Vibrations of Structures

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Lecture No. # 39

Vibrations of Circular Plates

Today, we are going to discuss the vibrations of circular plates. So, in the last lecture we have seen the vibrations of rectangular plates. So, today we are going to discuss the circular or the polar geometry.

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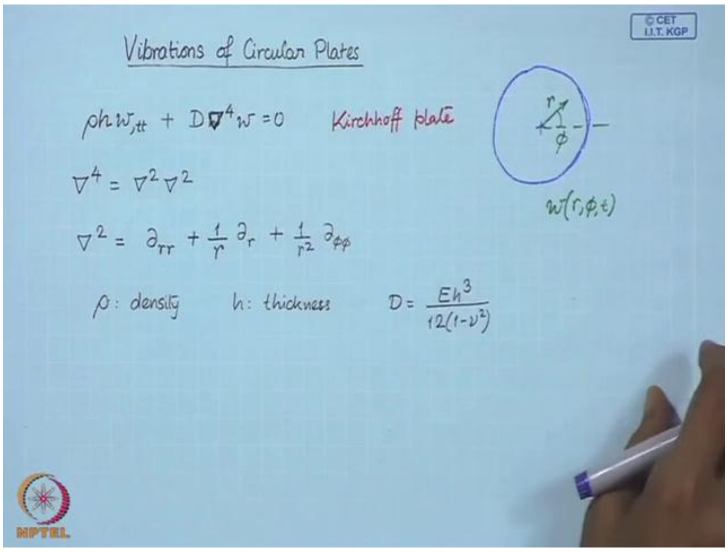

So, let us consider this plate. We are using the polar coordinate r phi. So, at time point r, phi at any time t, the field variable is represented by w(r,phi,t). Now, the equation of motion reads… So this is the equation of motion of the plate, where this nabla's square is of course square of the Laplacian and the Laplacian in the plane polar coordinates is given by this operator. So, the square of this operator is nabla power four which appears in this; and here rho is the density; h is the thickness; thickness appears here as a constant; and D… So, E is the Young's Modulus and nu is the Poisson's ratio. So this is the Kirchoff's plate model. So, we are going look at the vibration of circular plate; so essentially we are going to solve the Eigen value problem for various kinds of boundary conditions. So, let us have a look at, let us say, the boundary conditions of…

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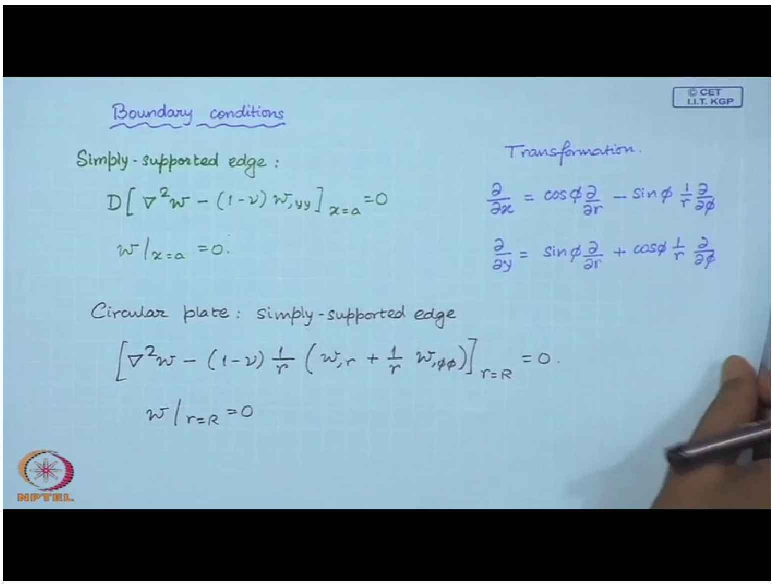

So, in the case of rectangular plates we have seen for example, for a simply supported edge, we have… So, for the rectangular case, this actually simplifies. So, this is the natural boundary condition; this along with… So for rectangular plate, this was the boundary condition. Now, this of course because there is no variation in y at x equal to a, we actually get the second derivative of y at x equal to a as zero. But when we come to the simply supported condition for the circular plate, what we can do is we can use the transformation from the rectangular to polar coordinates. So, we replace the derivatives in all these terms by the derivatives with respect to r and phi. So, this is the transformation. Now, when you use this transformation on these boundary conditions, then for the circular plate these… Now, if the periphery is simply supported at r equal to R, then this term will vanish; but still then there is a contribution from this term. So, for the circular plate, the natural boundary condition is given here; and the other boundary condition of course is a zero displacement condition.

