Vibrations of Structures

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Lecture No. # 04

Module No. # 01

Variational Formulation-I

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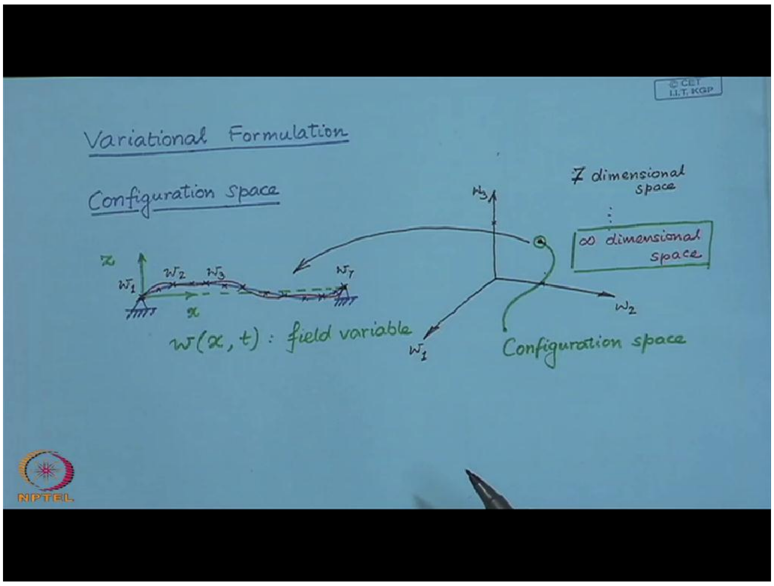

In this lecture, we are going to initiate some discussions on an alternate formulation of dynamics of continuous systems. This formulation is known as the variational formulation. Before we get into this variational formulation, we have to understand a few concepts based on which this variational formulation has been made. So, this concept is of configuration space. Now, what is this configuration space? Let us picture a string. A taut string, which has been displaced from its equilibrium position, which is the x axis. Now suppose, I want to represent or track the configuration of this string. So, the simplest thing that I can think of is, track certain particles on this or material points on this string. So, here for example, I have seven points. So, let me call them as, the displacement corresponding to this point as, w1, w2, w3 etcetera up to w7. Now, to visualize this configuration of the string at this instant, I can think of an Euclidian like space. Here what I have done is, I have taken three axes and that is the best think I can draw. w1, w2 and w3 and then there are axes like this up to w7. So, here I cannot draw a space like this with seven axes. But I will appeal to your imagination that you think of a space in which there are seven such axes; and then mark on this axes the displacement of each of these points.

Say for example, w1 is zero, w2 has a certain displacement, w3 has a certain displacement. So, like this in this space, which is now a 7-dimensional space, there is a point which represents this configuration. So, in such a space, point represents a configuration. So, this point for example, represents this configuration of the string. But suppose now this string is actually like this, then also, I mean, these seven points have the same locations. I have drawn this red configuration of the string very carefully, so that this point in the 7-dimensional space is the configuration of string. But we can see from this figure that it is not exactly the same as the previous configuration, the blue configuration.

So, what do we do? So, we increase the number of points. And like this if you go on increasing the number of points to capture more and more configurations or infinitely many possible configurations, finally what you come to, is an infinite dimensional space, which can track the configurations of the string. So finally, we require an infinite dimensional space to represent all possible configurations of the string. Then, as you can see such infinite possibilities therefore if you want to represent them, we require what are known as field variables; and that is what we have been using till now.

So, w the transverse displacement of the string from the equilibrium position is therefore represented as a field variable, which can capture all these infinite possible configurations of the string. So, this space in which a point represents any configuration of the string is known as the configuration space of the string. So therefore, you can clearly see that for a string and for any continuous system, the dimension of the configuration space is infinity. So, such systems, continuous systems, are represented as points in an infinite dimensional configuration space; and the number of coordinates of the configuration space represent the degree of freedom of the system. So, the degree of freedom of a string or any continuous system is infinity. What is very important now to move on is to remember that in the configurations space, any configuration of a continuous system is represented by a single point. So therefore, as the string vibrates, as it moves through configurations, there are trajectories in this configuration space through which the string is going to move. So, with this basic definition, let us look at the variational formulation of dynamics.

