Vibrations of Structures

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Lecture No. # 35

Vibrations of Circular Membrane

So, we have been discussing about vibrations of membrane in the present lectures. So, today we are going to look at the vibrations of a circular membrane. So, we have, in the previous lecture, looked at the rectangular membrane. So, today we are going to look at a different geometry, which is the circular geometry.

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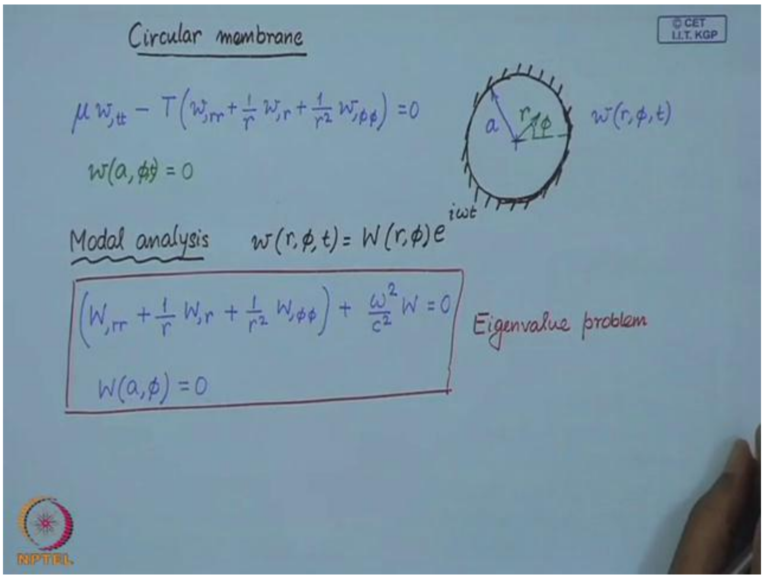

So, consider this as a circular membrane. We are going to consider a membrane with fixed edge. So, the radius of the membrane, let it be small a. Any point on this membrane is denoted using the coordinates r and angular coordinate phi. So, using this and our field variable, which is the displacement of the membrane from its equilibrium position measured at a coordinate location r comma phi at time t; so, this is the transverse displacement of a point at r phi at time t. So, the equation of motion, so if mu is the areal density and T is the force per unit length, so in the polar coordinates, so the Laplacian operator… along with the boundary condition, so the displacement on the boundary is zero. So, we will be looking at the modal solutions of the circular membrane. So, we are interested in solutions of the form of this structure. So if I substitute this solution in the equation of motion and in the boundary condition, and I do a rearrangement by removing this exponential i omega t… so this along with the boundary condition; so then this defines our Eigen value problem. So, this is our Eigen value problem, which we must solve in order to determine the circular natural frequency and the modes of vibration of a circular membrane.

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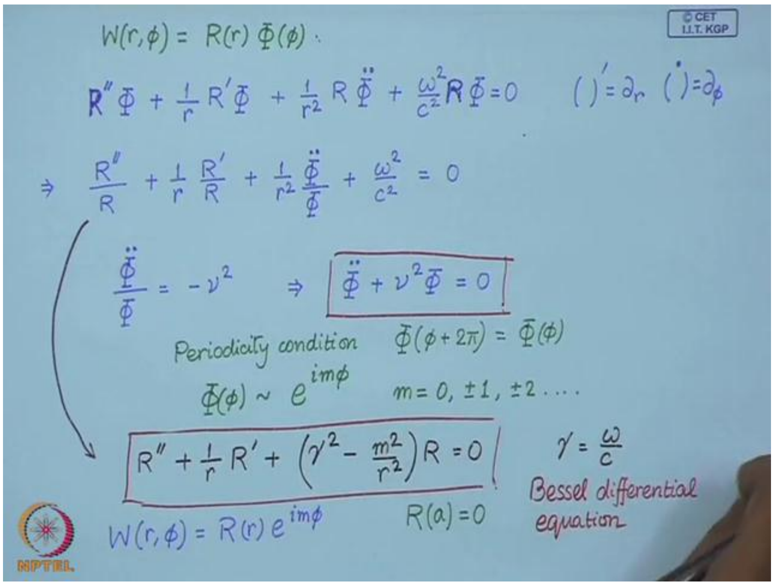

So, we search for separable solutions; once again as we did for the rectangular membrane, we look for solutions, which are separable in r and phi and this is motivated by the fact that these are independent coordinates. So, they can be separated out. So, we are searching for solutions of with this structure. So, let us see what happens when we substitute this in here. So let me indicate del del r with the prime and derivative with respect to phi with dots. So, here I have use del del r by prime and dot indicates del del phi. Now I do some rearrangements; I divide this throughout by this product R into Phi. So, then now, you see that this capital R is only a function of the coordinate r, while Phi is only a function of the coordinate phi.

So, if this which is the function of R, and this which is purely function of phi; so, and that must add up with constant, which is independent of R and phi and that it has to be equal to zero, for any R and any phi; then it is natural that these are all constants, so, which means… I will write it as minus of nu square, for reasons that will be clear later;

since I want, you see that this is going to give me, this equal to zero. So, this is purely a function of the angular coordinate phi; and we must have periodicity in this function, otherwise at phi equal to zero and phi equal to 2 pi, this function will not match. So, we must have… Now that can be ensured, when phi is proportional to an exponential i m phi, where m can be zero, plus or minus 1, plus or minus 2 etc. So, we must have essentially, what did this mean is we must have a solution in terms of cos m phi and sin m phi. So, that can be written in this complex form.