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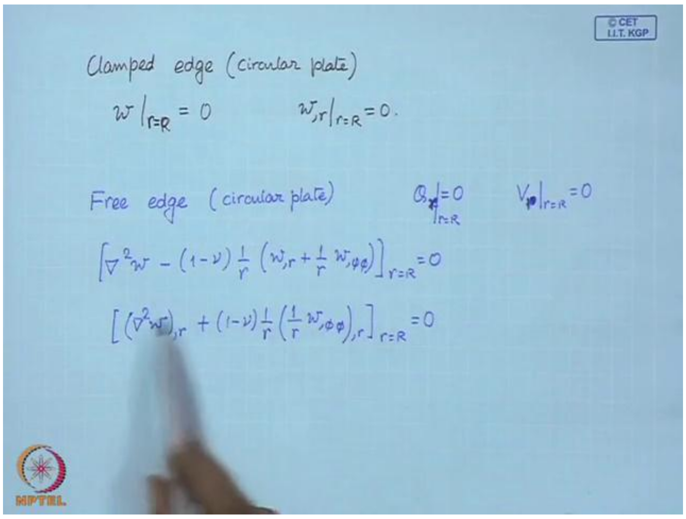

Now, if you look at the clamped case, for a circular plate then for this boundary conditions are simple. The displacement at r equal to R is zero and the slope is also zero. Now, for the free edge, again for the circular plate, you can derive this once again from the boundary conditions of the rectangular plate using the transformation. So, for the free edge we have the stress resultant because of out of plane shear equal to zero and the edge force; so that must also vanish. So, corresponding to these boundary conditions of the circular plate, they are… So, this comes from the out of plane; so this should actually be R; this comes actually from the first condition and from second condition this is the boundary condition, zero edge force at r equal to R.

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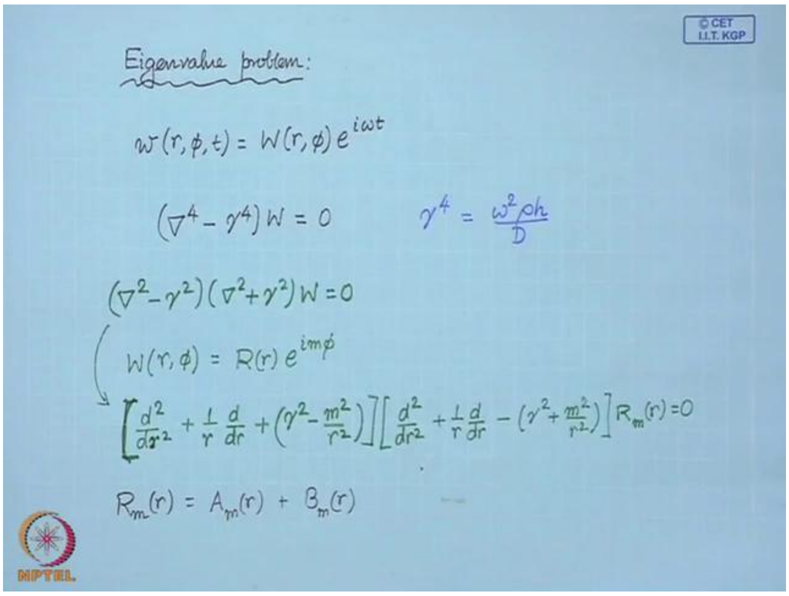

Now let us look at the Eigen value problem that we obtain when we do the modal analysis. We will be looking at the solutions of the form separable in space and time. When you substitute this solution in the equation of motion; so this is the differential equation of the Eigen value problem; along with this we will have the boundary conditions where this gamma is… Now, this can again be decomposed as we have done for the rectangular plate. We try a solution again separable in r and phi, because for a complete circular plate this function must be periodic in phi. So we already know that this then must have a structure form like this So, we substitute this solution form in here, in this equation then we can write… Now, because of this m, I can put an index m. Now, this is the differential equation. Now, this function R, we can consider as we have done for the case of rectangular plate, if you consider this function R constructed out of two functions, A is a solution of this operator and B is a solution for this operator then we can construct R using these two operators.