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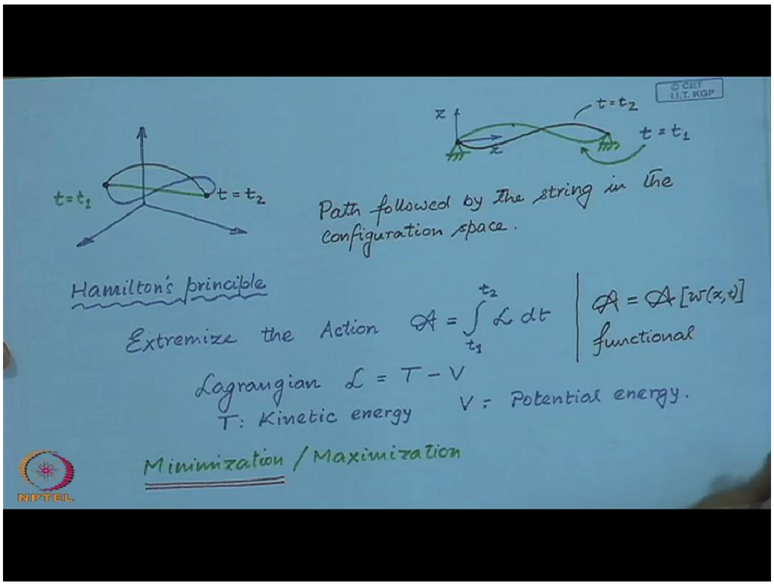

So, imagine an infinite dimensional configuration space; and at time t equal to, let us say t1, I observe the configuration of the system is represented by this point, t equal to t1. So, this point in the infinite dimensional configuration space represents the configuration of the string, let us say at time t equal to t1, which may for example, look like this. So, this is the configuration of the string that I have observed at time t equal to t1. Now, suppose I close my eyes and allow the string to move; and suppose yet at time t equal to t2, it attains a configuration like this. So, this was the configuration at time t equal to t1; and the black configuration is the configuration of string at time t equal to t2.

So, at time t equal to t2, I have opened my eyes and I observe the configuration of the string to be this, which is another point in the configuration space of the string. Now, the question is, how did the string move from this configuration at time t equal to t1 to this configuration at time t equal to t2. Well, it could have moved in this manner or it could have moved in this manner or may be this. The question is, can I say which path did the string follow in the configuration space. So, this is our question. So, is there a way of knowing? Because I am not seen how it has evolved from this state to this state. Is there a way of knowing which path it followed?

So, this question is answered by something known as the Hamilton's principle. So, what is this Hamilton's principle? So, what Hamilton's principle says is that, of all this infinitely many available paths for the system to move from the configuration at time t equal to t1 to another configuration a time t equal to t2, the one the system follows will extremize the action, which is defined as an integral from t1 to t2 of a scalar known as the Lagrangian, which is again defined as a difference of the kinetic and potential energies of the system.

So, to read it again we have observed two configuration of the string; and we have not observed the intermediate configuration, how it went from configuration 1 to configuration 2. But this principle says that the path taken by the string or the intermediate configurations attained by the string in moving from configuration 1 to configuration 2, extremizes the action which is defined in this manner. Now, what is this extremization? So, extremization has a connotation of minimization or maximization. But in mechanical systems, such as a string or a bar, this is the connotation of extremization. It is actually minimization.

Now, let us see what is made by this extremization. Now, the thing that has to be understood is this; we are talking in terms of paths in the configuration space. We are talking in terms of paths in the configuration space. So, this action is a function of this path, which is function of time. So, this action… Well, so given the motion of the string or given the path taken by the string from configuration 1 to configuration 2, which is the function of time, we have defined this action as an integral over this scalar function, which is defined in terms of those paths. Such a thing is called a functional. So, it is a function of a function. So, this action is a function of a function. So, we have to minimize over functions. We have to find a function which minimizes the action. Now, this is slightly different from something we talk about in minimization of only functions.

