Vibrations of Structures

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Lecture No. #38

Vibrations of Rectangular Plates

Today, we are going to discuss the vibrations of rectangular plates. In the last lecture, we had seen the mathematical modeling of plate vibrations under small transverse displacements.

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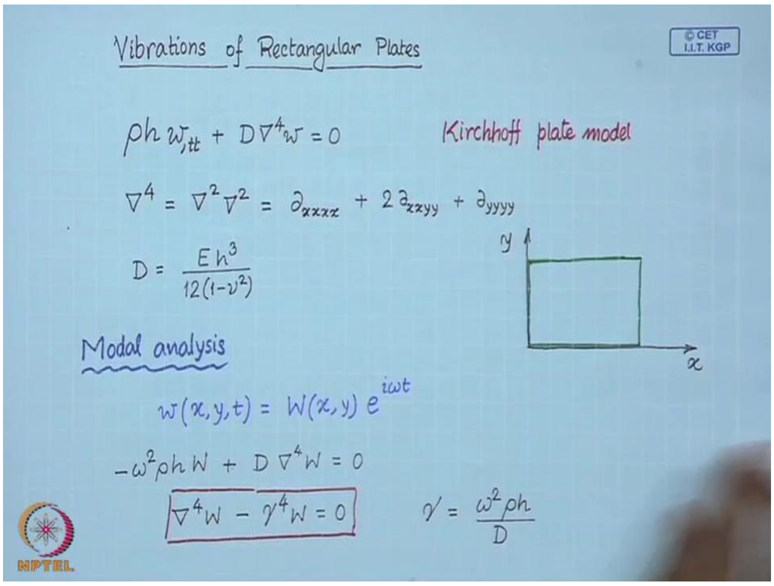

So, today, we are going to look at the vibrations of rectangular plates. Let us recall that the equation of motion, for Kirchhoff plate. So, this is the Kirchhoff plate model, where this nabla 4 is the square of the Laplacian, and that turns out to be in this form; And D is in terms of the Young's modulus E, thickness cube, thickness of the plate h cube divided by… where nu is the Poisson ratio. Here of course, rho is the density of the material of the plate.

This is a Kirchhoff plate model for a plate with constant thickness. Now along with this we will have of course the boundary conditions, which we will look at as we proceed. So, we are interested in the modal analysis. We will be a looking for solutions. So, we are looking for solutions for rectangular plates. This is in the Cartesian coordinates. So, we look for separable solutions, space time separable in this form. So, suppose we have a plate lying in the x y plane and the displacement the field variable w is measured in the transverse to this plane which means a perpendicular to the plane of the paper. So, if you substitute this solution form in the equation of motion, we can write this as… and we will make a redefinition of; so, we will rewrite this as… where we have defined this gamma as omega square, rho times the thickness divided by this constant D. This is our differential equation of the Eigen value problem. The complete Eigen value problem description will also have the boundary conditions along with this differential equation. Let us look at this differential equation first.

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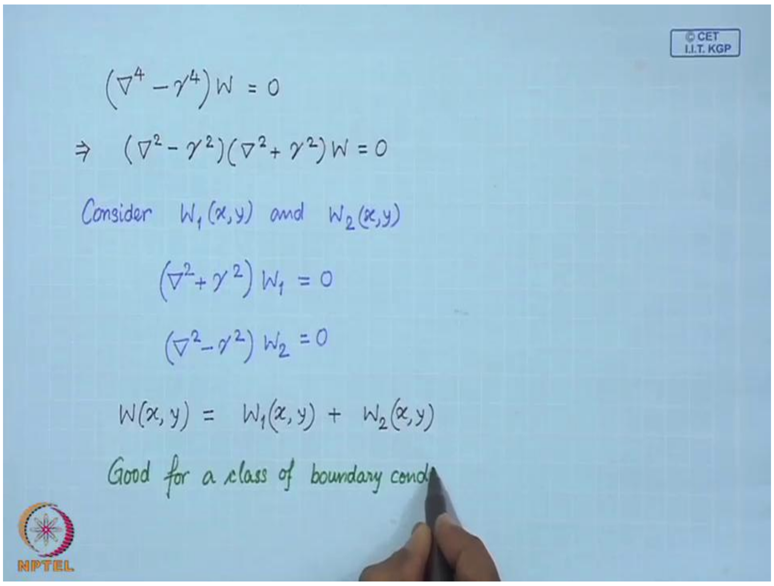

