Vibrations of Structures

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Lecture No. #05

Module No. #01

Variational Formulation-II

Let us resume our discussion on the variational formulation that we have started in our previous lecture. So, today what we are going look at is the transverse vibration of a hanging chain. So, this is the chain as you can see this is an inextensible continuum. So, we are going to look at the equation of motion of such a chain.

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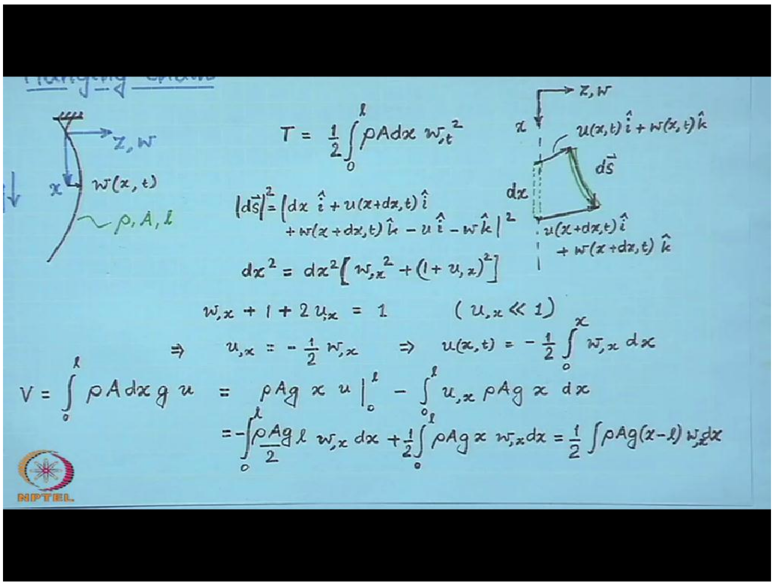

So, let us consider this chain hanging in a uniform gravitational field. So, we assume that this chain is made of a material of density rho, has a constant area of cross section A, has the length l or this is an inextensible chain. So, though it does not a anyway resemble a string in that sense, which is a elastic continuum one dimensional elastic continuum; this is this is inextensible yet as we will see where you soon the equation of motion at least upto the linear order is similar to that of a string. So, we are going to write down the energy expression. So, the kinetic energy, so, at any location x our field variable is w(x,t). So, the kinetic energy of an infinite assemble element can be written as ½ rho A dx is the mass of the infinite assemble element and velocity square. So, if I integrate this over the length of the string then what I have this is the kinetic energy now the potential energy now the potential energy of this chain now why this chain restores back to its vertical position it is because as it deflects or as it is disturbed the potential energy of the chain changes, so, it increases. Now to calculate that let us look at a small element of this chain. So, this is the undeformed configuration of the chain and this is the deflected configuration. So, this is infinitesimal element had a length dx. Now since this chain is has inextensible as we have assumed, so, the length of this inextensible element in the undeflected or the equilibrium configuration is same as in the deflected configuration. So, this is also dx. But now this is deflected. So, I will represented as a vector whose length or whose magnitude is dx and these vectors I will represent as… this way and this vector at (x+dx), this is given by this vector. So, if you write out this vector equation, so, we can represent this ds vector in terms of the deflection of the chain. Now if you take the magnitude of vectors on both sides, then as I mentioned that the magnitude of this is still dx because the change is inextensible. So, what I will do is I will take the magnitude square let's say and if you simplify the right hand side will have this expression form which we can write… Here I have made this approximation…

So, in this step I will coming from this step to this step I have made this approximation that del u/del x is much much smaller than one. So, the expansion of this, in this expansion, I have dropped del u/del x whole square. So, this implies…or…

So, I have represented the axial deflection or axial motion of the chain. Now this is very different from that in strings. So, in strings we had neglected the axial motion, but in a hanging chain, we must consider this axial motion, because that is the reason why the potential energy of the chain is changing, as the chain deflects, from its equilibrium position. Now once we have this expression of the axial deflection, we can write down the potential energy of the chain. So, rho into A is mass per unit length, dx that gives the mass of a little element, infinitesimal element of the chain into the acceleration due to gravity into u. So, that would be the potential energy; and if you integrate over the length of the chain that will give you the net potential energy. Assuming that the potential energy is zero in its equilibrium configuration. Now what I can do is I can integrate this expression by parts.

I can write this as taking u as the first function and rho A g as the second function. So, the first function integral the second function minus integral of derivative of the first function the integral of the second function; now these simplifies to. So, this boundary term can be written as u evaluated at l, so, that would mean this integral with x replaced by l; and one minus from here when I replace del u/del x here, and so, this finally simplifies to… So, we have the kinetic and the potential energy expressions of the chain.

