Vibrations of Structures

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Lecture No. # 36

Special Problems in Membrane Vibrations

In the past few lectures, we have been looking at the dynamics and vibrations of membranes, in particular flat membranes. Today, we are going to look at some special problems in membrane vibrations. So, the first problem that we are going to discuss is that of a membrane placed over an enclosure.

So, for example, you have seen drums. So, if you have a drum or a tabla for example, it is an enclosure; it is completely closed by a rigid surface and on one flat surface you have the membrane, which encloses, closes this enclosure. So, what you have is the air inside this enclosure is now trapped. So, what as a result of this when the membrane vibrates you can imagine that the volume of the air inside is going to change. Now, when that happens it is expected that pressure is going to change and that is going to force the membrane. So, till now we have not considered such a situation where the membrane can be forced, because the pressures on the two sides of the membrane are different; but in this case, it becomes different because of the enclosed volume of air.

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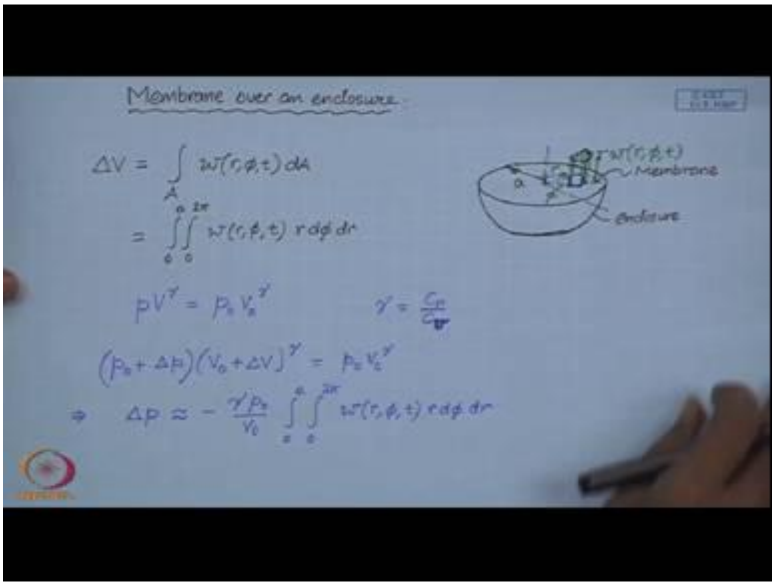

So, let us look at the problem. So, we have a circular membrane enclosing a rigid vessel or a container. So, this side is the membrane and this is the enclosure. Now, because of this air that is enclosed within this enclosure, when the membrane vibrates let us see what happens. So, let me draw a portion, an element of this membrane in the initial configuration; and when it deforms, so this little element; now this is an exaggerated diagram; so, this little element has displaced a height at a location r and phi at time t. So, this is phi and this is r. So, at a location r and phi at time t, this is the element. So, what is the change in volume because of this? So, let us write that down. So, because of this small element, so it can be written as w at… into dA and that integrated over the area of the membrane. So, that is going to be the additional volume that this enclosure is now going to have, the air inside the enclosure is going to occupy. So, if we use polar coordinates, then we can write this as… So, here a is the radius of the membrane.

Now, so this is the small change in volume of this enclosure. Now, corresponding to this change in volume, we must now calculate the change in pressure. Assuming, that the membrane is going to vibrate sufficiently rapidly, so that we can consider the adiabatic process for the air, where gamma is a ratio C_p over C_v is a ratio of the molar specific heats, so at constant pressure and at constant volume. So, these are standard definitions. So, gamma is C_p over C_v and the pressure times, new pressure times, new volume power gamma must be equal to constant which must be equal to then the initial pressure times initial volume power gamma.

So, if the volume, changes to V_0 plus delta V and pressure changes to p_0 plus delta p. So, if you approximate this, because the change in volume in relation to the initial volume will be very small; we are going to consider very small displacements of the membrane. So, then this can be written as you can calculate delta p approximately. So, that is the expression of a delta p. Now why did we calculate delta p: So, that we can, now, find the force that will be acting, because of this pressure difference. So, delta p is the pressure difference between the ambient air and the air inside the enclosure. So, that the small pressure difference is this delta p. So, this has been calculated in terms of the initial pressure, so, which is the ambient pressure actually, initial volume of the enclosure and this is a constant for the air which is gamma C_p over C_v and this is that displacement of the membrane.

