out certain characteristic solutions of the system. By characteristic solutions, I mean that things that will not change, a solution will not change that those things will not change with change in initial conditions or certain properties of the system that will not change with change in initial conditions.

So can we find such characteristic solutions? So, the answer to this question is an affirmative and we know this from our study of vibrations of discrete systems. So, in there, we solve for what are known as the natural frequencies of the system. Natural frequencies, so, how fast the system is going to vibrate; so, that is, I mean an idea of that is given by the natural frequencies of the system. By natural, we mean that the system is in free motion. It is not being forced or disturbed from outside as such. The other is, how, the mode of vibration. So, one thing is the natural frequencies and the corresponding modes of vibration of the system.

So, these two constitute the characteristics solutions of the system and finding them out is known as modal analysis. So, when we do modal analysis, what we are doing is actually searching for solutions of very special form. So, if you recall the equation of motion of a string, so, suppose its fixed-fixed string of length l . So, we have the equation of motion and the boundary conditions. Now, our field variable is this, $w(x,t)$. Now, this is the general function of the spatial coordinate x and the temporal coordinate t . Now, when we do modal analysis, we are searching for solutions which are of very special form, which look like this; where this i is the square root of minus 1, ω is known as the circular frequency, and W is the amplitude function.

So, we are searching for solutions of this very special structure, in which, in a certain way, the special function and the temporal function are somewhat separated. Now, they look separated, but, remember that we have introduced. So, now, the solution is in the complex form. So, this w may, in general, be a complex function. In that case, the solution is not strictly separated or separated in space and time, but, we will encounter such solutions later in this course. So, for the time being, we introduce this complex solution, complex solution structure and we say that the actual solution, because our equation and the boundary conditions are all real, so, the actual solution is obtained by

taking either the real part, so, the actual solution will be obtained by taking either the real part of this quantity or the imaginary part of this quantity or a linear combination of the real and imaginary parts.

So, if you consider this possibility and we will initially, most of the systems that we are going to study will have this capital W , the amplitude function as real, in that case we can rewrite this solution in the form. So, if you consider that W is a real function and as I mentioned, that you can take the linear combination of the real and imaginary parts of this complex solution form as the general solution.

Then your solution looks like this. Now, from this structure, we can deduce immediately a few properties of this solution. So, here you can immediately see that the solution is strictly separable in space and time. So, space part and the time part, they are separated. Then, this, the temporal part of the solution can become zero at a certain time instant. In that event, all points for all x , the solution is zero, which means, that the motion of the string or the continuous system passes through the equilibrium point; all points pass through the equilibrium point at the same time. So, the system passes through the equilibrium point, so, all points of the system pass through the equilibrium point at the same time instant. Then, there are points, there can be points at which $w(x)$ is zero; such points are called nodes.

So, there is the possibility of existence of nodes, where $w(x)$ is zero. The fourth property of this solution structure is that, the phase difference between any two points of the system is either zero or π . So, if you take any two points, x_1 and x_2 and if you observe the product of this amplitude function, if this is positive, then the phase difference is zero. If it is negative, the phase difference is π . There is another interesting property of the solution that is, the ratio of amplitudes at x_1 and x_2 , that is, independent of time; that is, to say this ratio, so, essentially, this is independent of time as we can see. So, we will start with this structure of the solution and see or try to find out the characteristic motion of the system.

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$\rho A w_{,tt} - T w_{,xx} = 0 \quad w(0,t) = 0 \quad w(l,t) = 0$
 $w(x,t) = W(x) e^{i\omega t}$
 $W'' + \frac{\omega^2}{c^2} W = 0 \quad W(0) = 0 \quad W(l) = 0$
 $W(x) = D \cos \frac{\omega x}{c} + H \sin \frac{\omega x}{c}$
 $\begin{bmatrix} 1 & 0 \\ \cos \frac{\omega l}{c} & \sin \frac{\omega l}{c} \end{bmatrix} \begin{Bmatrix} D \\ H \end{Bmatrix} = \vec{0}$
 For non-trivial solutions $\sin \frac{\omega l}{c} = 0$ Characteristic equation.
 $\omega_n = \frac{n \pi c}{l} \quad n = 1, 2, \dots, \infty$ Circular natural frequency
 Characteristic freq.