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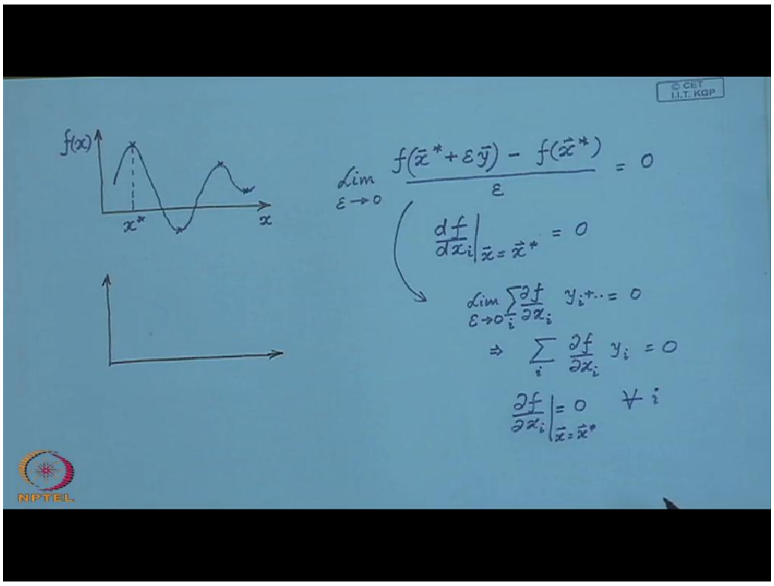

So, let us see. So, suppose we have a function f of x. How do you find out the extremem points? So, these are extremum points. So, what we say is that, suppose x star is an extremum point, in this case a maxima, then how do we detect the maxima? And how do we detect this point, an extremum point? So, we make a test. So, let us suppose that x star is a solution of an extremum point. Let us make this test. We disturb or perturb this point, this x star, by a small amount. Sere I have written this small amount as epsilon times y, where epsilon maybe a small quantity. Now, if I compare, take this difference, divided by epsilon and take the limit epsilon tends to zero, for arbitrary pertubations y; and if this turns out to be zero, then we say we have found an extremum. So, I mean this the standard definition, this is in terms of the derivative. This is what we do. But remember this x is a variable. Now, suppose we want to draw an analogy from here to understand what is Hamilton's principle. So, let me draw an analogy.

So, here by the way, this way of doing things will work even if x is a vector. If x is a vector, then let us see what happens. If you Taylor expand, then the first term… So, this is the first term. There will be further terms. So, finally, those terms will have epsilon. So, when you take this limit, they will vanish. So, this is what we will get; and now as I said that this should vanish for arbitrary perturbations y. So, this is to vanish for arbitrary perturbations y, then we must have… This should vanish for all i.

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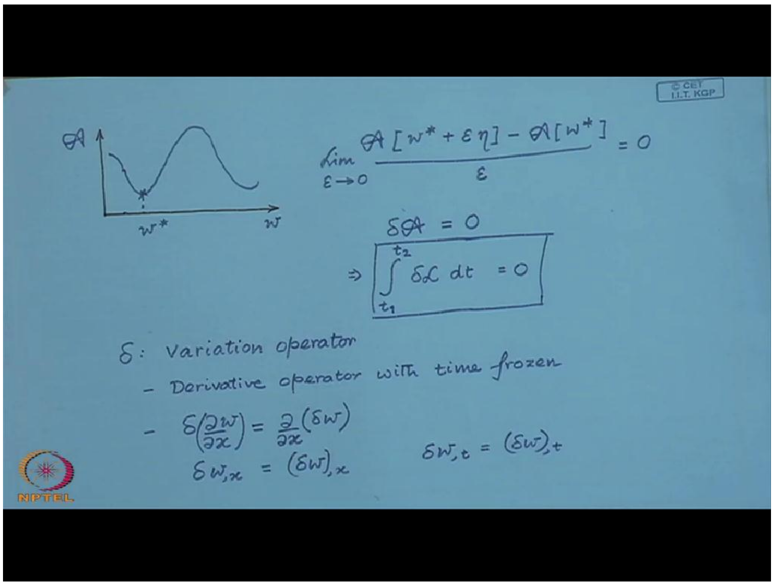

Now, let me extend this to understand Hamilton's principle. So, let we make a rough analogy, I mean, this definitely cannot be representation of Hamilton's principle per say, but with a little abuse of this notations, let me imagine that this action is being calculated for different functions. So, on this x axis now, I have different functions. So, a point here is the function and for which I am going to calculate A.

