Vibrations of Structures

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Lecture No. # 40

Special Problems in Plate Vibrations

We have been discussing about the vibrations of plates. We have looked at the rectangular and the circular plates in our previous lectures. Today, we are going to discuss some special problems in vibrations of plates. Before we do that now as, you have seen in the previous lectures that even for some simple problems the analytical solution becomes very complex and sometimes with increasing complexity of the geometry or the boundary conditions etc., the analytical solution becomes intractable. In such situations we would like to have approximate methods for discretizing the dynamics of plates and performing modal analysis which we obviously, expect them to be approximate. But then, we can; there is always a scope of improving the accuracy of these methods and we have seen such approximate methods in this course. Today, we are going to look at two examples based on which we are going to solve using the approximate method. Before we start discussing the examples, let us look at the variational formulation of plate dynamics. Till now we have not discussed this variational formulation. We have discussed only the Newtonian formulation of plate dynamics. Now this variation formulation is little cumbersome for the plates. Let us see at least how it is formulated? But then if you go through the variation formulation of plate dynamics then you will also see that the boundary conditions are obtained naturally in the process. So, you can cross check, that the boundary conditions that we have discussed based on the Newtonian formulation and based on Von Karman's boundary condition for free edge; you can obtain them directly from the variation formulation. Let us a start with a little bit of discussion on the variation formulation for dynamics of plates.

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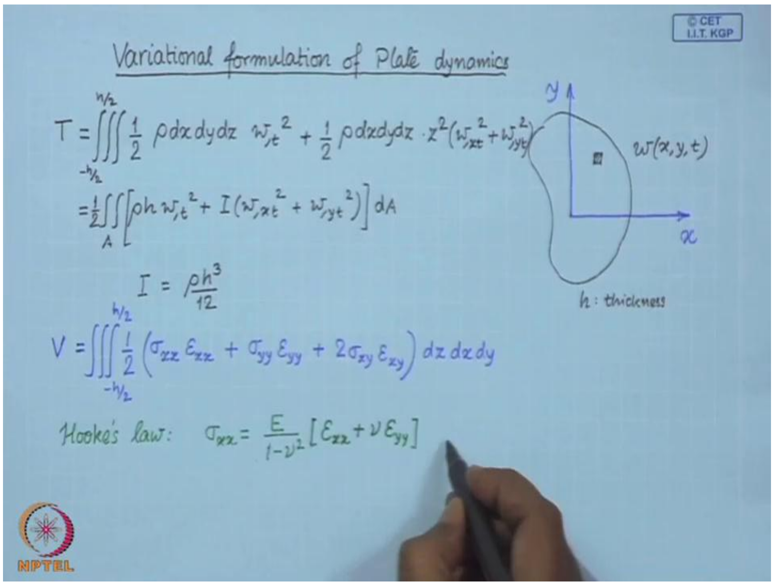

So, let us consider a plate. Now, here when we do the variational formulation, we first write down the kinetic energy of the plate. So, the kinetic energy of this little element is given by one half, if rho is the density then dx dy dz, where z is the coordinate perpendicular to the figure that is the plane of the paper; so, this is the mass of this little element times its transverse velocity's square. Now, in addition if you also want to consider the rotary inertia then you have in addition to this half rho; now this is the mass times z square; so mass times the distance from the neutral plane square; so that will give the moment of inertia of this little element at height z from the neutral plane times the angular velocity square; now this angular velocity can be written as, in the y it is written as del/del t of del w/ del x whole square; so this is the contribution from the rotary inertia and if you now integrate this over the complete volume of the plate, then we can obtain the total kinetic energy of the plate. Now, since this field variable tracks the transverse displacement of the neutral plane, so this is independent of z; and if density is uniform etc., with all those nice conditions, there is nothing in the integrant that is function of z. So, z can be integrated out and written as… So, this z goes from minus h by two to plus h by 2. So, this z coordinate goes from minus h by two to plus h by two, where h is the thickness of the plate. So, then if I perform this integral, I can simplify this expression; so now integral over A; that is the kinetic energy. Now, this I, which is the moment of inertia per unit area, so that is the moment of inertia per unit area of the plate. Next, we will write down the potential energy expression for this little element. Now, we have considered that this plate is subjected to in-plane stresses. If you have in-plane stresses then the strain energy stored because of deformation is given by half times the stress times the strain. Now we have the normal stresses times the strain in the x plus normal stress in y times the strain in the y and plus two times the in-plane shear stress and the corresponding shear strain of the element; now this is per unit volume. Now, I have to again integrate. The thickness goes from minus h by two to h by two; and x, y go over the domain of the plate. Now, here finally I obtain this expression in terms of the displacement field variable which is w; to do that we first write down the constitutive relation which is Hooke's law. From Hooke's law, we can write; you have seen this before; so, the stress, normal stress in terms of the normal strains in x, similarly in y, and the shear stress in terms of the shear strain.