So now, we are going to consider that this, because of this pressure difference, there is going to be a force on the membrane. So, this is going to be the... So, if this is the pressure difference, so, that is going to force the membrane dynamics.

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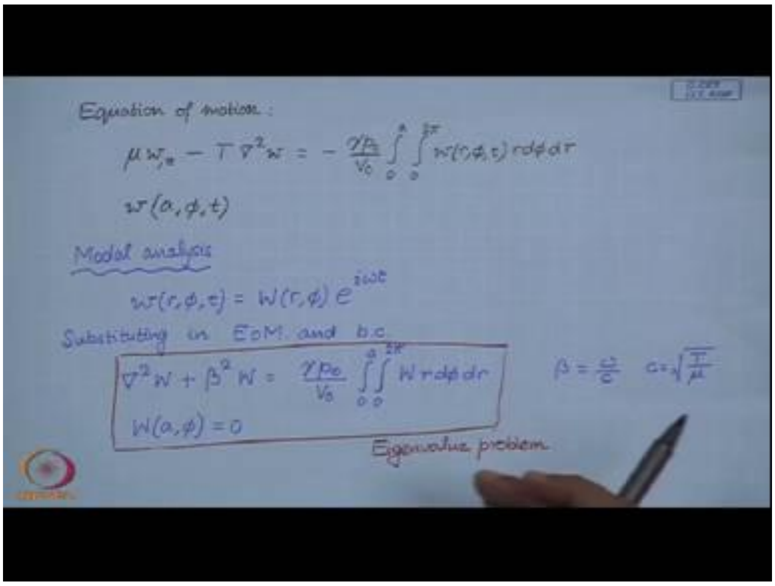

So, the equation of motion of the membrane may be written as… Since, the forcing on the membrane must be force per unit area which is the pressure, so, we have this pressure. Now, there is an assumption when we write this equation of motion. So, of course along with this let me first write the complete problem. So, this is the boundary condition. So, this completes the formulation for this membrane vibrating on an enclosed, over an enclosed volume. So, when I write this pressure directly in the on the right hand side of the equation of motion, see we are assuming that this pressure is uniformly getting distributed over the membrane. So, that is the, that is how we are actually calculating this pressure, and putting it in the equation of motion; that is the reason why we do what doing it this way. We are assuming, therefore that there will not be any change of pressure or the variation of pressure because of the dynamics of the air itself inside the enclosure.

In other words, the cavity, the frequency of the cavity that is inside the enclosure is much much higher than the frequency of vibration of the membrane, so that the pressure, the air inside has enough time to equilibrate uniformly over the surface of the membrane. So, this is an assumption that we are making. So, the acoustic frequency of the enclosure is much higher than the membrane frequency. Only in that case we can calculate this delta p in this way and put in the equation of motion.

Now, so we would like to find out the natural frequencies, and the modes of vibration of a membrane over an enclosure. So, we are going to perform the modal analysis for this problem. So, we once again assume a space time separable solution for this problem and if we substitute this in the equation of motion… and of course the boundary condition; we obtain on simplification… and the boundary where, I have used this beta as omega over c and c is under root the force per unit length in the membrane divided by the mass per unit area, so, that is the wave speed in the membrane and this along with this we have the boundary condition. So, this therefore defines our Eigen value problem, so which we are going to now solve. So, we have solved in our previous lecture the problem with homogeneous equation of motion; so, no right hand side. So, we already have the Eigen functions for the problem. So, let us see what we can do with the Eigen function expansion that is the first thing that we can of course try. So for this Eigen value problem let us… Here we are assuming that the radial function is as yet unknown. So, W_m^c is… Let us see what will this turn out to be. So, when we substitute this kind of an expansion, so, we have this structure from what we discussed in the previous lecture. So, when we substitute this in the equations, then we observe that; see here on the right hand side there is an integral from 0 to 2 pi.

