Vibrations of Structures

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Lecture No. # 37

Dynamics of Plates

Today we are going to discuss the dynamics of plates. So, we will be, in the next few lectures, we are going to actually discuss the vibrations of plates. So today we are going to initiate some discussions on the modeling of plates. Now, what is the plate? So, we have seen that membrane is a two dimensional continuum which does not transmit any bending moment. Now when you think of a plate, it does transmit bending moment or it can resist bending moment; so, which means a plate is a two dimensional elastic continuum which resists or transmits bending moment. So, in order to; so, first where do we find plates? So, plates are found in various machines in civil structures etc. So, we are interested in first the dynamic model for; we are interesting setting up the equations of motion, modeling the dynamics of plates. So, as it happens with any dynamic modeling, we make some simplifying assumptions so that we can have two dimensional theories for plates, so a continuum in two dimensions. So, what assumptions to be make to simplify our models?

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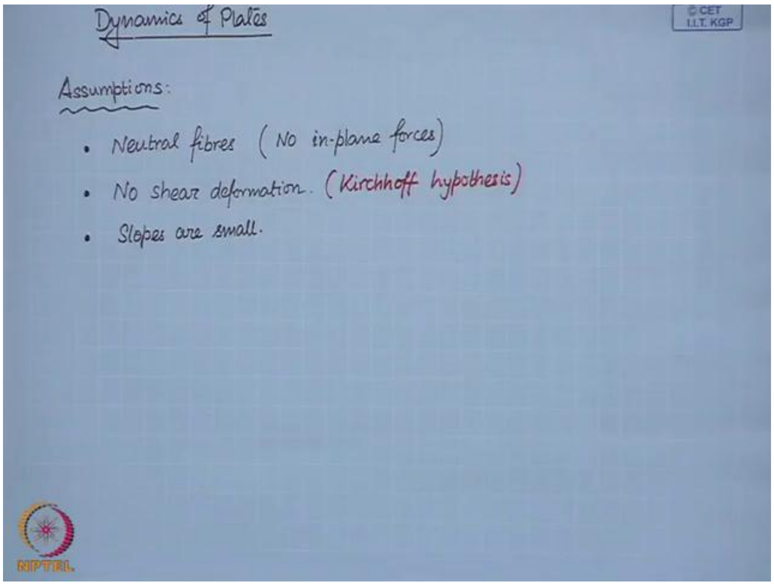

So, the first thing that we assume, we assume that there is a plane or there are fibers which are unstressed; so these are called neutral fibers. So, we assume the presence of neutral fibers which occurs if you do not have, you do not have in-plane forces in the plate. So there is a, so the plate is not subjected to any in-plane forces, then you have and when the plate undergoes transverse small transverse vibrations then there are fibers which remain unstrained. The second assumption we make is that there is no shear deformation; so this is known as the Kirchhoff hypothesis. So, this corresponds to the Euler-Bernoulli hypothesis for beams. The third assumption we make, to keep our model linear, is that the slopes are small. So, when the plate deflects the slope because of this transverse deflection they are small. So we are going to model under these assumptions.

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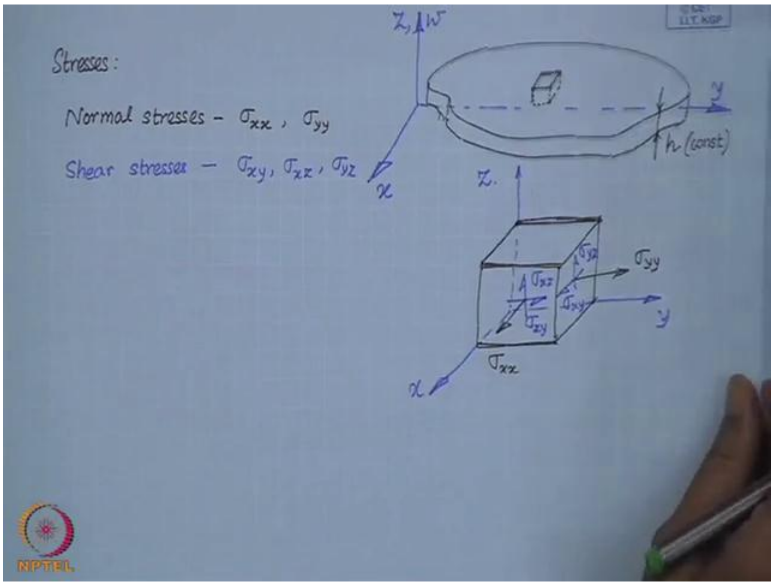