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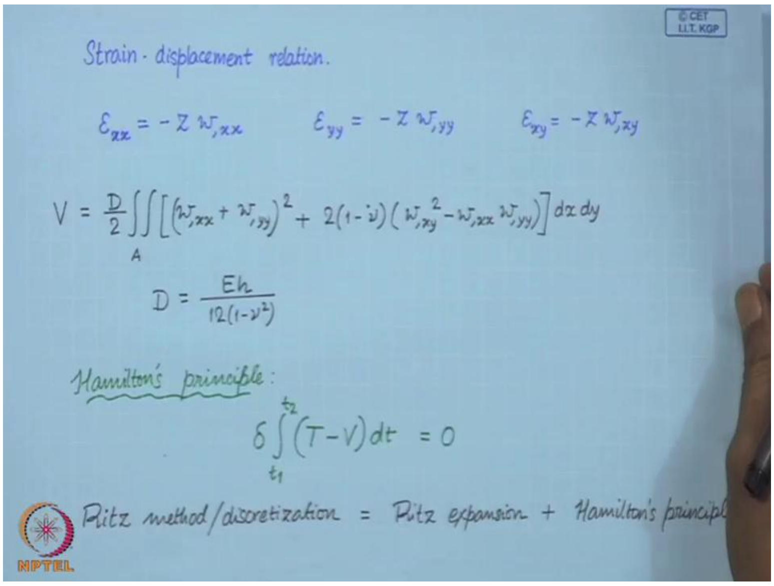

So, if you use these expressions in the potential energy x and of course, we also have to write the strain displacement relation which we have also written out before. So, from the displacement kinematics… So, we have seen these expressions before. Now, we will substitute these strains in terms of the displacement field variable in here. Finally, the stress terms will be put here. If you do that and make the final simplification, then upon integrating over the thickness direction, this is the expression of the potential energy, the strain potential energy, where this D is… here of course E is the Young's modulus and nu is the Poisson's ratio. Now, with the obtained expressions of kinetic and potential energy, we can now write the… use the Hamilton's principle to derive the equation of motion. Hamilton's principle says that, this variation must vanish. We have these expressions of the kinetic and potential energy and if you want to derive the equation of motion, then you must follow the procedure that we have discussed in case of other structure elements like a strings, membranes etc. This will closely follow that of the membrane; only the terms are more complicated or complex here and because of that the procedure is straight forward, but cumbersome. So, one can derive the equation of motion and the boundary conditions are also obtained along with this. So, that is the advantage of using the variation formulation. So, now, using this then we can also use the discretization using Ritz. So, based on based on this Hamilton's principle and Ritz expansion; so Ritz method requires, the Ritz expansion plus the Hamilton's principle; these two lead us to the discretization of the dynamics. Let us look at two examples. So, the first example that we are going to discuss today is that of a square plate supported on a circular boundary.

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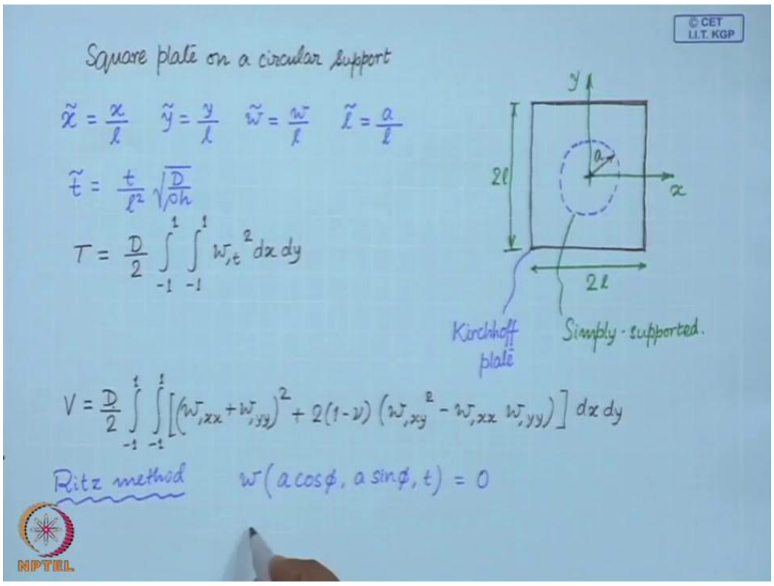