We have, I can write this as in the equation of motion, this operator in the differential equation of motion; I can write this as nabla power 4 minus gamma power 4 operating on W, and that is zero; and I can factorize this in this form. Now, these two operators they commute. So, this can operate first or this can operate first; it does not matter in which order. Then if I consider two functions such that: this operator operating on W_1 is zero and W_2 is such that… So, these two functions satisfy these differential equations. Then I can say that the solution W can be written as a combination of W_1 and W_2. So, this can be very easily checked that if you construct solution like this, then this is going to satisfy our original differential equation of the Eigen value problem. Now, but it is so happens this is as we will discover very soon that this is not the most general solution structure, that is possible. So, this solution is valid or good for a class of problems. So, by class of problems, I mean a class of boundary conditions. So, this structure can be used to satisfy a class of boundary conditions.

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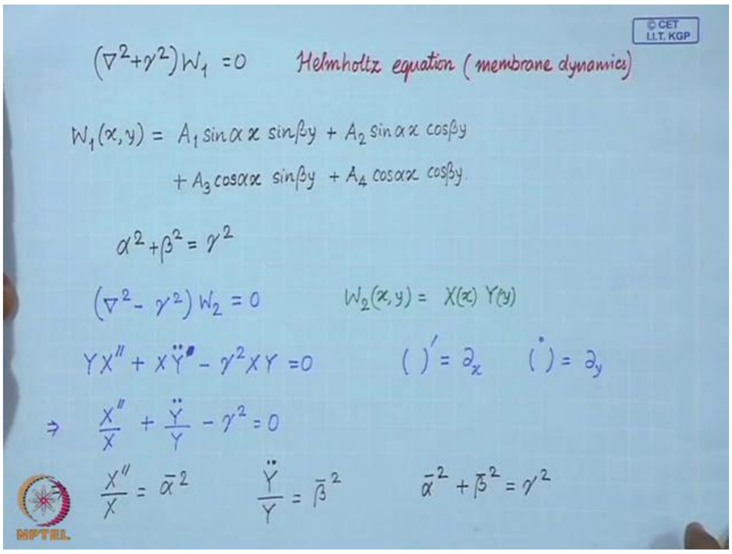

Now, let us look, let us search for the solutions of this class; then we have to look one by one at these two differential equations. So, let us first take this differential equation, which… so, this differential equation. Now, this differential equation is also known as the Helmholz equation, which we have also come across when we discussed dynamics of membranes. So, when we studied the Eigen value problem for the membrane, we have encountered this differential equation; and the solution, we can recall, is in this form. So, the general solution of this differential equation is…So, that is the general solution of the Helmholz equation, where these alpha and beta, they satisfy the condition that alpha square plus beta square must be equal to gamma square; that we have already discussed. Now, let us look at the other differential equation. So, this nabla square minus gamma square operating on W_2 is zero. Now, let us look for the solution with the structure, which are separable in x and y. So, if you substitute this solution form here, then I can write… So, let me indicate this by dots, so what I have used is; this is del del x or d d x and dot indicates d d y. So, if I divide this equation throughout by x y then, I have this structure of the equation. Now, you see that this term is only function of x; this term is only a function of y; and this is a constant. So, for arbitrary x y, if this equation has to be satisfied then each of them must be constants. So that would imply… Let me indicate this constant by alpha bar square and y, this constant beta bar square. In that case, what I have is alpha bar square plus beta bar square must be gamma square.