So, let us first look at; so, let us consider a plate; and we look at a little element in this plate. So what are the stresses on this element? We will make the further assumption that the plate thickness is constant. So this plate is lying in the x y plane, and its deflection is transverse to this plane. Now we consider that the stresses that are acting are the normal stresses; so we have stresses sigma x x, sigma y y; these are normal stresses; and we have shear stresses, so sigma x y, sigma x z and sigma y z. So, I am showing the stresses only on these two surfaces. Now there can be a distributed force on the plate, but at present we are going to drop that. So, essentially we have only these stresses which are non-zero. Now when we want to construct theory in two dimensions, then we integrate over the thickness of the plate. We integrate over the thickness of the plate; and we define what are known as the stress resultants; the force and moment resultants also known as the stress resultants.

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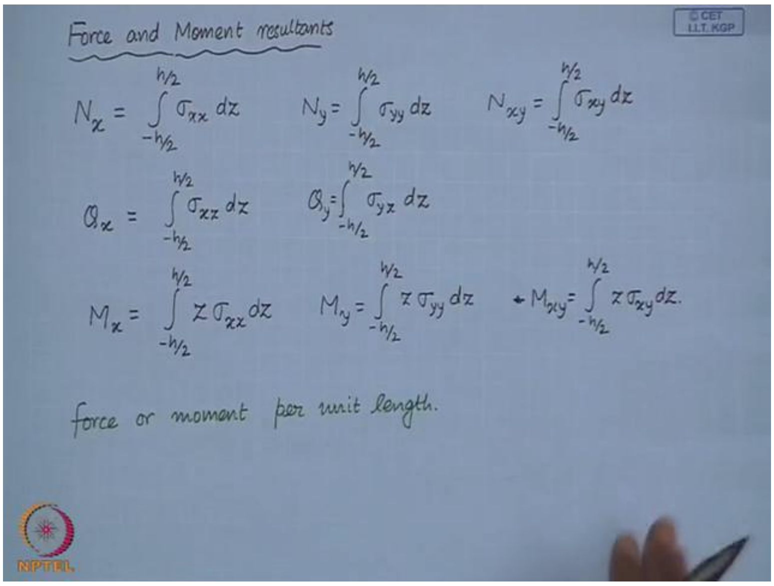

So, the stress resultants due to the normal stresses, so these are because of normal stresses, and this is because of the shear stress, so this is in-plane, and then we have the out of plane shear stresses, which we denote by capital Q_x and capital Q_y; and we have the moment resultants, because of the moments due to the normal and shear stresses; so these are the in-plane. So we have defined here M_x as z times sigma x x, but actually this moment is along the y axis, but still we call it M_x because it is a moment with sigma x x. Now these resultants, they have the units of force per unit length, or moments per unit length.

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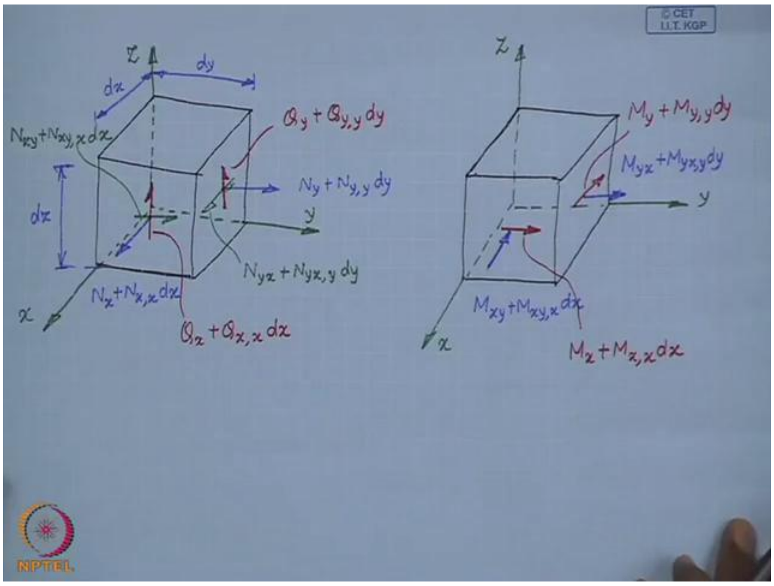