So, when we have a cosine and the sine functions integrated from zero to 2 pi because of periodicity that is going to go to zero for m not, so, for m not equal to zero. So far m not equal to zero, this is going to be our equation, if we consider an expansion like this.

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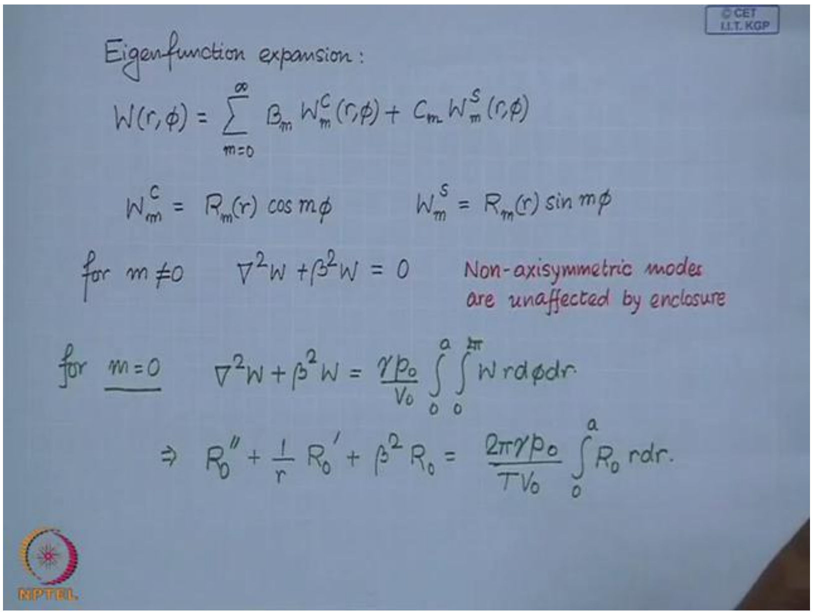

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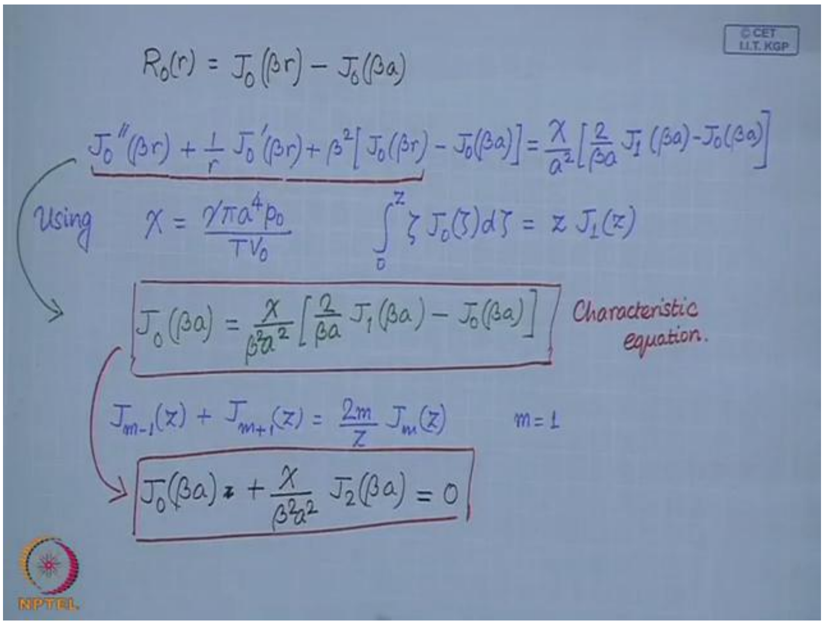