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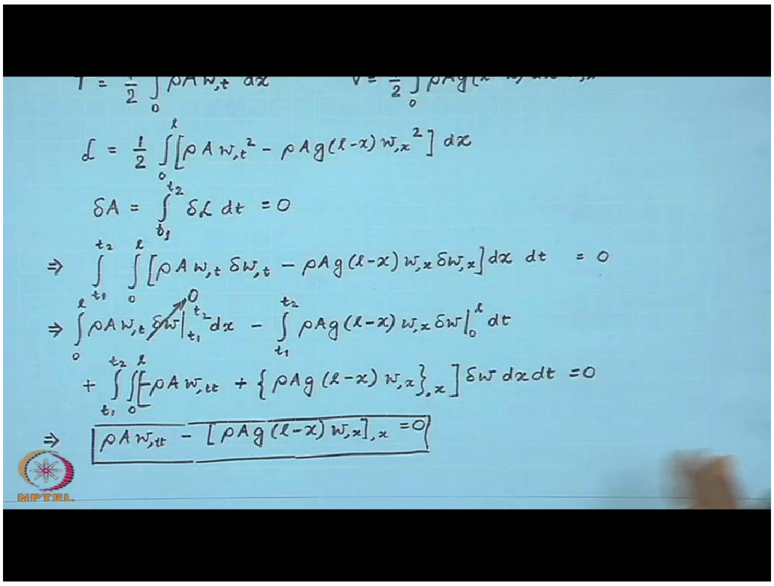

So, let me write them again. So, if you look at these expressions, this kinetic energy expression, this resembles the kinetic energy of the of the normal elastic string and this potential energy expression also resembles that of a string expect for this term. Now in a normal elastic string this, here this term is the tension which is constant, but in the case of hanging string or hanging chain as we know the tension varies with the location. So, our Lagrangian now may be expressed like this and from Hamilton's principle which is the variation of the action is zero, we can write… Now here once again as we had done before, we will integrate by parts this term with respect to time and this term with respect to space. To obtain, so this is what we obtain finally. Now here as we have stated before since we know the configuration of the chain at the time instance t1 and t2, so there cannot be any variation of the configuration. So, these at both times t1 and t2 this variation must vanish now here in the term which is evaluated only at the boundaries and here it is evaluated or in its over the full domain, is integrated over the complete domain 0 to l; now I can always hold the boundary variation fixed and change the inner portion arbitrary. So, therefore, if this sum has to vanish they have must vanish separately. So, for arbitrary variation, therefore, if this integral has to vanish then you can write this must be zero, which is our equation of motion of the hanging chain. Now these boundary terms must also vanish. So, this will give as the boundary condition. So, let us have a look at the boundary conditions. Now at x equal to 0, so at this boundary either this must be equal to 0 or now in our case if you consider that the chain is fixed at the top end then this is the boundary condition.

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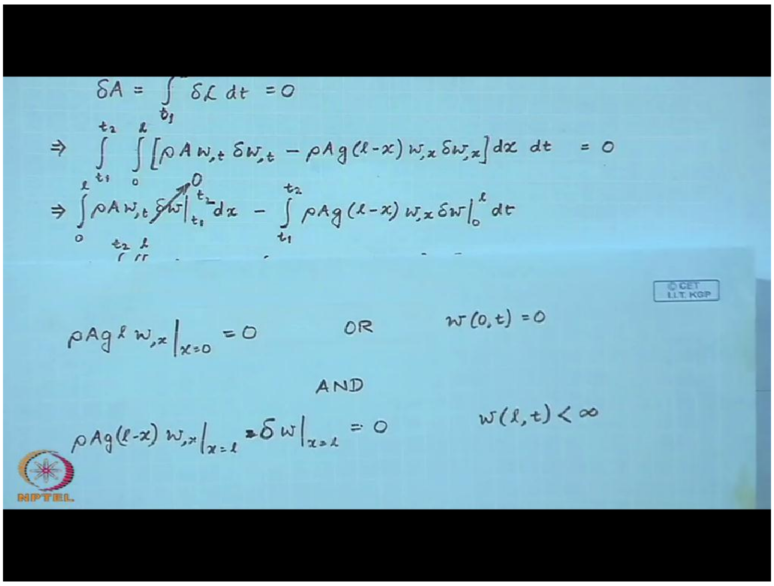

If it is free then this is going to be the boundary condition. So, these are the boundary condition; these are the possible boundary condition at x equal to 0 and we can have, so, here as you can see that at x equal to l, this term vanishes.

Now this remember was a tension in the string; and tension at the free end of hanging chain is definitely zero. So, if you, so this of course, this is the product. Now here in this case of a chain with free end, this vanishes. Now in order to have zero contribution from the boundary at x equal to l, therefore you must have these variation to be finite because this part of the boundary term is zero at x equal to l; so therefore, we usually write, we say that, the deflection of the string, of the chain, at the free end must be finite; as the implications of these boundary condition will be seen when we discuss the solution procedure for the string, the hanging chain.