So, let us look at the geometry of the problem. Suppose this is a square plate of uniform thickness, which supported on a circular support; it is simply supported. We consider that the length is 2l and this is at the geometric centre. Now, we are looking at square plate on a circular support. This radius of this support circle is a and the side is 2l. We already have the kinetic and potential energy expressions; but I will make some simplifications, some non-dimensionalization. This x coordinate is non-dimensionalized with respect to the half length of the plate. Similarly the field variable w which tracks the transverse displacement of the neutral plane is also non-dimensionalized with l and we define non-dimensionalized l tilde. Now time is also non-dimensionalized in this form. Now with this non-dimensionalization scheme the kinetic energy; we will consider an Euler Bernoulli plate, the Kirchhoff plate. So, we will consider a Kirchhoff plate. In the Kirchhoff plate we do not consider the rotary inertia terms; the kinetic energy expression simplifies. These are all simplifying assumptions. I have dropped the tilde for convenience. Here now, x and the y coordinates go from minus 1 to plus 1. Similarly the potential energy expression… So that is the potential energy expression for the plate. Now, let us look at the support condition. Now in there we will be using the Ritz method. The advantage of this method is that you have to generate only admissible functions for the problem. Now, we know that when this is simply supported; the only condition at this boundary is that the displacement is zero. So, if I parameterize this circle in terms of phi, this is the boundary condition. In terms of x, y I can write, I can write in any one of these ways.

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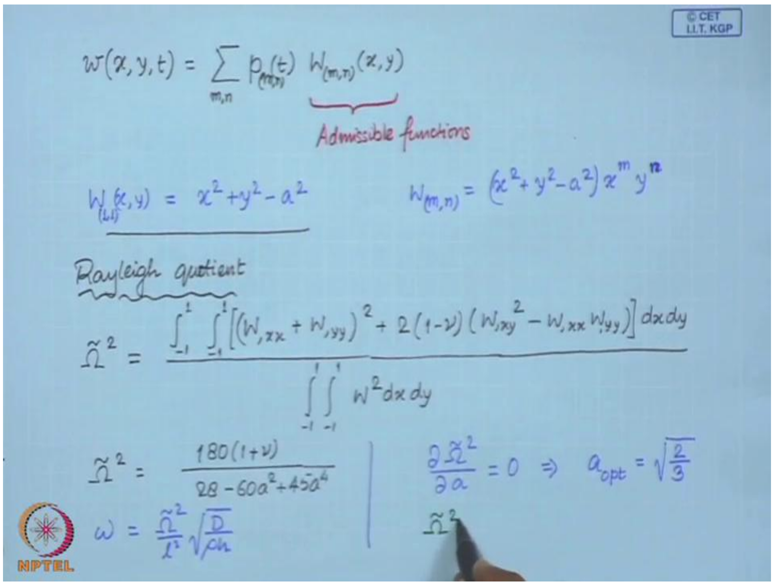