So, to do this we will start with the example of the uniform taut string. So, the equation of motion and the boundary conditions look like this. Now, we are going to substitute this solution structure; and if you do that and with a little rearrangement, you can easily see the equation reduces to this, because exponential $i \omega t$ will never be zero for any time. So, the remaining part, the coefficient of exponential $i \omega t$, which is this, must be 0 and along with this, we must have the conditions at x equal to 0 and x equal to l . Now, this differential equation is very familiar and the solution can be written directly. Now, this is the general form of solution of this differential equation. Along with this, now we have these boundary conditions at x equal to 0 and x equal to l . So, we will substitute these in the boundary, this solution form in the boundary conditions and we can rewrite the boundary conditions in this form. Now, if you want to have non-trivial solutions of D and H , which is what we desire, then the determinant of this matrix must vanish.

So, that implies, so, determinant of this is nothing, but, $\sin(\omega l/c)$ and that must be zero. This equation is known as the characteristic equation because it yields, this condition yields certain characteristic solutions or properties of the system and they are the circular frequencies, which are also known as the natural frequencies, circular natural frequencies of the system. So, from this condition, we can immediately write that ω , now there are discrete solutions of this characteristic equation, but, there are infinitely many of them. So, there are countable infinite many points at which this will be satisfied

for special values of omega. So, we will put an index n and write this as, where n, this index n goes from 1 to infinity.

So...So, there are countably infinitely many natural frequencies of a string, of a fixed-fixed string, which are given by these values. They are called the circular natural frequency; also sometimes known as the characteristic frequencies of the system.

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$D_n = 0 \quad H_n = 1 \quad n = 1, 2, \dots, \infty$
 $W_n(x) = \sin\left(\frac{n\pi x}{l}\right) \quad n = 1, 2, \dots, \infty$

Eigen functions described modes of vibration

$W(x, t) = \sum_{n=1}^{\infty} (C_n \cos \omega_n t + S_n \sin \omega_n t) \sin \frac{n\pi x}{l}$

$\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = \frac{l}{2} \delta_{mn}$

$\omega_1 = \frac{\pi c}{l}$
 $\omega_2 = \frac{2\pi c}{l}$

Modal space Configuration space

Now, once we find these special values of omega in this matrix for which we have non-trivial solutions, then we can really find those non trivial solution by finding the corresponding D and H values, which look like, so, if you substitute these values of omega n into the boundary conditions, now it can be easily seen that the solutions are or can be written in this form. So, anything proportional to this will be a solution, where again n goes from... So, corresponding to each circular natural frequency, you have this vector which gives you the actual solution which is obtained as these are called the modes of vibration or they are actually the Eigen functions. Again, they are, they correspond to the natural frequencies of the system. So, for every natural frequency, there is an Eigen function which defines or describes a mode of vibration. So, the total solution now looks like... So, everywhere we have this index n. So, here also I must have this index n. So, this is the solution. For each n, I can have a solution and since our system is linear, a super position or a summation of this solution is also a solution. So, I

can finally, sum all these solutions and construct the most general solution of a vibrating string with fix support.