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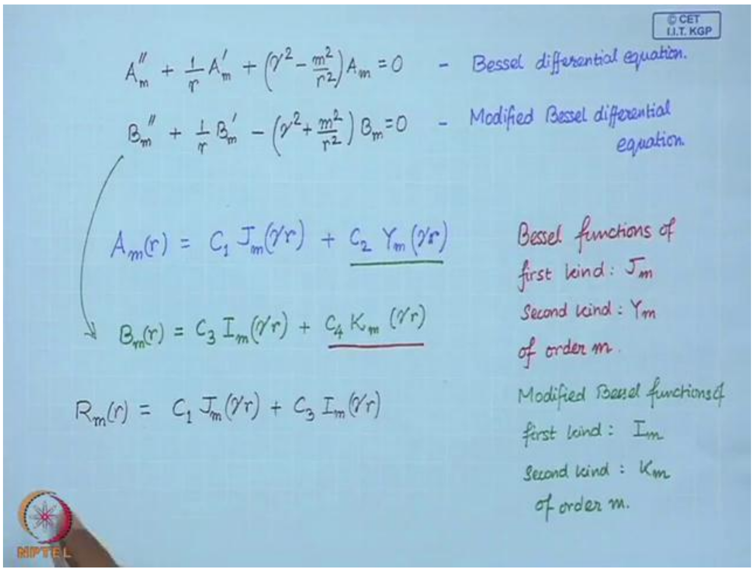

So, then what we have; so A satisfies differential equation and B satisfies this differential equations. Now, we are going to look at the solutions of these differential equations. Now, let us start with the first one. We immediately recognize this is the Bessel differential equation and this actually is the modified Bessel differential equation. So, the solution for the Bessel differential equation we can straight away write where these C_1, C_2 are constants and these are the Bessel functions of first and second kind. So, these are of order m. Now, the solution of the modified Bessel differential equation, this is written in terms of the modified Bessel functions; so these are the modified Bessel functions. So, we have come across the Bessel functions. Bessel functions J and Y and we already know that Y has a logarithmic singularity at zero argument. So, at r equal to zero it has a logarithmic singularity. So, this function cannot appear in the solution of a complete plate, complete circular plate. If it is an annular plate then this function can be present. But for a complete circular plate, this function, this cannot appear in the solution for A_m.

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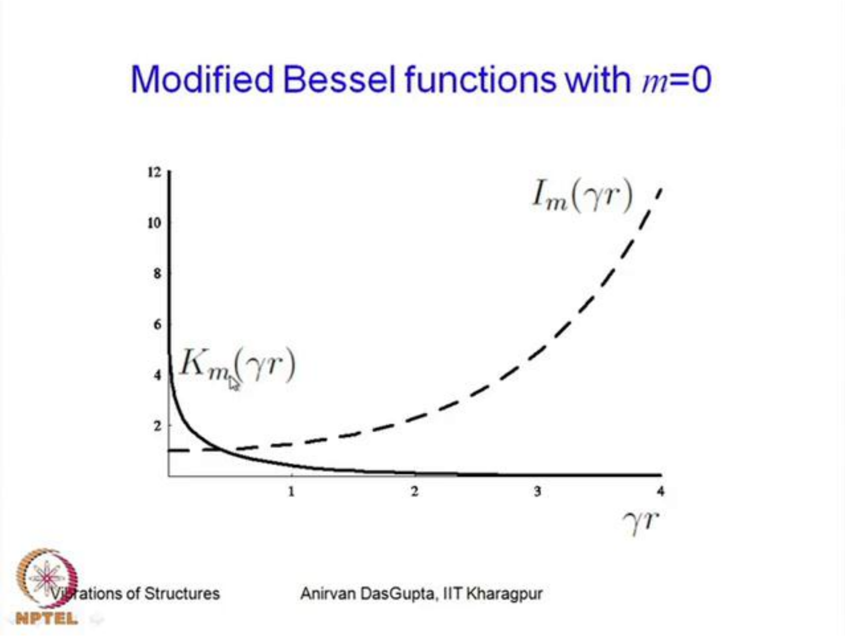

Now, let us look at these two functions, the modified Bessel… So, this figure shows the modified Bessel function for m equals to zero; so this is K zero; this is I zero. So, you can see that this K zero and all modified Bessel functions of second kind, they have a singularity again at this argument zero; and I is this dashed curve. So, in a complete circular plate again, this function cannot appear for a complete circular plate. For an annular plate, this can appear. So, since we are considering a complete circular plate, so our solution for R, therefore, is…