Now we will; what we have to do is we have to write this as an expansion. There will be of course two indices in this form. So, we have to write the solution as an expansion in terms of some known functions. These are our admissible functions, which need to satisfy only the geometric boundary conditions of the problem and this boundary condition is geometric boundary condition. We have a geometric boundary condition in this problem and of course, there is a natural boundary condition also; but and on this, so, here on the edge; these are free edges; we have again natural boundary conditions at the free edges. Now what could be a good choice of admissible functions for this problem? So, let us see; if you have, suppose you choose W (1, 1) the first term in this expansion as this. Now you can see of course, that on this boundary this is going to vanish. So then I can generate the expansions here as multiples of or products of this function and other monomials, say x power m and y power n. I can a think of this construction; but before we do the general situation, let us look at only one term expansion and do the Rayleigh's, determine the Rayleigh quotient. We are going to determine the fundamental frequency of the square plate which is supported on this a circular boundary. This non dimensional frequency square then turns out to be… So, this is the assumed Eigen function for the fundamental mode which we are assuming in this form. So, this is going to be the Rayleigh quotient. If you substitute this as the assumed mode shape function in the quotient then this happens, turns out to be… if you perform these integrals… I have this a as it is. I have not assumed any particular value of a. Now we may then think of for example, improving the support; by improving the support I mean that increasing the natural frequency of the plate. So if you; so, here of course, this is the non-dimensional frequency. The dimensional frequency will be obtained using this expression. Now, if you want to improve the support, suppose you want to, suppose you ask the question that what should be what should be good radius for support, this circular support so that the plate is a well supported. In that case, what we are asking is what radius will have give a very high natural frequency? High natural frequency would mean the support is quiet stiff. So, what would result in a very high natural frequency? So to do that, you can make this stationary with respect to a and that gives a value of optimal radius as square root of 2 over 3 and corresponding to this the optimal basically, this is maximum you can check that by taking the second derivative. This is actually maximized, which can be checked by taking the second derivative with respect to a and looking at the sign of the second derivative. So, this happens to be the optimal non-dimensional radius. Now, let us go back to our Ritz discretization problem and now we can take this optimal radius and discretize the problem.

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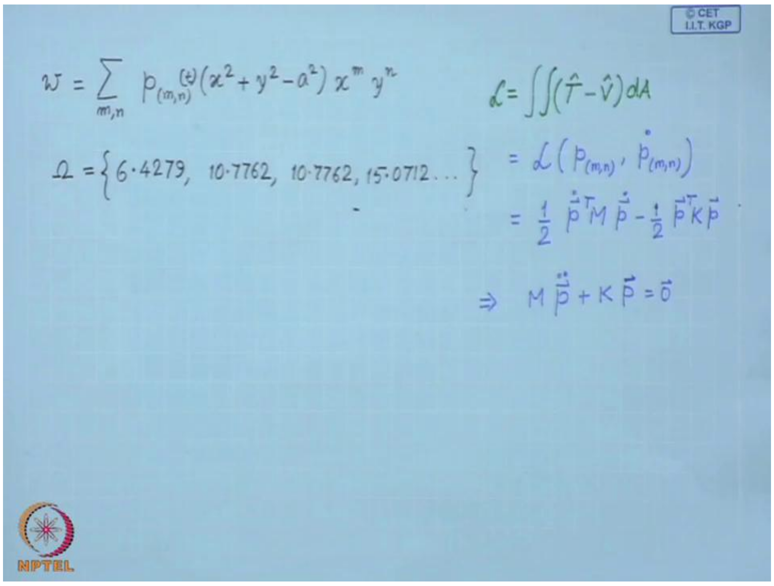

So, the expansion that we are using; so these are the coordinates and that multiplied by the admissible functions; this is an expansion which we will substitute in the Hamilton's principle. We will calculate the Lagrangian that is obtained as an integral over the area which now is actually dx dy. So, we will integrate over x and y to obtain the Lagrangian. Now, since we know all these functions and these are polynomials. It is very easy to perform these integral. We obtain the Lagrangian. This Lagrangian is a function of these coordinates and the derivatives. Once we have that we have actually discretized. So, the problem, this Lagrangian will have this structure. The equation of motion will follow… So, that is straight forward. So, these are the mass and the stiffness matrices of the discrete problem. Now, if you perform this and do the modal analysis of this, then the natural frequencies, the first few of them etc.; so, you see these two frequencies are the same. So, there is a modal degeneracy. You can check that the Eigen vectors that you will get corresponding to these two Eigen frequencies are distinct. You will have, from here you can construct back the Eigen functions and these will be orthogonal.