Now, let us look at these force and moment resultants on the infinitesimal element. So this is an infinitesimal element of the plate. So, I will mark out the resultants, so the first is N_x, so I will show the... So, this face at x equal to zero, there you have N_x in along the negative x direction, and on this face you have N_x plus N_x derivate with respect to x dx; so this length is dx; similarly this is d y. So, this is the normal stress resultant on this face; similarly you have on this face. Now the resultant because of the shear stress on this face is up, on the other opposite face it will be down; that will be Q_x, and this is Q_x plus del $Q_{x/}$del x dx; similarly here up, this is Q_y plus… and the in-plane shear stress resultant which is N_{xy}; so this of course is similarly this will be… Now, on the other two faces you can imagine that they will be without this additional conditional part, and they will be opposite in direction; so these are the stress resultant on the force resultant. Now let us look at the moment; so we look at the moment resultants now. So here we have this moment in this direction on the normal to this face is… on this face… So, this is the moment because of the normal stress sigma x x; so that is going to follow this right hand rule; so this is the moment on this face, on the face x equal to zero. You have just M_x in the opposite direction; and on this face it is because of sigma y y. So you have these as the moment resultants. Now we will come back to this figure again.

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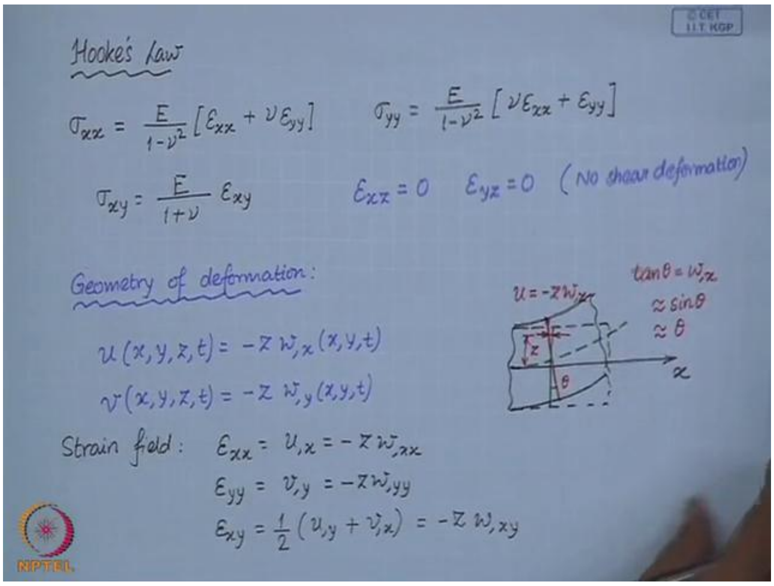

Now, let us write down the constitutive relation for the materials. So these stresses are related to the strains. Now, since we had considered that the plate is infinitely stiff in shear, there is no shear deformation… So there is no shear deformation of the element in any z. So, now… So, therefore, the corresponding stresses actually cannot be calculated from any material constitutive relation; we have to determine them from the equations of motion. Now, let us look at the geometry of deformation, so that we can calculate the strain in terms of the deflection of the plate. So, let us consider the plate initially undeformed, represented by this dashed curve, let us say along the x axis and this deflects in this manner; so if this is a line which is initially along the z, so it is vertical, then this line deflects to this configuration. So the displacement in the direction of x can be written approximately, if you call that displacement as u, is minus z times del w/del x, so it is a as you know that del w/del x is tan of this angle; so, tan… and for small theta this is also equal to… and that is equal to theta. So, this z is the location of this point from the neutral line, neutral surface, then the deflection in the direction of x is given by minus z del w/del x. Similarly, for the y, in the y direction we can write… Now, using this we can calculate the strain field. So that is the Strain field. Now, if you have these strains, so these are linear in z; now this w is independent of z. So, Strain is linear in z, so epsilon x x and epsilon y y are linear in z. So therefore, sigma x x is also linear in z, as we can see from these expressions; so that will be linear in z. So, if you go back to this calculation of the resultants, if this is linear in z, then z integrated from minus h by 2 to plus h by 2 is actually zero; so which means that these terms are going to vanish. So, these are N_x, N_y and $N_{xy,}$ they are all zero. So, the non-zero resultants are Q and M. So let us calculate then M, because Q will ultimately come from the equations of motion.