So, let us just imagine for a movement that this is what our calculations result in. So, if I have to find out the extremum points, then I must calculate this action; and I want to understand in the similar manner, what is this extremization.

So, suppose w star is the actual path taken by the particle or by the string as it moves from configuration 1 to configuration 2. So, this I can perturb with another function eta and a small quantity epsilon and take the difference, divide by epsilon and take the limit epsilon goes to zero; and if this turns out to be zero, then I say I have found a path that the string has taken to move from configuration 1 to configuration 2. Remember this action is being calculated as an integral from t1 to t2 of the Lagrangian. Now, this is represented in this manner, just as in the case of functions we say del f/del x vanishes, here we write delta of the action is equal to zero.

So, this therefore implies this condition. And this is what we will use in this formulation. Now, what is this delta? Delta is known as the variation operator. It is very similar to a total derivative operator, except that it does not differentiate time. So, time is frozen. So, when this operator is applied, it is assumed that time has been frozen. All these thing can be understood if this limit that we have calculated is analyzed. So, what we are looking at is paths. So, we will be perturbing is paths. So, time will be held frozen and paths will be perturbed. So, this variation operator is like a total derivative operator with time frozen.

The second property of this operator that we will assume is that, this operator commutes with partial derivatives, which is to say… So, suppose you have partial derivative of w with respect to x and you operate delta over this; and this can be written as… So, in our notation we will be writing… And similarly…

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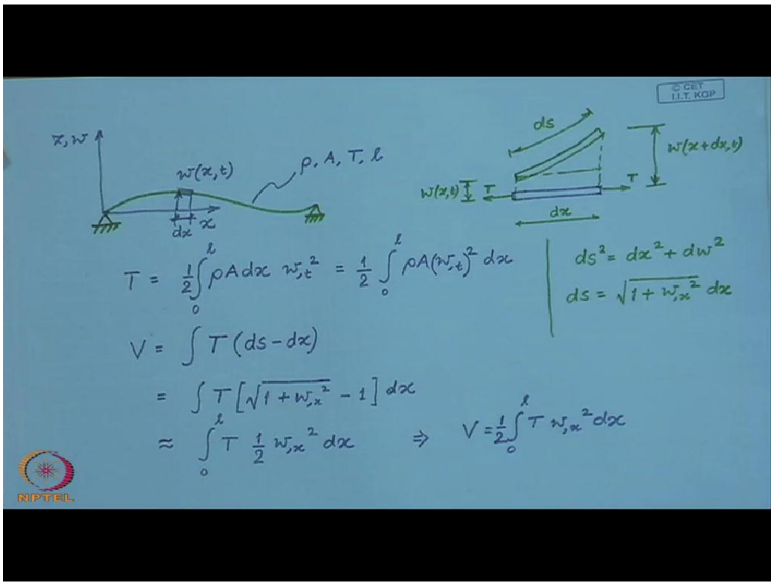

Now, let us go over to some examples and see the application of Hamilton’s principle. So, here we have a string whose transverse displacement is represented by this field variable w. We assume that this string is made of a material of a density rho, area of cross section A and is under tension T, has a length l. So, the first thing that we need to write is the Lagrangian and construct the action integral on which we will apply this variational formulation or Hamilton’s principle. So, what is the kinetic energy of this string?