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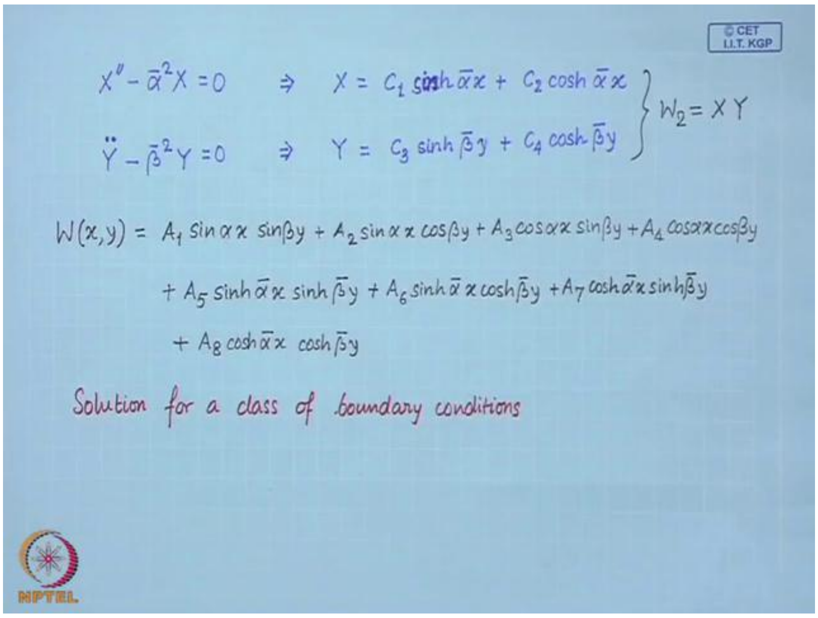

So, now let us look at these differential equations for capital X and capital Y. So, these differential equations they read… so double derivative of capital X minus alpha bar square x must be zero and we know that the solution of this differential equation may be written as… Let me write this as, first term as a sine hyperbolic… Similarly from the second equation… so the solution of this can be written as C_3 sine hyperbolic beta bar y plus C_4 cos hyperbolic beta bar y; and this of course with a condition that alpha bar square plus beta bar square equal to gamma square. Now, then let me write down the solution so far. So, our W is W_1 plus W_2; therefore we have all these terms. Now, you see the actual solution of this W_2, so W_2 is X multiplied by Y; so now this product I can write, if C_1 C_3 I write as A_5, then thus is A_5 sine hyperbolic alpha bar x sine hyperbolic beta bar y, then I can… Now, you can see here that we have product of the trigonometric functions in x y direction and product of the hyperbolic functions separately again in x y directions. In this class of solutions, there is no product of trigonometric and the hyperbolic functions. So, this solution, we can intuitively have an idea that this is not general enough. So, but this solution; this nevertheless is the solution which can solve a class of problems. So, we can use this solution for a class of boundary conditions. Let us look for certain examples.

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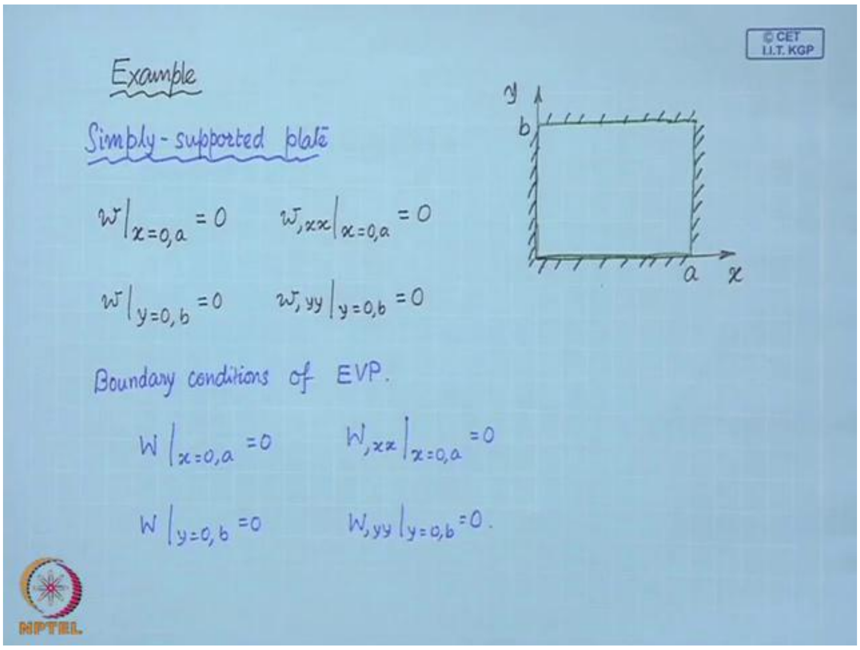