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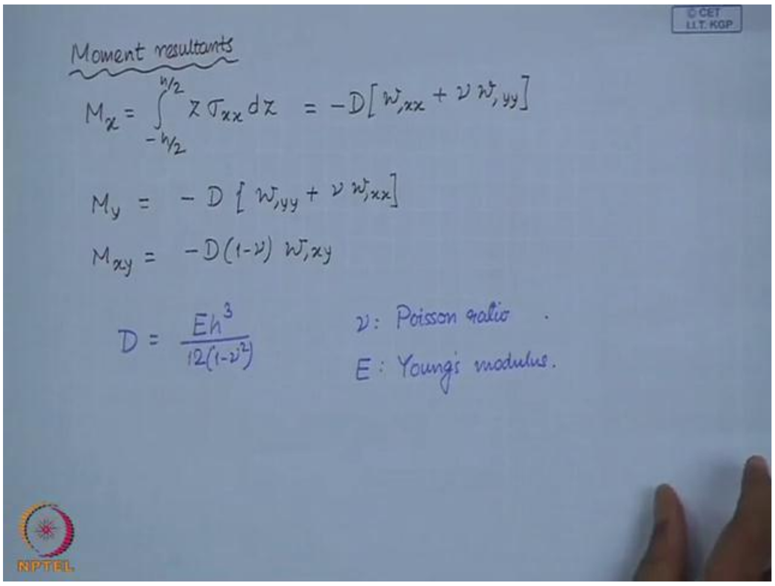

So the moment resultants, so, recall this is the expression. So if you substitute, so use this expression of the stress, where these strains are written from here, and if you substitute in this expression and simplify then you can see that this leads to… similarly… So here D and of course nu is the Poisson’s ratio, and E is the Young’s modulus. So these are the moment resultants.

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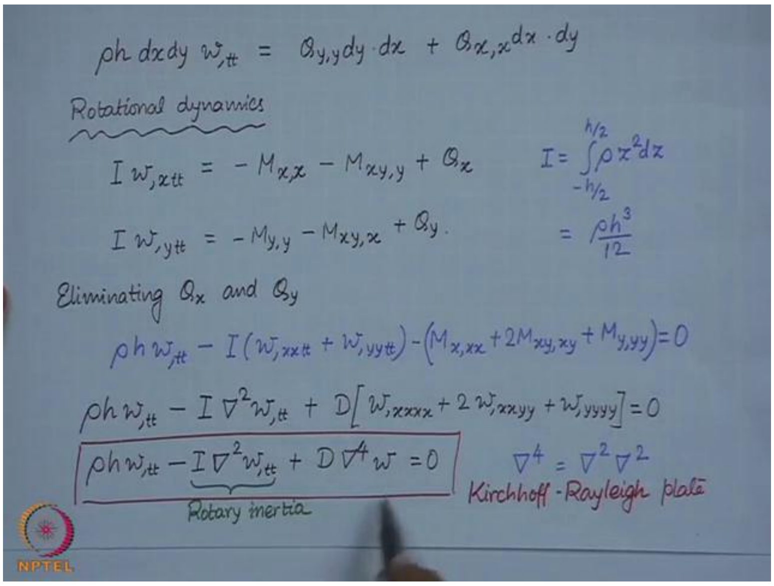