So, here of course W is this expansion. So, for m not equal to zero, we have our homogeneous system. So, which means these are the non-axisymmetric modes. So, the non-axisymmetric modes are unaffected by the enclosure or the change in volume. So, the non-axisymmetric modes actually do not lead to any change in volume. So, that is what we conclude. Now for the other modes, so, for which means for modes with m equal to zero, this our equation is going to… and that is going to lead us to with this kind of an expansion; if you substitute this expansion in here and simplify… So this equation is going to lead us to this, for m equal to zero. For m not equal to zero, we are going to get back our normal circular membrane Eigen value problem; so, which immediately tells us that the non-axisymmetric modes of the normal circular membrane are same as that of this membrane over the enclosure. The only change comes in the axisymmetric modes. Now, let us consider the solution of this; so, this form of solution is guided by two factors, one is that this operator corresponds to the Bessel's function, and we have this boundary condition which I must write here. So, the boundary condition at R_0at a must be equal to zero. So, you can see that this at a is zero. So, this structure of a solution actually satisfies the boundary condition and is motivated, the structure is motivated also by this operator which is the Bessel's differential equation. So, let us now substitute this in the equation of motion and if you do that and simplify… So, this is what we are going to obtain. Now, here I have defined these new terms, which was in the equation of motion. So, this term was there and I have called it chi and I have used this property of Bessel's function. So, I have used this property. So, using these two things I have obtained this differential equation. Now, you can very easily see that this is zero; this is nothing but the Bessel's differential equation; this is the solution for m equal to zero. So, therefore, this is zero. So, we are left with… this and this is nothing but our characteristic equation. Now, using another property we can simplify this little further. If you use this property, so, this is the standard property of the Bessel's function; so, I can replace this J_1 in terms of… So, if I put m as 1 then I will get J_1 in terms of J_0 and J_2, and then my characteristic equation simplifies to… So, that is our characteristic equation. Now, this can be solved numerically or by using may be properties of Bessel's functions you can approximate the solutions.

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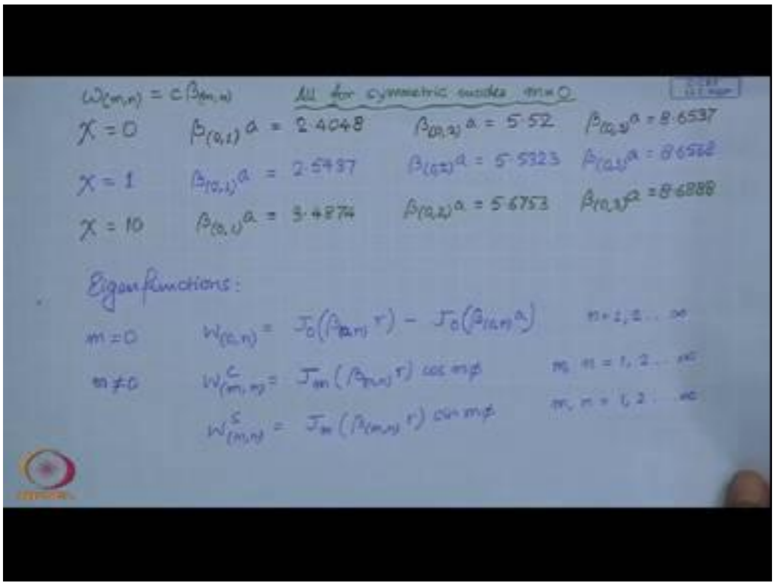

So, if you calculate these… so, what we are going to solve is beta a, now for different values of chi; now remember this chi has this constants and this pressure and the volume.

So, for different values of let say enclosed volume, enclosure volume or tension in the membrane, force per unit length in the membrane, you can have different values of chi. So, if you solve for beta for different values of chi, let say if chi is zero, which means that we have a normal membrane. So, chi can be zero, if the enclosed volume is infinity; so then chi goes to zero. So, for this you can check that we get back the frequencies of the normal membrane.

Now, of course, this beta… So, remember that omega m n; so, of course we are discussing the case for m equal to zero, but this was the general relation. So, that will give us the circular natural frequencies. Now, for chi equal to zero, then you can have chi equal to, let us say when you take chi equal to one, and if you increase chi to 10, so what we can observe is that as you increase this chi, so, which means the effect of this enclosure you are increasing, the effect of the enclosure as you can see here. So, this is our characteristic equation. So, if chi is zero, so, we get back the characteristic equation of a normal circular membrane; and if chi is not equal to zero, then we have this effect of the enclosure. So, as you increase the effect of the enclosure by increasing chi what gets so, these Eigen frequencies, since beta is proportional to omega the Eigen frequencies are all increasing; but then the increase is much larger for the lower modes, and as you go to higher and higher modes, the change is getting diminished. So, this has change from 8.65 to 8.68 when chi goes from 0 to 10, whereas this has gone from 2.4 to 3.4 for the same increase of chi.