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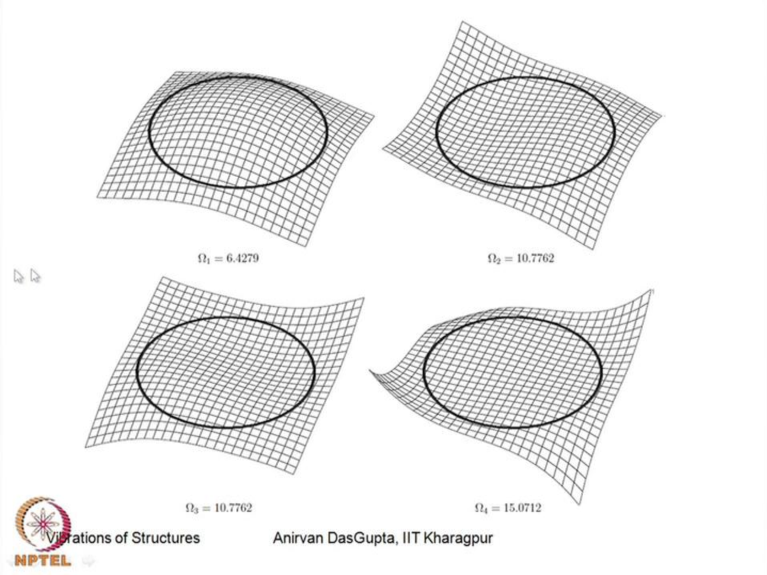

Now this figure shows the modes of vibration of the plate. So, this is the optimal radius which happens to be around 0.8, non-dimensional values. So the radius is 0.8. This is the fundamental mode; these two are the degenerate mode. As you can see that, this is nothing but a rotation of pi by 2, which is the symmetry of this problem because this is the square plate; rotation of pi by 2 is not going to change the problem. You can see that this mode is rotated version of this mode and vice versa. Then this is the third, distinct mode; these are the degenerate modes; the second mode is degenerate. Now it may be of a interest to see what happens when a changes when we vary a.

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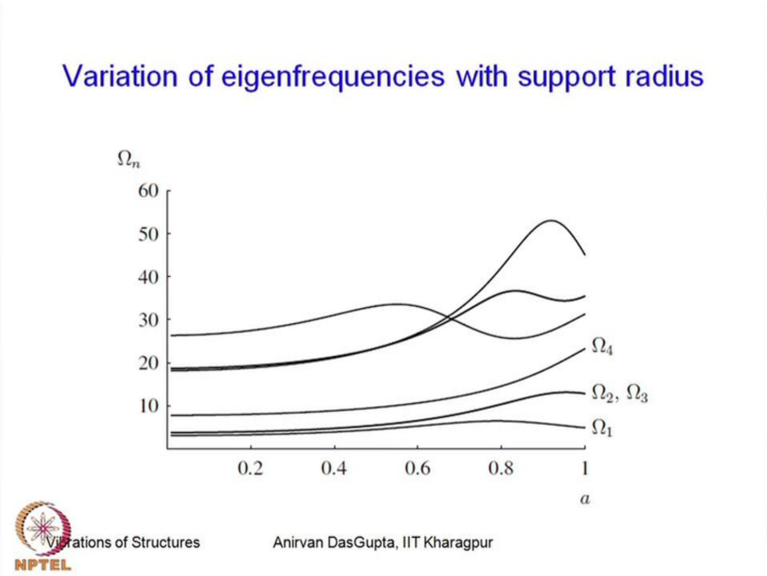

This figure shows, the variation of the first few non-dimensional circular Eigen frequencies with a. So this is the fundamental frequency of the plate with variation of a. So, you can see that the maximum occurs somewhere here; these are degenerate; there are two frequencies here; two modes corresponding to this branch and this is the next higher mode and then these are some further higher modes. You can see the variation of thus Eigen frequency of the plate with a. So, we have seen how we can optimally support square plate or now you can attempt any other form of plate with a circular support and trying to find out the optimal support radius for example, that gives the maximum stiffness to the structure. So, this is a very important problem in structural engineering.