Now, this structure looks very familiar, if you think of the Fourier sine series. So, this represents a function, which is periodic and so, the periodicity is l . So, l is the, where l here refers to the length of the string. So, this structure of the solution, finally what we have is like a Fourier sine series. So, which means that any shape of the string, since we know from Fourier series theory, that any shape between two supports, if I have this shape, by continuation I can construct the Fourier or represent this as a Fourier series, Fourier sine series like this. So, any shape of the string is now, at any time instant, if you think about the shape of the string at any time instant, then this is a constant, which is represented here as a n and I mean this is expanded in terms of this function $\sin n \pi x/l$. Now, we also know from Fourier series theory, that these form the basis functions which are orthogonal in the sense that... So, if you integrate, perform this integration, so, if you take any two Eigen functions with different index n and m , multiply them and integrate from 0 to l , the length of the string, then it is l over 2 the Kronecker delta $m n$. So, which means, if m is not equal to n , this is zero. If m equal to n , this is l by 2. So, these are orthogonal. So, $\sin(\pi x/l)$ is orthogonal to $\sin(2 \pi x/l)$, etc. So, if I make a graphical representation of this solution, I can think about it with the slight stretch of imagination. Suppose, I draw axis which are orthogonal to represent this orthogonality conditions and I call it $\sin(\pi x/l)$, call this axis as $\sin(2 \pi x/l)$, etc. I cannot draw all these infinitely many axes, but, I appeal to your imagination that you can consider this to be an infinite dimensional space, where axes are orthogonal to a represent the orthogonality of these functions. Then a solution of this form or a representation in this form for particular shape of the string, is actually a point in this infinite dimensional space. This is a_1 , this is a_2 , this is a_3 and like this you can have all these coefficients a_1, a_2, a_3, a_4 up to a infinity. So, in this space, the configuration of the string at a time, at time instant, at a particular time instant, is nothing but a point in this infinite dimensional space. So, as the string moves, so, is nothing but the motion of this point. So, if the string executes a periodic motion in this space, then should be a perfectly a closed curve. Now, in this space, then, what is the simplest motion that is possible? So, simplest motion would be for example, the string moving only along this axis, only along this axis. This is a modal solution, the motion of the string in the first mode of vibration.

So, this motion is nothing but only $\sin(\pi x/l)$, is represented by $\sin(\pi x/l)$. All the other coefficients are 0, a_2 , a_3 etc. are all zero, except a_1 . So, this is the first mode of vibration of the string which looks like. Similarly, if you consider motion only along the second axis $\sin(\pi x/l)$, then the motion of the string looks like this. So, essentially the string is vibrating between these two extreme configurations. The corresponding frequencies, corresponding circular natural frequencies are given by these values, ω_1 as $\pi C/l$ and ω_2 as $2\pi C/l$. This infinite dimensional space is sometimes called the modal space or sometimes also can be called the configuration space of the string. So, in this space as the configuration of a string at any time instant is represented by a point; and it is nothing but, when it moves from one configuration to the other, it is nothing but the motion of this point in this space. We have seen that the elemental motion or the motion along these axes, they represent the modal solution.

Now, I am going to demonstrate with the small experiment, these solutions. Before that, let me just state, point out here, that as you can see in this solution, there is no node within the domain of the string, whereas, here there is one node, where the string remains fixed. This point is not moving out from the equilibrium position whereas, other points are vibrating. Here, all points are vibrating in the same phase. So, the string essentially moves through intermediate configurations like this and then again comes back and passes through the equilibrium points. At the equilibrium points, the string again becomes straight and then starts moving down and reaches this extreme; again repeats; starts afresh.

Here, the intermediate configurations look like this. So, you see a point here and a point here, they are simply out of phase. So, when this is moving up, this is moving down and vice versa. So, but two points on this, in this lobe, they are moving in phase. So, either the phase is either zero, phase difference between two points of the string is either zero or it is π . So, now I will demonstrate using a small experiment these modes. So, here I have a string, which I have made taut by pulling and here this is an exciter which is nothing, but a simple electric shaver. Now, as I pluck the string, you see the motion ceases after some time. This is expected, since all real systems, they have internal damping. But, in our model that we have considered till now, we do not have damping. So, to cancel the effect of damping, I must have an exciter, which must pump the same

amount of energy that is being dissipated, with the due to the internal damping in the string.