So, the first example that we are going to take is that of a simply supported; so let us consider a plate which is simply supported on all four boundaries. So, let us consider this as a, and this a s b. So, the boundary conditions for this plate, as we have seen in our last lectures; so the boundary conditions, so at x equals to zero, which means this edge, and at x equals to a, which means this edge, the displacements are zero; and since they are simply supported, the moments are also zero; and the moments I this case happen to be double derivative of w with respect to x. Similarly at these two edges, the displacements at y equals to zero and y equals to b that means at these two edges, displacements are zero, and bending moments are also zero. So, these are the boundary conditions for the simply supported plate. Now, the corresponding boundary conditions for the Eigen value problem, so they can be easily determined from here. So, these are the corresponding boundary conditions for the Eigen value problem.

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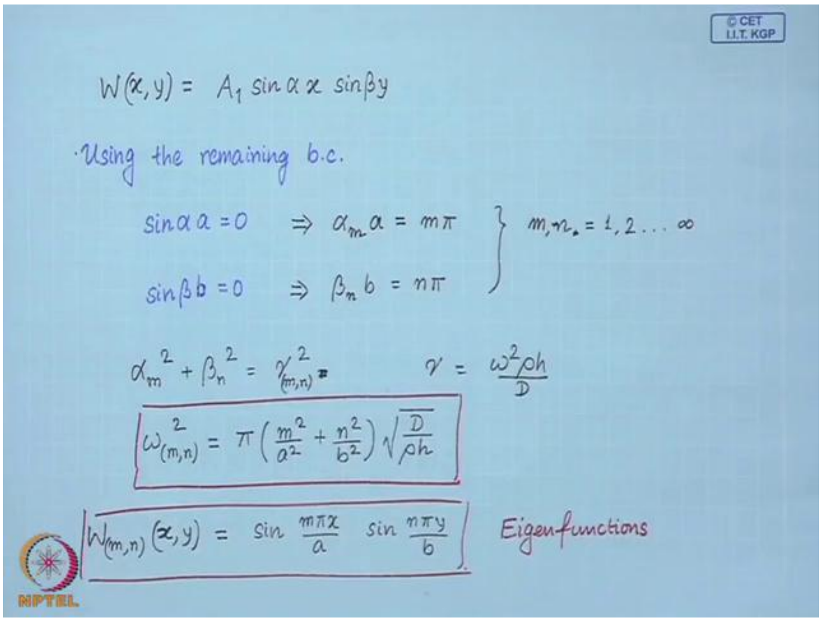

Now, let us look at the solution and these boundary conditions. So, this was our general solution, and these are our boundary conditions. Suppose when x equal to zero, these terms vanish. So, we are left with these four terms. Similarly, when you take double derivative with respect to x and looks at x equals to zero, then again you will find that those terms will vanish. Similarly, when you consider y equal to zero, the displacement and curvature, double derivative with respect to y; so if you look at all these conditions that you obtained, then finally using these conditions you will come to the conclusion that the solution will boil down to… Now, here we have used the conditions at x equal to zero for the displacement and the double derivative of the displacement; and at y equal to zero the displacement and the double derivative of the displacement; so, we have some further boundary conditions, which we must satisfy with these equations. Now, then if you use those conditions, then so the remaining boundary conditions; so if you use the remaining boundary conditions, then you will find, you will obtain these conditions. So, for example, when you use W at x equal to a equals zero, then sine of alpha a for all y must be zero; and when you take the double derivative with respect to x, then again you will get sine of alpha a equal to zero. So, then you have this condition; and you have another condition imilarly sine beta b equal to zero; this is for W at y equal to b and W double dot at y equal to b vanishing. So, that will give us this condition. So, that implies, the first condition implies alpha, now this get indexed, because there are countably infinitely many solutions; so alpha m times a must be equal to n pi; and from here, another index; so m and n can have values from one to infinity. So, if you recall the definition of, the constraints on these alpha and beta, that must be equal to gamma square; and if you look back, then this gamma square, so you have this gamma defined as… so this gamma is now also get indexed because of this m and n. So, omega is also get indexed as m and n, so omega m n. If I now use these expressions of alpha m and beta n, so this is m pi over a, and… So, that is what we are going to obtain as; so, this is; so, omega m n square is given by this. Now, so we obtain the circular natural frequencies of the plate, and the Eigen functions are obtained from here, they also get indexed. These are the Eigen functions.