Now, we can write down the equations of motion. So, the first we write down the transverse, equation of motion in the transverse direction. So if rho is the density of the material and h is the thickness, so this is mass per unit area times the area of the little element, times the acceleration. So, if you now look at this figure. So, we have in the transverse direction these forces Q; so here it is upwards. So here there will be a downwards which will be minus of Q_y in the equation of motions. So you are left with, and since remember these are all forces per unit length. So, this and from this the contribution is… So, these are the forces in the transverse direction. Now, let us look at the rotational dynamics. So, for that we will refer to this figure. So I will directly write… So, this is the, del w/del x is the small angle; so this is the rotation about the y axis. So this double derivative with respect to time will give us theta y double dot, so rotation about the y, so angular acceleration about the y axis. So this equals, so we have these moments, so because of, so it is the rotation about the y axis; so, we have first this moment, then we have the moment this one because of y x. Lo let me write this term, and because of this force $Q_{x,}$ you have another moment. So this is the rotation about the y. Similarly you can write down the rotational dynamics about the x axis, and this is the… So, I is the moment of inertia per unit area. So for this element, we have I equal to rho h cube over 12. So these two equations correspond to the rotational dynamics, whereas this equation of course I will divide this whole thing by dx dy, so this corresponds to the transverse dynamics. Now, I am going to eliminate Q, Q_x and Q_y in the transverse dynamics using these expressions. So if I do that… so, this is what I obtain. Now here I will replace these moments using the expression that we have derived, the moment resultants in terms of the field variable. So, if you do that then you can simplify the equation, this can be written as… So, the Laplacian of del square w/del t square plus D, now this turns out to be… So, this is written as, so we can write this in a compact form. So this is the equation of motion of the plate. So, here nabla 4 is the square of the Laplacian. Now we have, now we need to talk about the boundary conditions. So, this model, this is known as the Kirchhoff-Rayleigh plate model. If you this is the rotary inertia term, this is because of the bending and this is the inertia term. So, if you drop this rotary inertia term assuming that the moment of inertia is very small in that case if you drop this term then you have the Kirchhoff plate model. So if you drop the rotary inertia term then you have the Kirchhoff plate model; with the rotary inertia term it is called Kirchhoff-Rayleigh plate model. So, now we will discuss about the boundary conditions.

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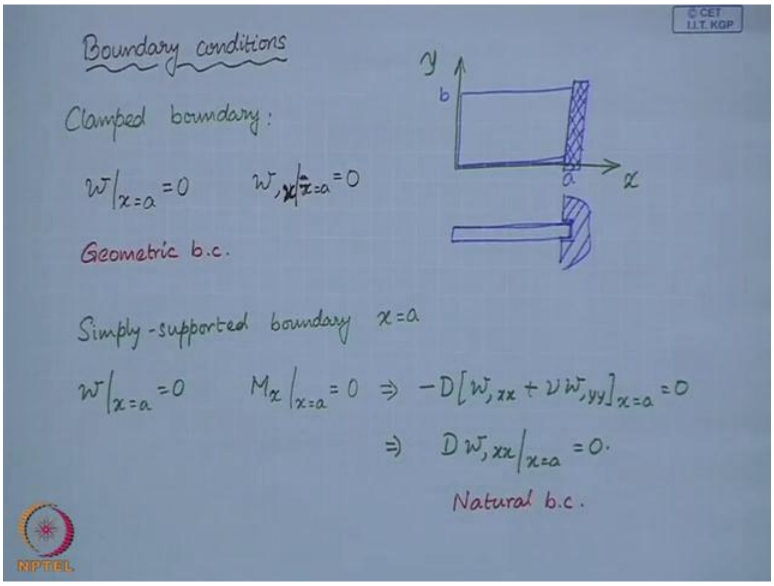

So, we can have various kinds of boundaries. So, suppose, let us consider a clamped; so suppose you have plate and you have boundary here which is clamped boundary. So, if the boundary is clamped this is actually quite simple; then the displacement must be zero, and of course the slope must be zero. So these are these are geometric boundary conditions. Then if you have simply-supported edge at x equal to a, so we have the displacement as zero, and the moment equal to zero. So this implies, if you use the expression of the moment… but if the edge is simply-supported and is straight, so that w is at x equal to a for all y is zero, then the there is no curvature. So, this term is also zero. So this will imply… Now this is the natural boundary condition. Now we come to this interesting case of a free boundary.