So, if I consider a small element of this string at a certain location x, then the mass of this little element is rho A dx, where dx is the length of this element, this infinitesimal element. Now, with this I multiply the velocity square of this little element; and if I integrate this power 0 to l and multiply by half, then I obtain the kinetic energy of this string. When I write del w/del t, so w,t whole square. So, this is assumed that it is like this. So, that is the kinetic energy of the string. Now, we have to write the potential energy. Now, to write the potential energy, let us look at the string, an infinitesimal portion of the string as it goes. So, this is the equilibrium configuration. So, the length at the equilibrium configuration was dx, after it has been displaced its length changes to ds. Now, this change in length is taking place under a tension T, which we have assumed not to change with displacement. This was one of the assumptions of our model. So, tension of course actually changes, but then that change is assumed to be negligible. There is no actual force on the string.

So, what is this deformed length of this string? So, this is the transverse displacement at x and this whole thing is w(x+dx) at time t. So, this length can be written as…, which may be represented in this manner. So, let us at the look at work done by this tension as the string stretches. So, tension which is almost constant times (ds-dx). Now, I integrate this over the length of the string. So, that should give me the work done and which is stored as potential energy in the string. So, this…

Now, I expand this assuming that del w/del x is small, which we have assumed. So, then this turns out to be… So, this will be (1+1/2del w/del x) whole square. So, that 1 cancels of. So, here I am left with only this term. So, that is my expression of potential energy of the string.

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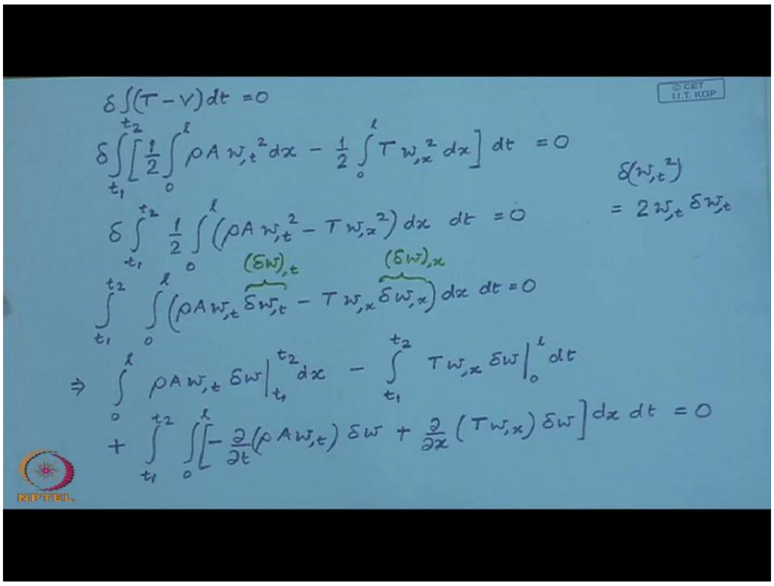

Now, let us look at the variational principle. So, what Hamilton's principle says. So, let me first write. So, this the statement of Hamilton's principle. So, which now can be written as this... And I can simplify this further in this manner. Now, I am going to apply this variational operator on the integral to obtain… So, del w/del t whole square, when I operate that will give me 2 (del w/del t) delta(del w/del t). Now, this 2 will cancel with this half, this 2 in the denominator. So, I am writing out the expression after such a simplification.

Now, this here as I had mentioned that, this variational operator can commute with this time derivative and here the space derivative, and this delta w is the small perturbation on the function w. Now, since we want to separate out this small perturbation, arbitrary perturbation, arbitrary variation over w, I would like to have something in terms of delta w. Now, to obtain that, I integrate by parts these terms, this will be integrated by parts with respect to time and this will be integrated by parts with respect to the spatial coordinate x.

So, if I do that and I assume that this integral will commute, then I can write... So, I will take this as the first function; and here this is nothing, but these expressions. So, first function time integral of the second function. So, this is the first part obtained from this term in the integral. Similarly, this is going to give me, this has to be integrated by parts over space. So, my time integral will remain. So, first function and the spatial integral of the second function. Now, minus the time derivative of this. And once again, the integral and in the same manner…

Now, if you look here, this variation over the configuration at two time instances. This variation has to be calculated at two time instances. Now, if you remember the formulation of the problem in the first place, I know the configuration of the string at these two time instance. So, there cannot be any variation. I am not trying to vary this two configurations, which are a time t equal to t1 and time t equal to t2. So, I am not looking at variations of the initial and final configurations. But I am looking at variations at the intermediate times. So, these variations must vanish.