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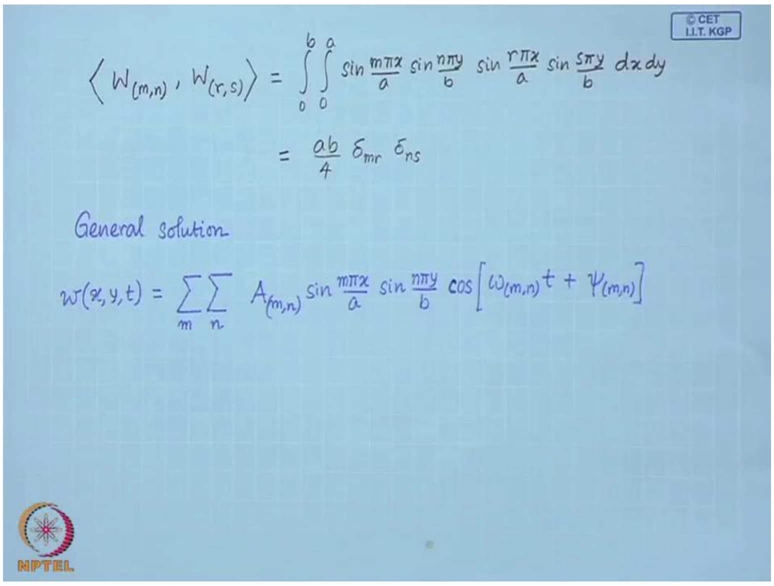

Now, you can quickly see that these Eigen functions satisfy the orthogonality conditions; so that is a b over 4. These are the Kronecker’s delta functions; they take the value one when the two indices are equal. These Eigen functions, they are orthogonal and the general solution, we can write down the general solution of the plate using these Eigen functions, where I have converted this to an amplitude and phase form, the temporal function. So, we can have this general solution. Now, let us look once again at these Eigen functions. So, these are, we have obtained these Eigen functions even for membranes; and the modes look very, the modes of vibration are the same for the simply supported plate and that of the membrane. Now, let us look at another example of the plate with mixed kind of supports; so on two edges we will consider simply supported edges and the other two opposite edges we will consider as clamped.

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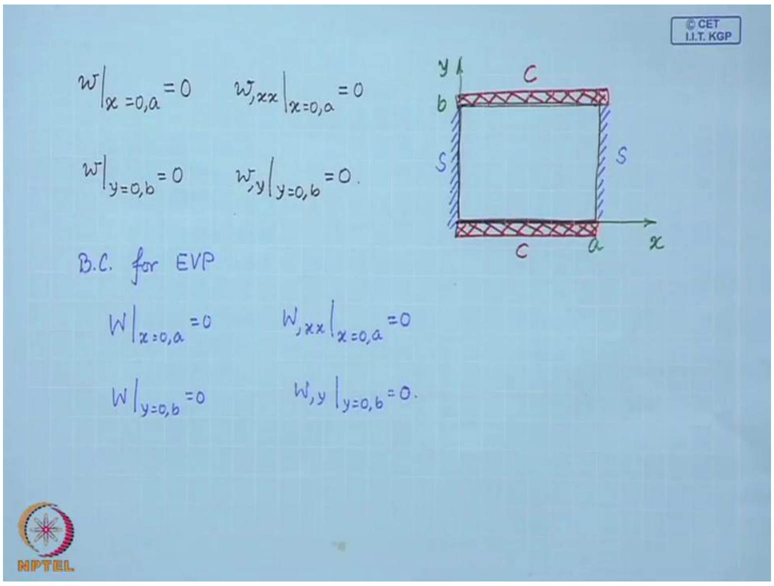