So, the enclosure increases the Eigen frequencies, but of course this is for the symmetric modes; this always for the symmetric modes; so, m equal to zero. So, it increases the Eigen frequencies but then that increase diminutions as you go to higher and higher modes. Now, if you now look back then the Eigen functions can be easily written; so for m equal to zero, which are actually affected… and for m not equal to zero, we have two kinds of modes, the cosine remains the same.

So, these are the Eigen functions for the membrane. Now, the next problem that we are going to look at is that of an elliptic membrane. So, we have looked at the circular membrane and we have seen how this, the symmetry of the circular membrane leads to modal degeneracy that we have discussed in our previous lecture.

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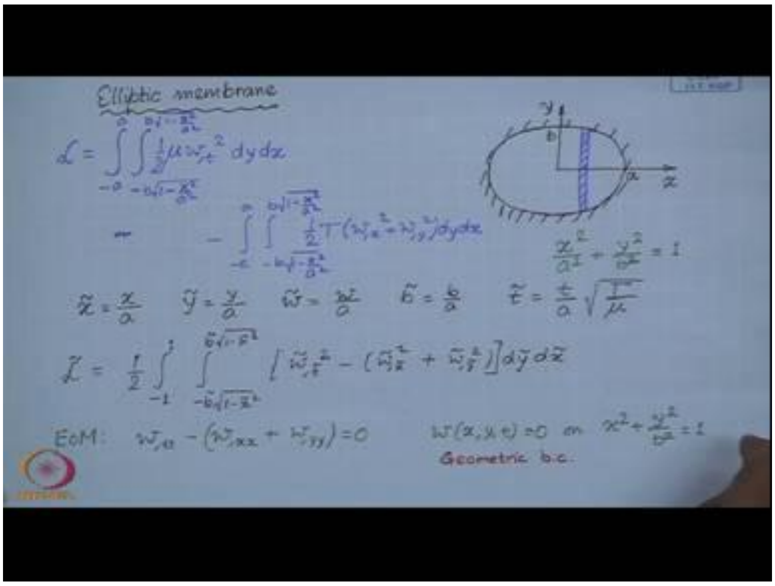

Let us now, see what happens if the membrane becomes elliptic. We are going to discuss this problem using the approximate methods. So, let us consider, so, this is roughly an elliptic membrane.

We will consider the fixed boundary case. So, the semi-major and the semi-minor axes are a and b. So, let me write down, since we are going to deal with this, using approximate methods, we are going to use the Ritz's method for the membrane. Now, let us see first the Lagrangian of the problem. So, if you consider a little portion of the membrane, so the kinetic energy one half mass per unit area times velocity square, and integrated over the total area of the membrane which I can write in terms of… So, y goes, at any given x, y goes from… since, the equation of this boundary… So, this is the equation of the boundary. So, for any given x then y goes from minus of b square root of 1 minus x square over a square to this value and x goes from minus of a to a. So, that is the total kinetic energy minus the total potential energy. Once again it will be integrated over the same domain of the force per unit length one half of this. So, this is our Lagrangian. Now, let us make some redefinitions and simplify this Lagrangian. So, let me define x tilde, the non-dimensional coordinate x using x over a, non-dimensional y as y over a, the field variable is W over a, b tilde as b over a. So, this is the ratio of the semi-minor to semi-major axes and I will also non-dimensionalize time; this is only for simplification. So, this is using the wave speed in the membrane. Now, so if you use this scheme then… Now, because this is going to be a little cumbersome to put tilde everywhere, so, from now on I am going to drop this tilde.

Now, we have to find admissible functions for this problem. Now, if you derive the equation of motion, so if you derive the equation of motion and boundary conditions; so, equation of motion is of course… and the boundary condition… so on this non-dimensional boundary. So, we have W as zero, so that these are geometric boundary conditions. So we have to choose functions which satisfy this boundary condition.