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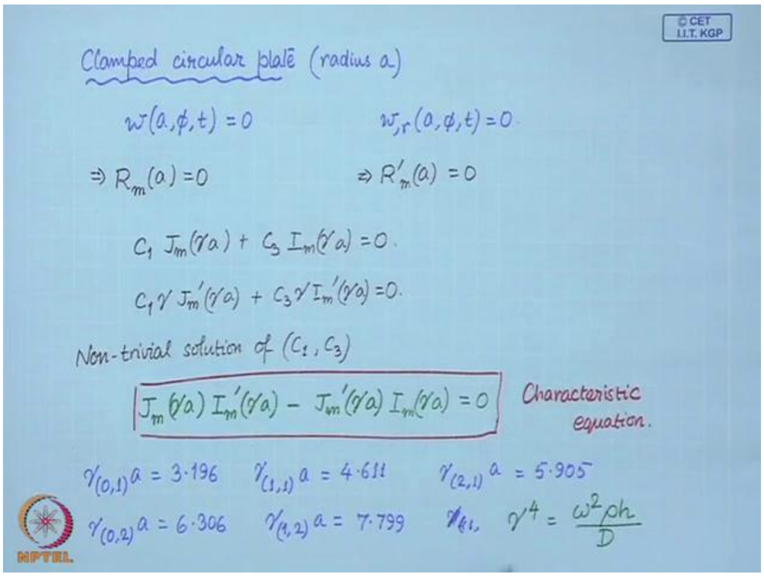

So, this is our solution for R. Now, we have to satisfy the boundary conditions in the case of a, let us say, let us consider a clamped circular plate. So, for a clamped circular plate, the displacement at r equal to a is zero and this is, the slope is also zero. So, if you use these conditions with; so R_m at a must be zero and the slope, the derivative of R at r equal to a must vanish. So, if you use these conditions, you have… and… these two equations. Now, for non-trivial solutions of C_1, C_3 we must have the determinant of this matrix; so that turns out to be… So, this is our characteristic equation from where we are going to solve for this gamma. So, gamma is in terms of the frequencies. If we can solve gamma, then we know the frequencies. This equation has to be solved numerically. If you do that then the first few values of gamma a… So, these are some of the initial values of gamma and we already know that this gamma power four is… So, if you know gamma, you can determine omega from here.

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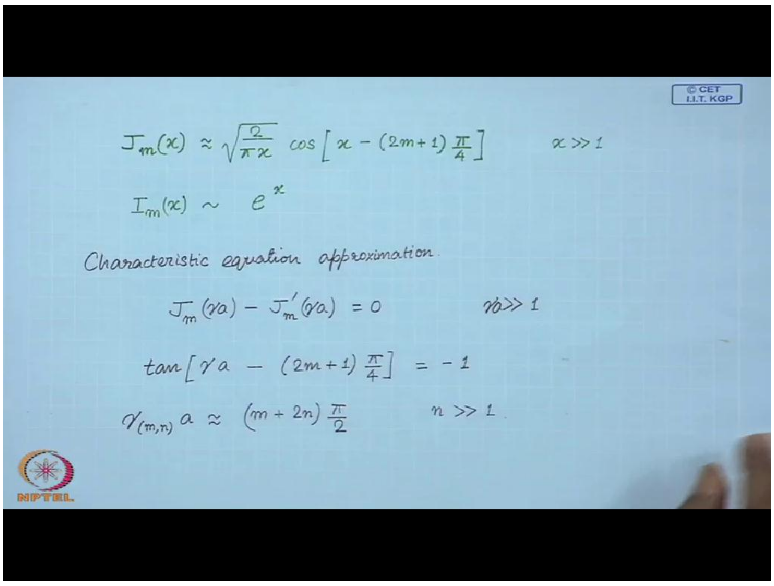

Now, this characteristic equation affords an approximation. So. as we have seen before that this Bessel function of the first kind, this has an approximation for large arguments. So, for the large value of arguments of the Bessel function, I can write… for x large. So, we have this approximation. Now, if you look at the function I, which is the modified Bessel function of first kind, then this is an increasing function. This is proportional to exponential x; so this function is dominated by this increasing function. So, slope of this is also therefore, some function of x time this exponential function. So, for large arguments of this modified Bessel function I, this derivative as well as the function itself, they are like exponential; so they can be dropped so that we can simplify the characteristic equation as… So, when gamma is very large, let me put it as gamma a is very large, now if you use this approximation, then this characteristic equation can be approximated, so again for large arguments and the solution, so the solutions of this gamma n times a… for n much, much larger than one. So, this can be used to approximate the circular Eigen frequencies of circular plate.