So, in the x direction, the length is a, and here it is b again. Now, we will assume that this side is simply supported; and these two sides are clamped. So, these two are clamped; and these two edges are simply supported. So, the boundary conditions for such a plate, with such boundary conditions, the mathematical representations are the displacements at x equal to zero at a, they must be zero; the moment these also must be zero at the simply supported edges. For the clamped edges, we have the displacements at y equal to zero and at y equal to b; and slopes at these two edges as zero. So, corresponding to these boundary conditions, the boundary conditions for the Eigen value problem… Now, if you look at these boundary conditions and also look at; so these are our boundary conditions now; and the kind of solution that we had here; so if you now use these boundary conditions for this solution, then you will find that this solution can not satisfy this set of boundary conditions. That can be checked. You have here at y equal to zero and once with single derivative of y. So, this structure being special, these boundary conditions can not be satisfied by this class of solution. So, we have to start a fresh for these boundary conditions; this set of boundary conditions, we have to look at the Eigen value problem at a fresh.

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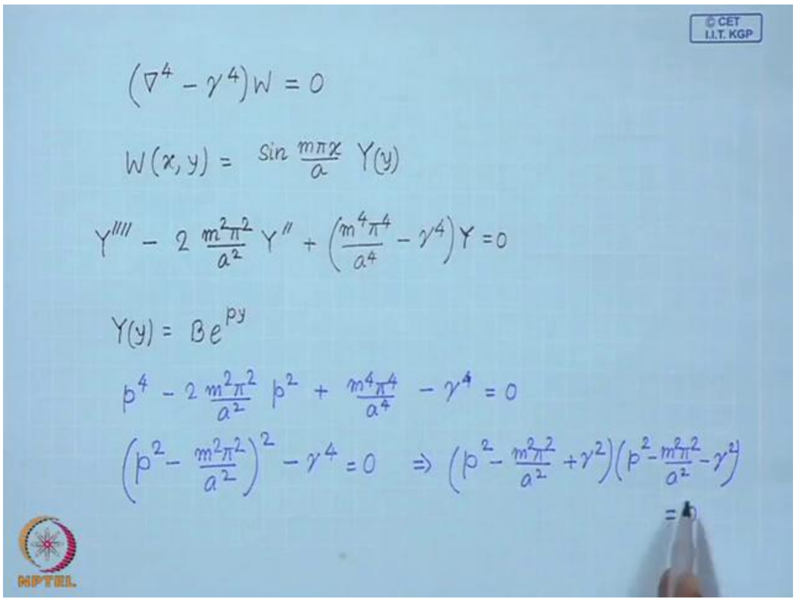

So, let us look at. So, this was the differential equation of the Eigen value problem. Now, here we have simply supported edges at x equal to zero and at x equal to a. Now, let us try some solution which already satisfies these two boundary conditions, at x equal to zero and a, the displacements and moments being zero; and we know that sine m pi x by a satisfies these four boundary condition, two on each edge. For the y coordinate, let us have this unknown function, function as yet unknown, Capital Y of y. Let us try this solution in the differential equation of the Eigen value problem. If you substitute in here and make some simplifications, then… So, this is what you are going to get. Now we can try a solution for this; let us say, if you try a solution like this, then p we will have as… You can see that this differential equation, so, this will reduce to… This can be decomposed as… So that implies that must be zero.