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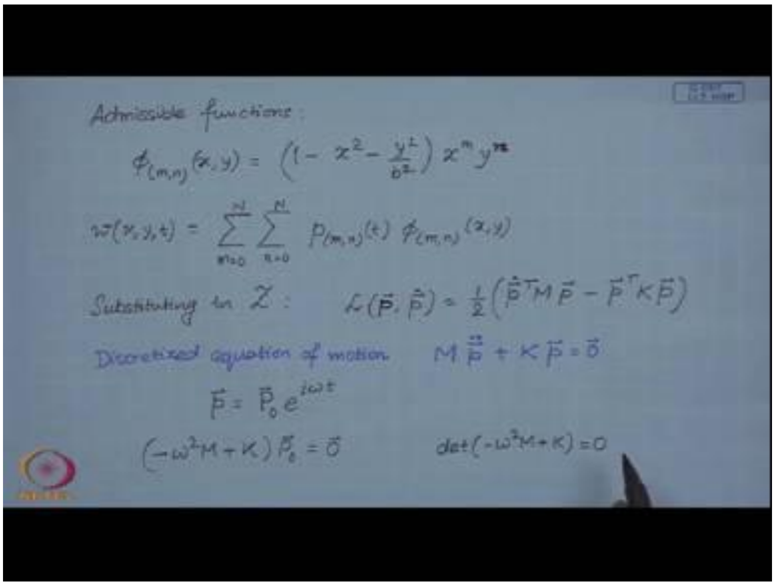

So, let us, so, you can easily see that functions of this form is going to… these functions can satisfy the geometric boundary condition of the problem. So, we use this expansion to a suitable number, for a suitable number of terms N. So, we have the coordinate times the admissible functions. So, using these functions, I construct an expansion; substitute this in the Lagrangian and integrate out over, since I now know x and y they are polynomials; so, I can integrate over x and y and finally, get… So, my discrete discretize Lagrangian will read, so this is in terms of these coordinates and their derivatives. So, you can write it like this from where the equation of motion, the discretized equation readily follows and you can now do a standard modal analysis to determine the Eigen frequencies. So, if you substitute this in here... So for non -trivial solutions of P_0… So, this is going to give you the circular Eigen frequencies of the problem.

Now, if you do that then finally, you can find out the, for say if you put b equal to 1 which corresponds to the circular membrane, then I have taken 6 terms in the expansion; and I have… Now here there are repeated Eigen frequencies; so, which indicate that there is modal degeneracy. The exact solutions are… So, this of course, this 1 over a square root of T by mu; so, these factors multiplied by this. Now, if you calculate the exact ones… So, you can see that the correspondents. So, this is the fundamental; this is the second, which is… so this is almost exactly calculated. So, for the circular membrane, therefore this formulation is giving us correct results. Now, let us look at b is equal to 0.6, which is an elliptic membrane. So, these factors are… and of course, there is this. Now you see there is no repeated Eigen frequency. So, since we have broken the symmetry of the membrane, the Eigen frequencies now they have split; so, these two have a split from here.

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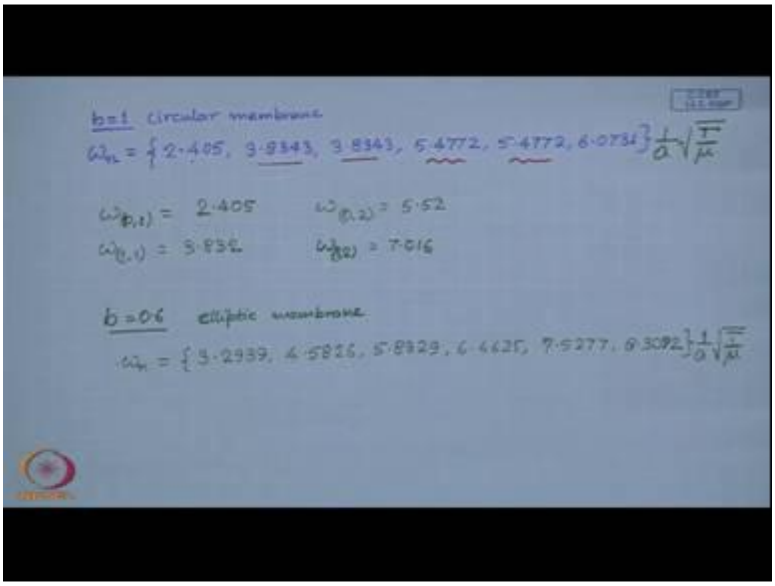