Now, let us see. So, this is the first mode of vibration of the string. This is how it will look like when it is vibrating in the first mode. Now let me try the second mode. This is the second mode of vibration of the string and I may also try the third mode, which may not be visible very clearly, because the amplitudes are so small. But, the second mode is quiet clear and the first mode as well. So, this is what we have observed. We have seen the first two modes quite clearly, third mode, because the amplitudes are so small, has been, is less visible and so on for the higher modes. But then, whenever I pluck a string, I actually excite a number of these modes. So, the motion of the string, if I pluck a string at an arbitrary point, then I am actually exciting a number of these modes. So, the solution is more complicated, but, I can, by special means excite only these individual modes and we have seen that in a demonstration.

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Uniform hanging chain

$$w_{,tt} - g[(l-x)w_{,x}]_{,x} = 0 \quad w(0,t) = 0 \quad w(l,t) < \infty$$

$$w(x,t) = W(x)e^{i\omega t}$$

$$\omega^2 W + g[(l-x)W']' = 0 \quad W(0) = 0 \quad W(l) < \infty$$

Eigenvalue problem

$$s(x) = 2\omega \sqrt{\frac{l-x}{g}}$$

$$\frac{dW}{dx} = \frac{d\tilde{W}}{ds} \frac{ds}{dx} = -\tilde{W}' \frac{\omega}{\sqrt{g(l-x)}} \quad \tilde{W} = \tilde{W}(s)$$

$$\frac{d^2W}{dx^2} = \tilde{W}'' \frac{\omega^2}{g(l-x)} - \tilde{W}' \frac{\omega}{2\sqrt{g(l-x)}}$$

Next example, that, we are going to look at is that of a uniform hanging chain. We have derived the equation of motion of a uniform hanging chain previously and it looks like... So, this is the equation of motion, these are the boundary condition at the fixed end. Now, once again, we are going to attempt a solution with this structure... and we obtain... with the boundary conditions...

Now, in the case of a string, of a uniform taut string, we obtained the equation previously. After substituting the solution structure, we obtained the differential equation and boundary conditions. Here again, we have obtained a differential equation and boundary conditions; this problem is known as the Eigen value problem.

So, in the case of a string, this was our Eigen value problem. Now, this structure of equation looks tantalizingly familiar because we have studied equations of this structure, when we discussed Sturm-Liouville problems in mathematics. Now, to convert this, because this is a particular structure, is very special, we can convert this into more familiar, into a familiar more familiar form by using transformation of the independent variable x . So, let us consider a variable s which is the function of x .

If you use this new variable s , then you can replace, which you can check very easily, where w tilde is a function of this new variable s . Similarly, you can work out the higher derivative.

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$$\tilde{w}'' + \frac{1}{s}\tilde{w}' + \tilde{w} = 0 \quad \tilde{w}(2\omega\sqrt{\frac{L}{g}}) = 0 \quad \tilde{w}(0) < \infty$$

$$y'' + \frac{1}{x}y' + (1 - \frac{n^2}{x^2})y = 0 \quad \text{Bessel differential eq.}$$