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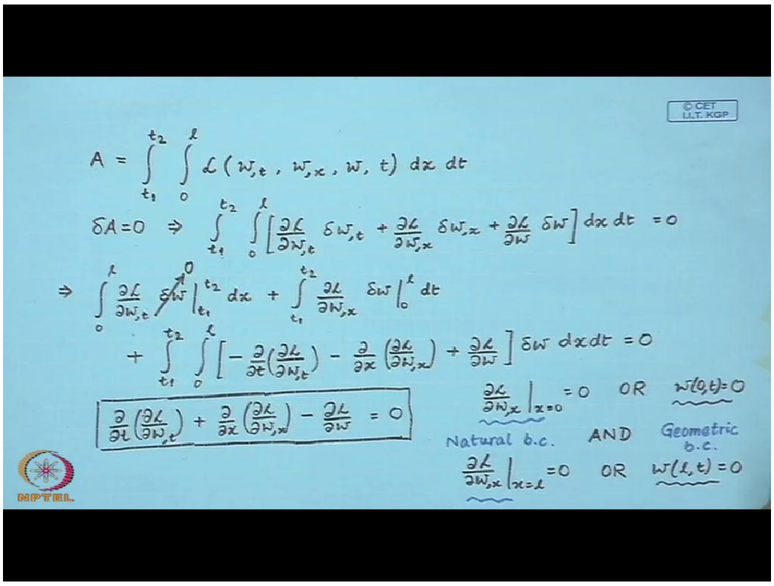

Next we will be going to slightly the generalize the procedure. So, we have been looking at action integrals in which the Lagrangian is a function of the velocity, the slope, it may be also function of the field variable w itself and it may be a function of time.

So, let us slightly generalize what we have been doing for a Lagrangian, which has this form, which is the function of the velocity, the spatial derivative of the field variable, and may be time. So, if you say that the variation is zero that could imply, so, delta A may be written as in this form; now I will integrate by parts, the first term in the integral the first two terms first term with respect to time and in the second term with respect to space and of course, this must be zero. So, that implies…

So, here we have minus of derivative with respect to time minus derivative with respect to x and delta w from some point that must be zero. Now as you know that at these two time instants I definitely know the configuration of the system as which means I completely know my field variable. So, that cannot be any variation on the field variable at these two time instants.

So, if the rest of the terms must vanish for arbitrary variation delta w, you see this term which is evaluated in the boundaries and this is dependent on the total domain. Now I can fix the variation at the boundary and change the variation over the domain. So, these two terms must independently vanish for this sum to vanish. So, for arbitrary variation if this integral is to vanish then it must be true; and this is our equation of motion and what we have from this term, we have the boundary condition. So, the boundary condition, so, the possible boundary conditions could be, so, del l/del w,x at x equal to 0 is zero or w itself at 0 must be zero and del l/del w,x at x equal to l must be zero or w at l must vanish. So, these are possible boundary conditions for the problem. Now it, which happens that, boundary conditions of this type they are the geometric boundary conditions; while these boundary conditions are the natural boundary conditions of the problem. So, let us look at some applications of this generalize formulation for vibrational bars.

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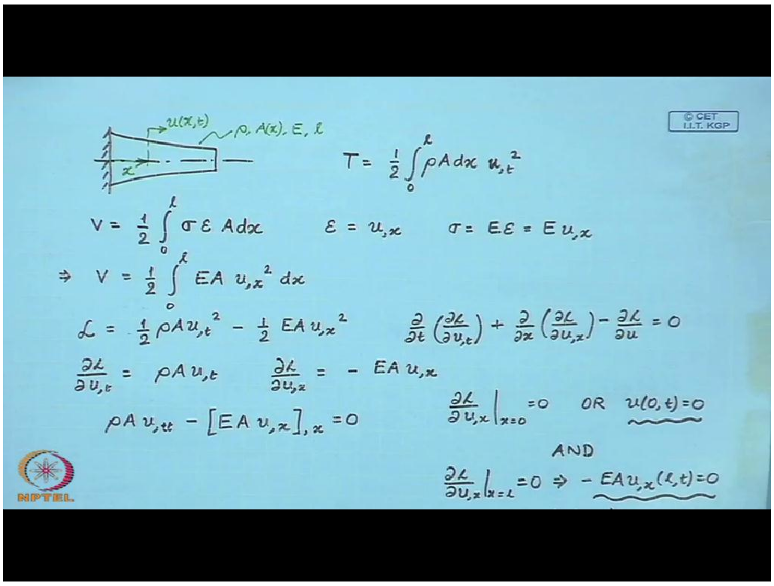