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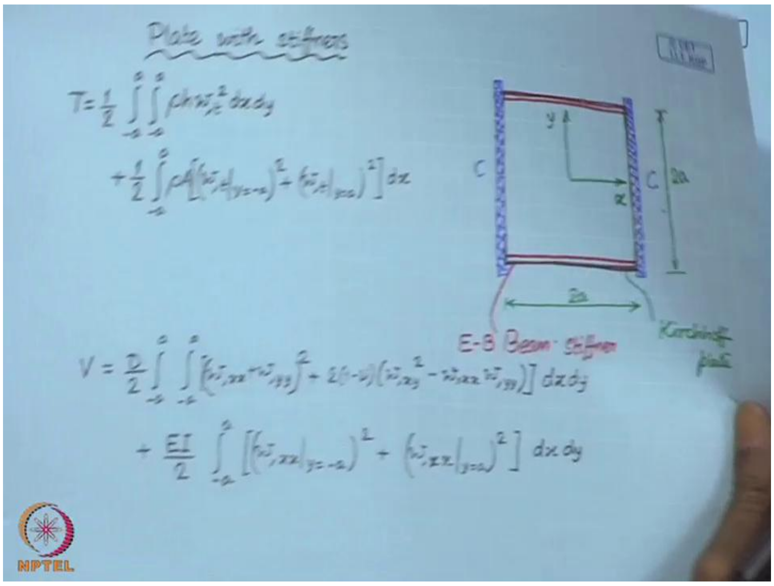

The next problem that we are going to look at is; we have a plate; we would like to see what happens when a plate is having stiffener, a stiffened plate, the vibrations of a stiffened plate. We consider again a square plate. We consider that this plate is clamped on these two opposite edges. So, these two edges are clamped. The coordinate system is located at the geometric center. These sides are 2a and this is stiffened on these edges are free, but they are stiffened by beams. We have two stiffeners on these two free edges.

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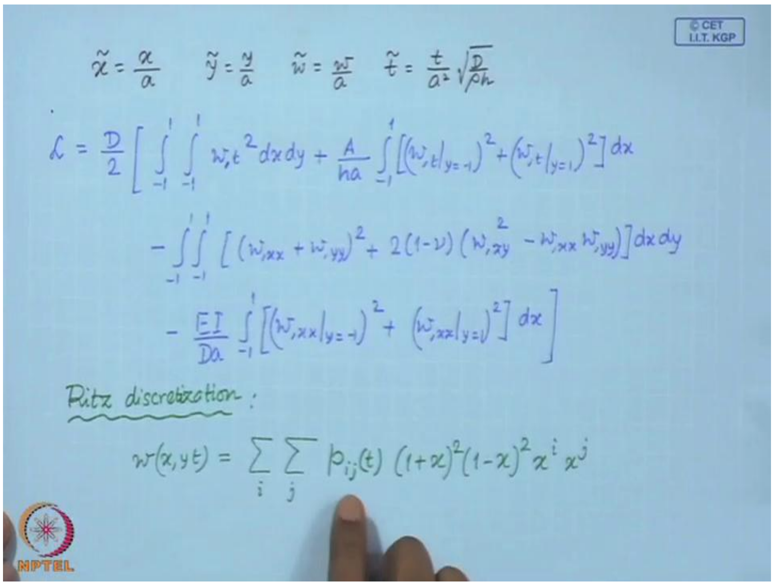

So, this is the description of the problem. Now, let us look at the kinetic energy. The kinetic energy is composed of the kinetic energy of the plate and the kinetic energy of these two beams. We assume, consider this plates to be this plate to be a Kirchhoff plate and this beams to be Euler Bernoulli beams. The kinetic energy of the plate is given by this and along with this, we have two beams on these two edges; rho a is the mass per unit length times the length times the velocity; now this velocity is a velocity of this point; so this has to be calculated at, so for this beam at y equal to minus a. We consider identical beams; so rho a is the same for both. So, that is the kinetic energy of the total system. Now the potential energy again will be the sum of the potential energies in the plate and in the beams. That is the plate part, and in addition to this we have, so everything being uniform… So this term brings in the strain energy stored in the beams. Now, once again to simplify these expressions, we use a non-dimensionalization. So, then the Lagrangian reads… This is the potential energy of the plate and this term is for the two beams. Now for the admissible functions, for the Ritz method, we will use once again the Ritz discretization. The admissible functions, this expansion is taken in the forms such that the geometric boundary conditions are satisfied. So, in this problem we have the geometric boundary conditions only on these two edges, where the displacement and the slope, they are zero. On these two edges, we have the displacements and the slopes as zero. So, the geometric boundary conditions, the functions that respect these geometric boundary conditions, these geometric boundary conditions can be written as in this form…

So, you can see that it starts with quadratic in this x at both boundaries so that the slope, the displacement is zero at x equal to minus 1 and plus 1, as well as the slopes are zero at x equal to minus 1 and plus 1. So, if you use this expansion in the Lagrangian and discredited and finally solve the Eigen frequencies, which are obtained as the non-dimensional numbers, so the first few circular Eigen frequencies are obtained like this; and the dimensional Eigen frequencies are obtained from here. Now, once you have the Eigen functions which are obtained once again from this expansion by calculating the Eigen vectors of the discredited system, you can determine the modes of vibration of the plate.