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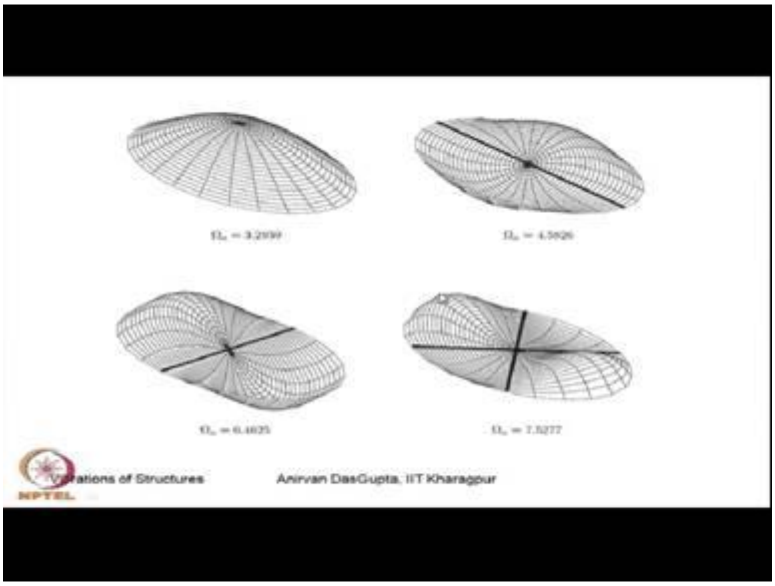

So, here in this figure I have plotted the Eigen functions the modes of vibration of the elliptic membrane for the first four modes. So, this is without any nodal line or circles; and these are with nodal lines.

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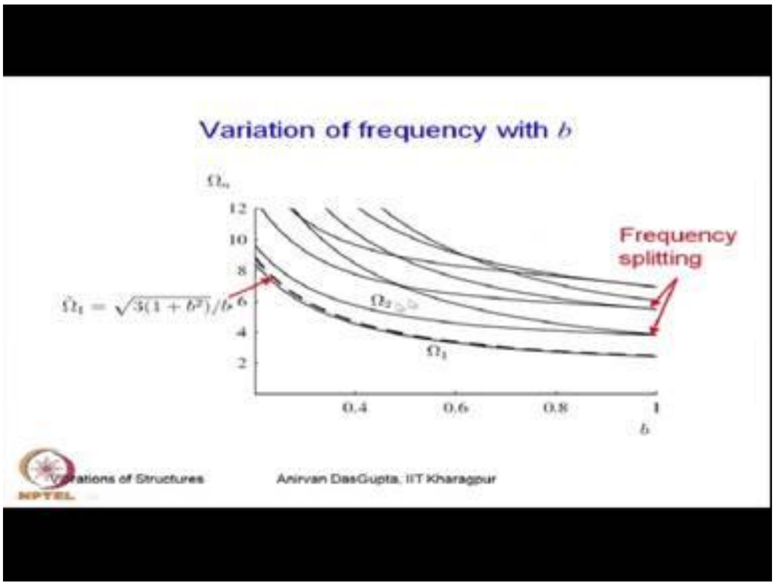

Now, here this figure shows the variation of the Eigen frequencies with the ellipticity of the membrane. So, when b is 1, so this is the circular membrane. Here you can see repeated Eigen frequencies and as you change b these frequencies are splitting. So, to recapitulate, we have discussed the some problems in membrane vibrations. The first problem, we discussed was that of a membrane over an enclosure, and a next we have use an approximate method the Ritz's method to study the elliptic membrane. So, with that I conclude this lecture.

Keywords: membrane-air coupling, modal analysis, Ritz method, elliptic membrane.