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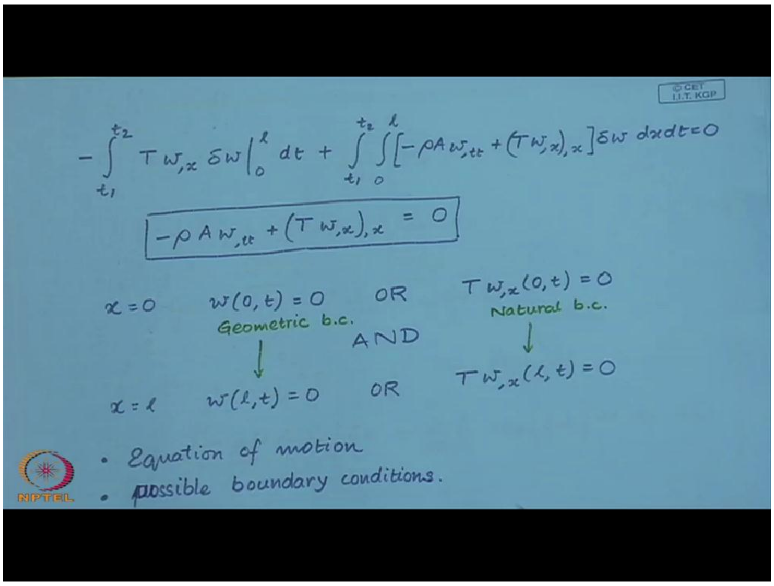

So, here I finally have… this. Now, the first term in this integral, if you see the variation is at the boundaries, delta w and all this terms will have to be evaluated at the boundaries. But here they are at the intermediate positions of the string, at all the intermediate positions of the string. Now, these two can be independently done. So, you can hold the variation over the string as fixed and just get the boundary or vice versa. So, therefore if this has to vanish, if this total expression to vanish for arbitrary variations, delta w, then I must have this integrant here in the square bracket must vanish. So, this must be zero, which gets as the equation of motion, as we can easily recognize now. And this must vanish at the two boundaries. Now, what this says is that, this can vanish in two ways, say at x equal to zero. Let us look at x equal to zero, this can vanish if delta w vanishes, which means that w is fixed; and for all times it maybe 0 or this may vanish at x equal to zero, if this term is equal to zero. Similarly, at x equal to l, this can vanish if w at l is fixed or this term is zero at x equal to l. Now, since this is being evaluated at x equal to 0 and x equal to l and both terms now must vanish. So, both these things must, so, either this equal to zero and this equal to zero; or this equal to zero and this equal to zero. So, various combinations are possible at the two boundaries. And you can now easily recognize that this condition on zero displacement, which is like a fixed end, is a geometric boundary condition, whereas this is the force condition and same thing here. So, what are the various possibilities? Possibilities are both ends of the string are fixed or one end is fixed or one end is sliding or both ends maybe sliding. So, what we find here is that, not only we get the equation of motion, but we also obtain all the possible boundary conditions.

So, when we use the variational formulation of the string or any continuous system, not only we get the equation of motion, but we also get possible boundary conditions. Now this is the extremely powerful method for formulating the equation motion of very complicated systems. Now, this variational formulation is not just an approach for finding of equations and boundary conditions. It is also, as we will see later, a method which will help us or lead to numerical methods for computational purposes; and very powerful methods have been based on these variational principles.

So, to summarize what we have discussed in today's lecture is the variational formulation of dynamics of a continuous system; and we have looked at the Hamilton's principle which forms the basis of this formulation; and before that we have looked at what are configuration space, what is the configuration space of the continuous system based on which the Hamilton's principle works. Then, we have finally taken an example of a string and derived its equation of motion and also obtained the possible boundary conditions for the string. So, we end the lecture here. We will continue further.

Keyword: configuration space, Hamiltons's principle, Lagrangian, transverse vibration of a string.