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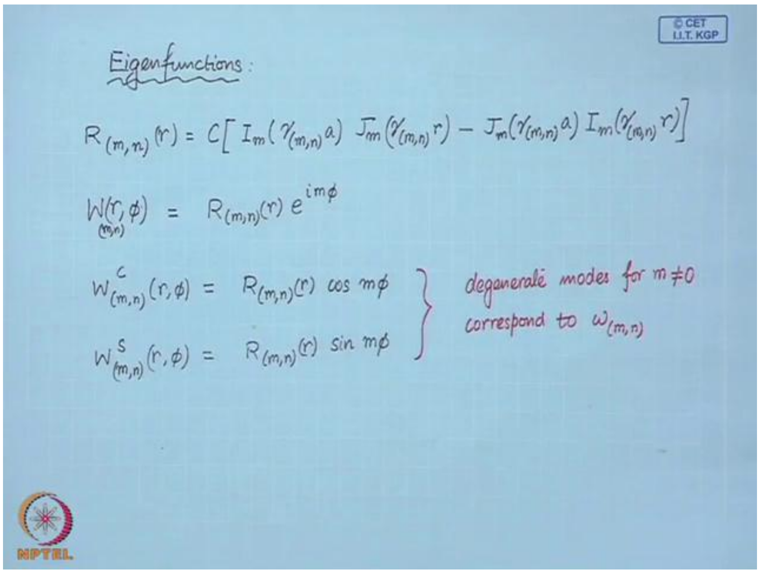

Now, let us look at the Eigen functions. So, first the radial Eigen function, it was; so initially we have an index m; now we have two indices m and n. We can solve; so essentially what we have to do is we have to solve for C_1, C_3 from these two equations and then we can write, so we can out these coefficients as some constant time J , and this coefficient as some constant time J m gamma m n into a; and we have this as I m. So, this is the radial Eigen function. Now, our original Eigen function, which also now actually be indexed, was written like this. Now, here we have the cosine and the sine now, both of them. So, we can take either the cosine part or the sine part or any linear combination of these two. So, we can say that we can have just like for the membrane, circular membrane, we have the cosine mode and we have the sine mode. Now, these two modes are of course orthogonal; if you integrate the product of these two then that you can very easily see. So, these are orthogonal and these Eigen functions, they are orthogonal and they correspond to the same Eigen frequency omega (m, n). So, these are degenerate modes for m not equal to zero. So both these modes, they correspond to the circular natural frequency omega (m, n). So, we have; so m equals to zero is axisymmetric mode; so with m equals to zero, the Eigen function becomes only a function of r, no dependence on phi; so that is the axisymmetric mode. When mis not equal to zero, then we have the unsymmetric modes.

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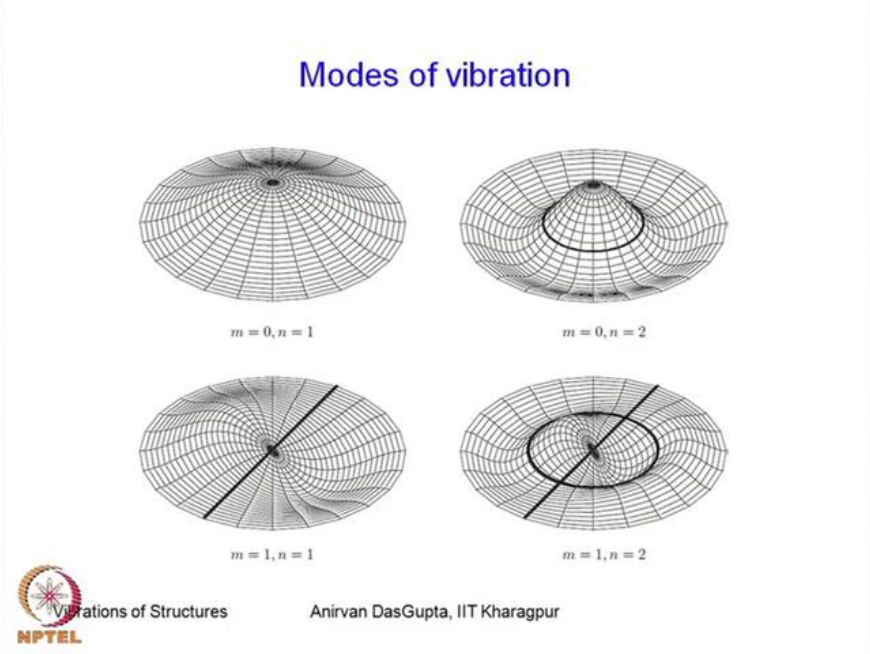