$$\tilde{w}(s) = D J_0(s) + Y_0(s)$$

Zeroth order Bessel function:
 of first kind: J_0
 of second kind: Y_0

$$\tilde{w}(s) = D J_0(s)$$

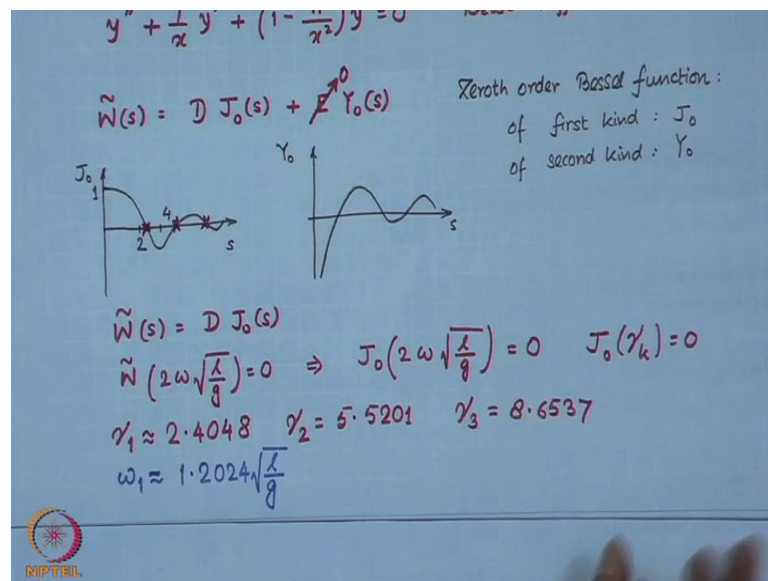
$$\tilde{w}(2\omega\sqrt{\frac{L}{g}}) = 0 \Rightarrow J_0(2\omega\sqrt{\frac{L}{g}}) = 0$$

Now, having done this, if you substitute back in the equation and convert this equation in terms of s , then what you obtain is... with the boundary conditions... This can be checked very easily. Now, this is a very familiar equation, which is the Bessel differential equation. So, this equation is the Bessel differential equation and in this equation if you put n as 0, then it reduces to this differential equation.

And, the solution of the Bessel differential equation with n equal to 0, which is the solution for our w tilde, where J_0 and y_0 , they are known as the 0th order Bessel function of the first and second kind, respectively. Now, if you look at these Bessel functions of 0th order, then you will find that this second kind has a logarithmic singularity at s equal to zero. So, if I make an approximate plot of J_0 , so at s equal to zero, y is in minus infinity. So, this will violate this condition, this boundary condition, which is the boundary condition at the free end. So, finiteness of the solution at the free end forces us to select E as zero. So, E must be zero. So, therefore, which can be written, so, I can write down, write this down in terms of my original variable x .

But, before I do that, let me look at the boundary conditions. So, we know that, so this implies this must be zero. Now, you can see from this figure, there are finite number of points, but infinitely many; there are discrete points, but infinitely many, at which this function J_0 is zero. So, these countably infinitely many solutions of this will give us the natural the natural frequency of the system.

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And, if I represent this by, something like Gamma k, then this Gamma 1 is approximately 2.4048, Gamma 2 is 5.5201, Gamma 3 is 8.6537 and like this you can find out the values and here from the these values of Gamma k. Therefore, you can find out say for example, you can easily write omega1 as 1.2024 under root l over g. So, you can see that compared to a mathematical pendulum, this is 1.2 times higher.

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$$w(x,t) = \sum_{k=1}^{\infty} [C_k \cos \omega_k t + S_k \sin \omega_k t] J_0 \left(2\omega_k \sqrt{\frac{l-x}{g}} \right)$$

$\omega_1 = 1.2 \sqrt{\frac{g}{l}}$ $\omega_2 = 2.76 \sqrt{\frac{g}{l}}$ $\omega_3 = 4.32 \sqrt{\frac{g}{l}}$

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Now finally, once you have this, you can write the general solution, once again using superposition. Where, so these are the, these represents the modes, these are the Eigen functions and the modes of vibration of a uniform hanging string are defined by these Eigen functions. So, the modes of vibration of hanging string are defined by these Eigen functions; and they look, so, this is the first mode, this is the second mode and this represents the third mode. Now once again, I will demonstrate to you the modes of vibration of a hanging chain.

So, if you see this chain, so, this is the first mode of vibration. Now, we will try the second mode. So, this is the second mode of vibration of the chain. Exciting the higher modes is little more difficult. So, in this lecture what we have studied is the modal analysis of continuous system. So, we started off with a modal analysis of a taut string and then we looked at the properties of this modal solutions and finally, we also looked at the example of a uniform hanging chain. So, with this we come to an end of this lecture.

Keywords: modal analysis, Eigen functions, natural frequencies, properties of modes, uniform taut string modes, hanging chain modes.