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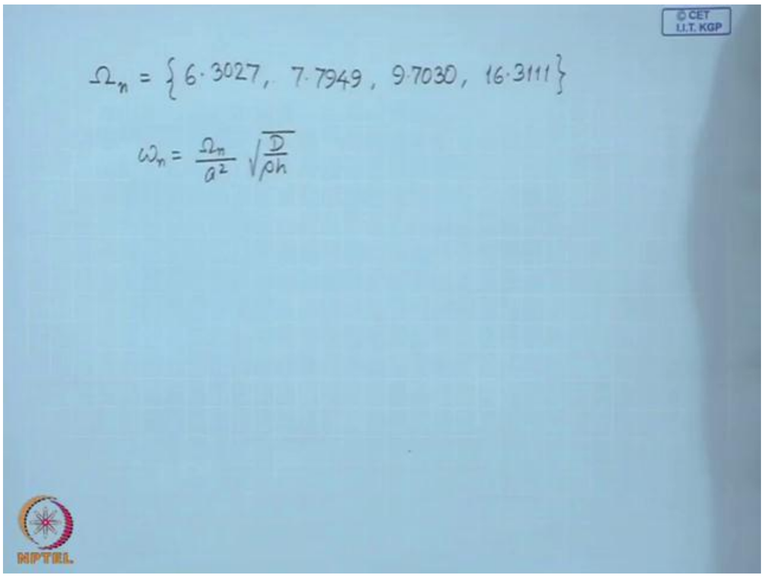

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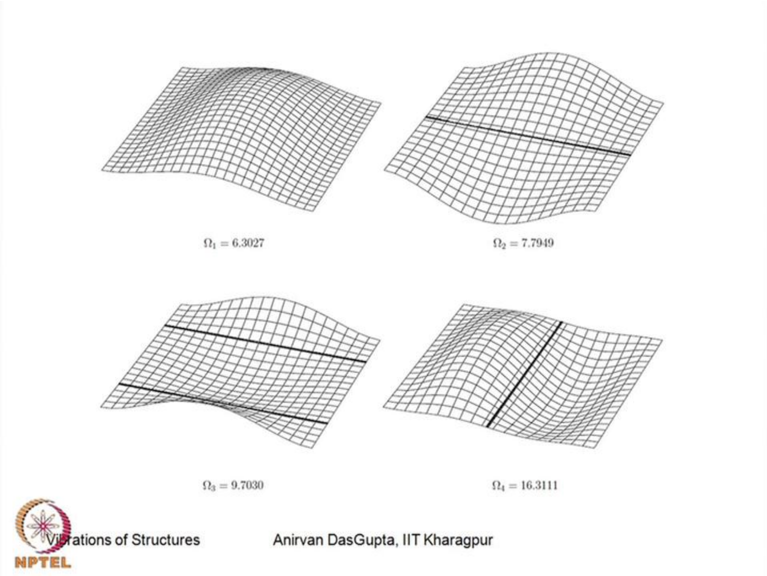

In this figure, I have plotted these first four modes of the plates. You can see these edges of the clamped edges; these edges are the clamped edges. So, you can see the displacement and of course, this displacement and the slope will be zero, because the geometric boundary conditions have been satisfied. Now here this is a stiffened edge, you can see that displacement here is maximum here it is not so high. Had there been no stiffeners then this would have vibrated like a beam, so the displacements here also would have been as high. Because of the stiffening effect of the beam, we have lower displacement at these edges. This is the second mode; this is an asymmetric mode with one nodal line at the center. Here this is the next higher mode with two nodal lines and here you have one nodal line parallel to the support. Here you can see the displacement of the stiffened edge is very low.

To summarize we have looked at some special problems, two special examples of a plate vibrations. We started off with the variation formulation of plate dynamics which can be used to derive not only the equation of motion, but also the boundary conditions. Then we have looked at square plate on a circular support and stiffened, edge-stiffened square plate. So, with that I conclude this lecture.

Keywords: variational formulation for plates, approximate modal analysis, Ritz method, circular support, edge-stiffened plate.