So, we consider first axial vibrations of a bar. So, the field variable is u(x,t). So, first we write down the kinetic energy. So, if rho is the density, A is the cross section, then this is the mass of infinitesimal element into the velocity square and this integrated over 0 to l is to be the kinetic energy of the bar. Now the potential energy of the bar may be written, from elasticity theory, as half times the stress times the strain and this the per unit volume. I must integrate this over the volume. So, if A is the area of cross section A dx is the little volume element and this integrated from 0 to l will give us the total potential energy of the bar. Now strain as we a derived before is del u/del x and sigma is…

So, therefore, this is the expression of the potential energy of the bar. Now what we have been describing as the Lagrangian or what is more appropriately called in Lagrangian density is given by…

Now, as derived the equation of motion, will be obtained from this equation; now as you can see that there is no explicit dependence of this Lagrangian density on u, so, this term here is zero. So, what we have only these two terms. So, del l/del u,t is and del l/del u,x. So, therefore, the equation of motion is obtained in this form which we have derived before as well and the boundary conditions are obtained; the possible boundary conditions would be, so, in this case of course, the boundary is fixed at x equal to 0. So, this is our boundary condition at x equal to 0 and at x equal to l. This is the boundary condition at x equal to l. There is another possibility which does not apply here. So, if the bar is fixed, if the bar is fixed at the other end as well then we have this boundary condition, but that does not apply here, so in our case, for these bar a fixed free bar these are the boundary conditions for the problem.

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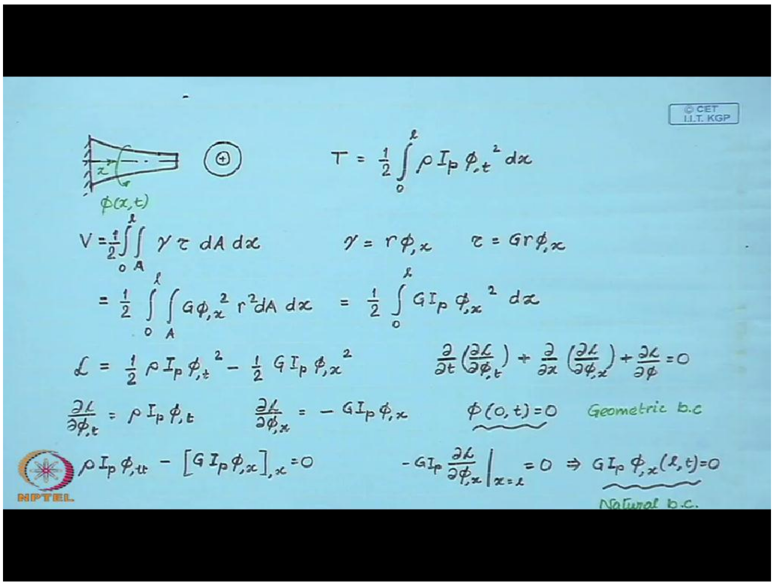

Now let us discuss this for torsional vibration of the circular bar as well. So, if phi is the field variable then again write the kinetic energy in this form. Now to write the potential energy of this bar, so again from theory of elasticity, now as we know that the stress, the shear stress is varying over the cross section, so I must first integrate over the cross section and then finally over the length. So, this is energy per unit volume and that I integrate over the total volume to obtain the total potential energy; and we know that the shear strain is given by r times del phi/del x and shear stress is G times the shear strain. So, if you substitute these two expressions in here, this will simplify to G times Ip, so, this of course comes with a square. So, that is the potential energy expression for this bar. So, the Lagrangian density is given by this expression. Now once again we apply our equation of motion. So, which means, once again we find that this Lagrangian density is independent of phi; so this term, the third term is zero. So, once I have evaluated these terms, I can now write down the equation of motion in this form; and the boundary conditions are obtained as at x equal to 0, since this bar is fixed to the wall, at x equal to 0 there is no twist in the bar and at x equal to l this the free end, so phi is not fixed, so therefore, so, this term must be zero which implies… So, these are our boundary conditions for the problem. So, this is the natural boundary. So, this is the geometric boundary condition and this is the natural boundary condition.

So, what we have looked at in this lecture we have started with the vibrations or the dynamics of hanging chain we derive the equation of motion and the boundary conditions. Then we generalized the variational approach to find the equation of motion and the boundary conditions and we have derived the general form of the equation of motion and boundary conditions and I have applied it to two examples. So, will carry forward this and see applications in other examples in the subsequent lectures. That is the end of this lecture.

Keywords: dynamics of hanging chain, Lagrange’s equation of motion and boundary conditions, axial vibration of bar, torsional vibration of circular bar