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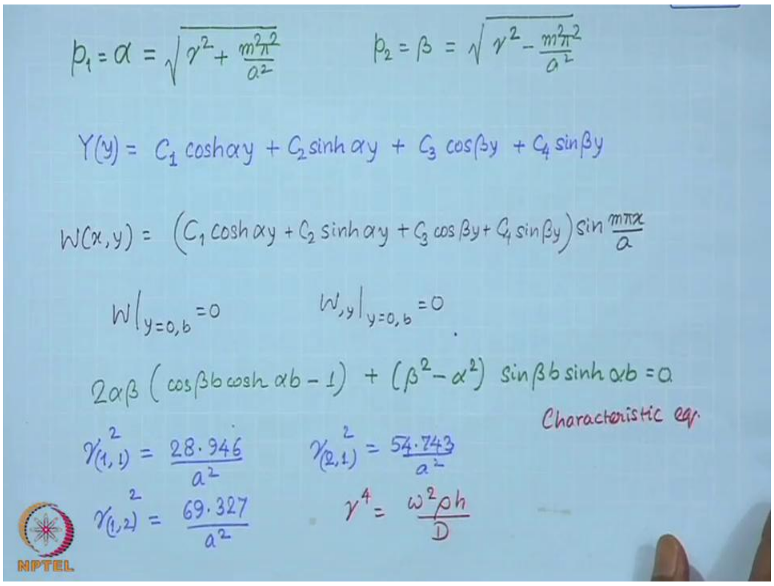

So, we have two solutions of p. Let me call them as; let me name these solutions of p as let say alpha, p_1 is alpha equals; so, let me consider this, gamma square plus… and p_2, I call that as beta. So, we have these two solutions and correspondingly we can write Y as… So, if you define beta in this form and alpha in this form, then I can define the solution… So because of this definition of beta which is; you have both real and imaginary solutions of for p and by defining p_2 in this form, I can write this in terms of trigonometric functions. So, then my solution stands as… Now, we have satisfied four boundary conditions by choosing the function in x; we are left with these four boundary conditions as yet. Now if you substitute in this solution form, in these boundary conditions finally, you will get the characteristic equation .For non trivial solutions of C_1 C_2 C_3 C_4… So, this is our characteristic equation. If you solve this equation numerically, and you already have the relation between gamma and the frequencies; so, you can find out the natural frequency. Now this… So, I have written other first three modes and if you look at this definition, this is also square. So, gamma is, so gamma power four is… So, from here you can determine the natural frequencies.

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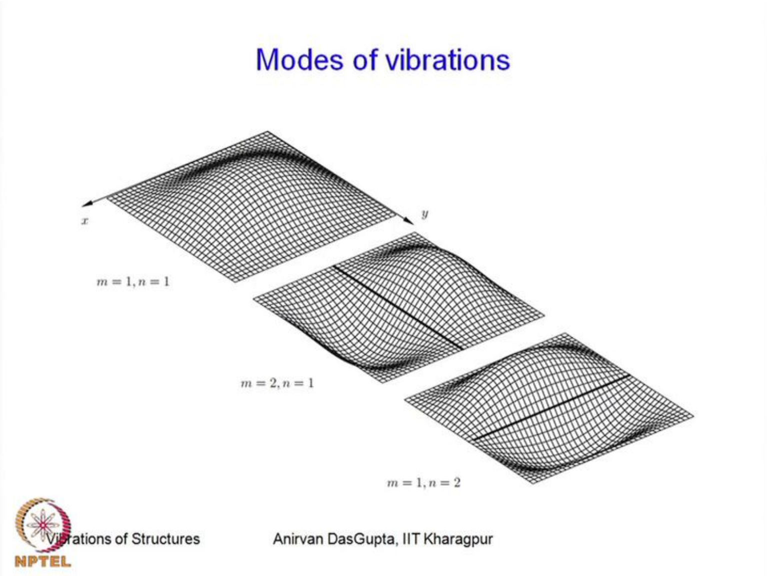

Now, this figure shows the first three modes of vibration of this plate. So, in the first mode there are nodal lines; whereas the second, and the third mode, they have these nodal lines. Now, here you can see these two are the clamped edges, so the slopes are zero; whereas, these are simply supported edges. So, to recapitulate, we have today discussed the vibrations of rectangular plates; and we have seen that the solution is, determining the solution is little complex. We have looked at two kinds of boundary conditions or two classes of boundary conditions; and we have solved these problems; and determined the Eigen frequencies and the modes of vibration. With that I conclude this lecture.

Keywords: rectangular plate vibrations, boundary conditions, modal analysis.