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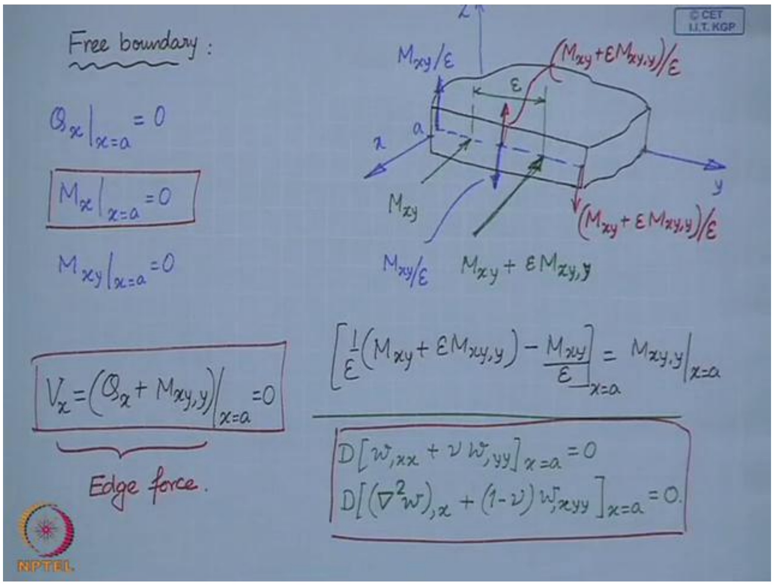

So, we have discussed clamped boundary, and simply supported boundary; let us discuss the free boundary. So, if a plate has a free boundary then intuitively we may guess that, so this is suppose this is x equal to a; so this boundary is at x equal to a, so intuitively we might guess that… So, this shear stress resultants, so this Q_x is going to be zero, so Q_x is along z on this face, so that should be zero; the moment equal to zero; and in addition we have another moment because of the shear stresses, in-plane shear stresses that also has to be put to zero; but you see this differential equation of the plate is fourth order. Now so it cannot support three boundary conditions at an edge like this; at a boundary it cannot satisfy three boundary conditions. So there must be something wrong about these boundary conditions, at least some of them. So there must be some combination. So, now we are going to discuss how they are actually combined. So, let us see this M_{xy}. So let me refer to this figure once again; so this was M_{xy} on this face, let us say at x equal to a, so this is M_{xy}. Now this is the moment because of the in-plane shear stress; so this moment can change as you move in the y direction. So, at another location this can be M; so I am considering two locations separated by a small distance epsilon. So, we are moving in the y direction. So this is the moment at a location, a distance epsilon along the y. Now this can be equivalently represented as a couple. So this can be replaced by a couple; and similarly this can be replaced by another couple. So, therefore, at this point you can imagine that the resultant, the force resultant here which is now a transverse force is given by… So, all these things are calculated at x equal to a. So this is an additional edge resultant force, which is in the transverse direction which is same as Q_x. So we can combine now… this must be zero, this is defined as the edge force, this is known as the edge force. So, this edge force must vanish. So, it is a combination of the force because of the out of plane shear stress and the moment because of in-plane shear stress. So these two they combine to give us what is known as the edge force. So we have the boundary conditions as this, and this; and if you write these down in terms of the field variable, so and if you calculate this edge force. So these are the boundary conditions for the free edge at x equal to a.

So let us summarize. So, we have looked at the equation of motion of small amplitude, small slope vibrations of flat plates. So, plates are two dimensional elastic continuums which can transmit or resist bending moment. So we have looked at the equation of motion, and the boundary conditions and we have for the standard boundary conditions which are clamped, and the simply supported, the boundary conditions are quite simple, whereas for the free boundary we have discussed about the edge force that must vanish. So, the boundary conditions for the free boundary have to be carefully determined. So, with that I conclude this lecture.

Keywords: transverse dynamics of plates, Kirchhoff-Rayleigh plate model, boundary conditions, edge force.