Let us now look at these modes of vibration of clamped circular plate. So, this is the first mode; so m equals to zero in this case. So, this is axisymmetric and similarly this is axisymmetric; this is the fundamental, the first mode. So, this has no nodal curve, nodal diameter or nodal circle, whereas this is the mode with one nodal circle. Now, this is (0, 2), but this is not second mode as such; so you can see here that (0, 2); so this is (0, 1) and this is (0, 2). So, this is the fourth mode. Now, we also have the (1,1) and (1, 2). So, these are the second and the fifth modes. So, on this figure once again you can see that (1, 1); so, this is a mode with one nodal diameter; and (1, 2) has 1 nodal diameter and 1 nodal circle. So, these two are unsymmetric modes. So, they have modal degeneracy. This is because you have one cosine mode and one sine mode, which will actually rotate this nodal diameter by pi by 2. So, if this is the cosine, then the sine one is with a nodal diameter orthogonal to this. So, once again we can understand this from the geometric symmetry or isotropy of this circular plate. Since there is no particular choice of any reference line from where phi can be measured, so, this reference line is actually arbitrary for a perfectly circular plate. So, you have this degeneracy. So, you have one cosine mode and the other sine mode.

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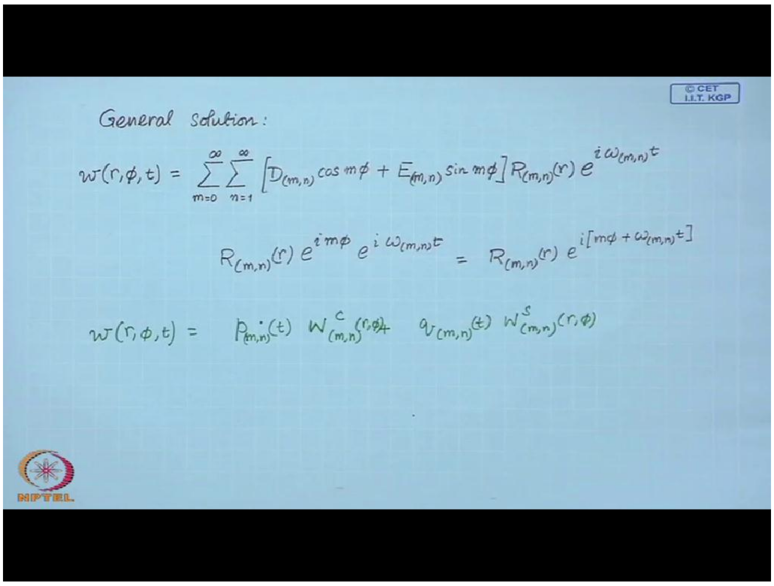

Now, you can write down, using this Eigen functions, you can write down the general solution. So, here we can have combination of cosine omega (m, n) t and sine omega (m, n) t. So, in general we can have linear combinations of the cosine and the sine omega n t terms. Now, you see we had this solution; we have this complex form of solution. So, this can be written as… So, because of this modal degeneracy, you can have travelling waves in the circumferential direction as this solution tells us. Also you can have depending, that depends on the initial conditions you can have completely separable solutions like this. Now, this occurs, because of this again modal degeneracy. So, if you look at the solution, the cosine solution and the sine solution they can come in arbitrary combination. So, for example, I can write… So, you can write the solution like this. So, this is of course… So, this is a cosine mode; this is a sine mode. Now, just as in the case of the circular membrane or the membranes with modal degeneracy, we have discussed that the degenerate mode, since they have the same frequency, the nodal structure, the nodal curves in this case, the nodal lines or the nodal diameters and the nodal circles, they need not remain steady. So, for example for the mode with one nodal diameter; so, if you have that the function p, as I wrote here, supposes cos and the function q as sin, then you can see that this nodal diameter will actually be rotating. So, there is no study kind of solution will be observed as we expect for in the case of a modal solutions. So, it might look unsteady. This is because of this modal degeneracy that is present in the case of the circular plate.

So, finally to summarize we have discussed today, the vibrations of circular plates. We have looked at some of the boundary conditions. Now, for the circular plate, they happened to be more complicated than the case of rectangular plate. Now, we have looked at some of these boundary conditions, and we have looked at the modal analysis of clamped circular plate. So, with that I conclude this lecture.

Keywords: circular plate vibrations, modal analysis, modified Bessel functions, modal degeneracy.