Now this solution is possible only if I choose this constant to be minus of nu square and nu cannot be an arbitrary we know that nu has to be an integer, so that, this periodicity condition is satisfied. Now if you have this as minus m square, so I can write, so this equation, then becomes can be written as… Let me define this as gamma square and this is minus of m square, so that must be zero, where gamma is omega over C. So, this is the equation governing our radial function. So, this this equation was for the angular function Phi. So, this is for the radial function R. So, for the angular function, we must have solution like this; so, our solution till now, what we have is like this… so this function.

Now we have to solve for this radial function. Now this has in addition the boundary condition; so if you use the boundary conditions, then you have… R at the periphery of the membrane at R equal to a, must be equal to zero. So, this equation has to be solved, with this as the boundary condition. Now this is the second order ordinary differential equation; but we have only one boundary condition. So let us see what happens, because this is going to lead us to something an interesting. Now this equation, this differential equation is known as the Bessel differential equation; and we have come across this differential equation, when we discussed the hanging string. If you recall, there also we had the Bessel differential equation except that this term was zero. So, let us see, then we can draw an analogy from the hanging string, and you can understand why only this one boundary condition supplies, the other condition comes from the finiteness of the solution.

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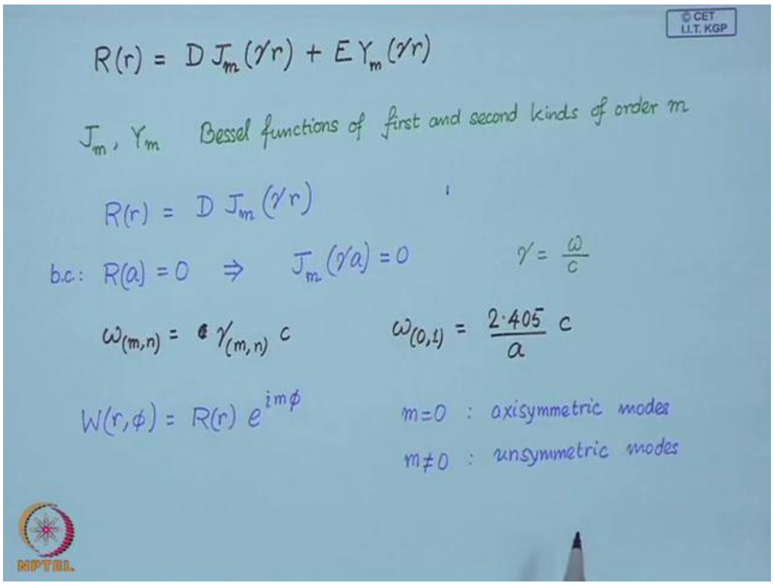

So let us see what is the solution of the Bessel differential equation? So, the solution of the Bessel differential equation; so this is a very standard differential equation in mathematics, and a solution can be written as some constant in this form. Here, so this is called the… They are the Bessel functions of first… and so these are of order m Bessel functions; this is the first kind and this is the second kind and they are of order m. Now, since we have the solution as a linear combination of these two functions, let us once have a look to just have an idea of these functions.

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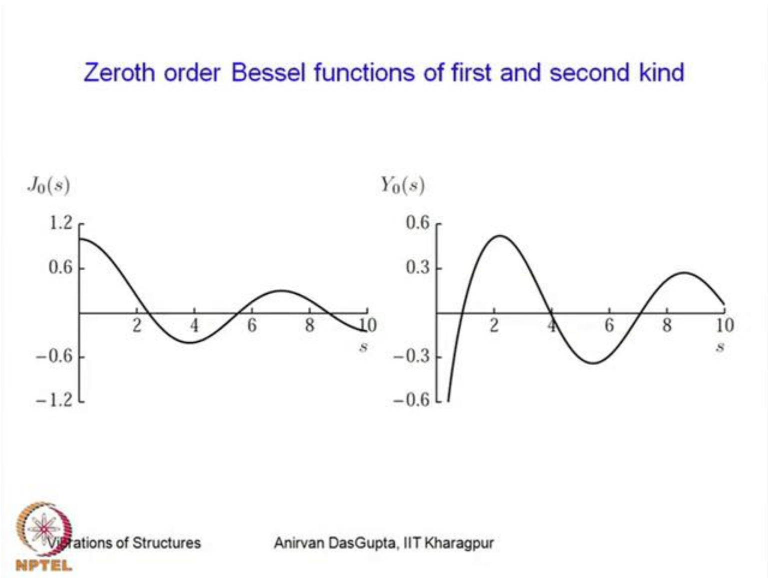

So, this figure shows the Bessel function of first kind. Here I have plotted only the zeroth order Bessel functions J_0 and Y_0 as a function of this argument s. So, here actually for integral values of the order, this Bessel function of the second kind has a logarithmic singularity at argument equal to zero. So, this is known from theory of Bessel functions. So what this means for our membrane is that at r equal to zero, this function is going to introduce a logarithmic singularity; that means, this is going to go to minus infinity, which we do not want, I mean physically we do not have the this kind of behavior.

So, we must drop this function from our solution; all and Bessel functions of all order of second time, they have this kind of singularity. So, we must write our solution or express our solution only in terms of the Bessel function of the first kind. Now if you do that now we have just one boundary condition; so, R(a) is equal to zero and that implies, so we must… So, a is the radius of the membrane. So, we must choose gamma, now remember that gamma is omega over c. So, we must choose this gamma so that this condition is satisfied.

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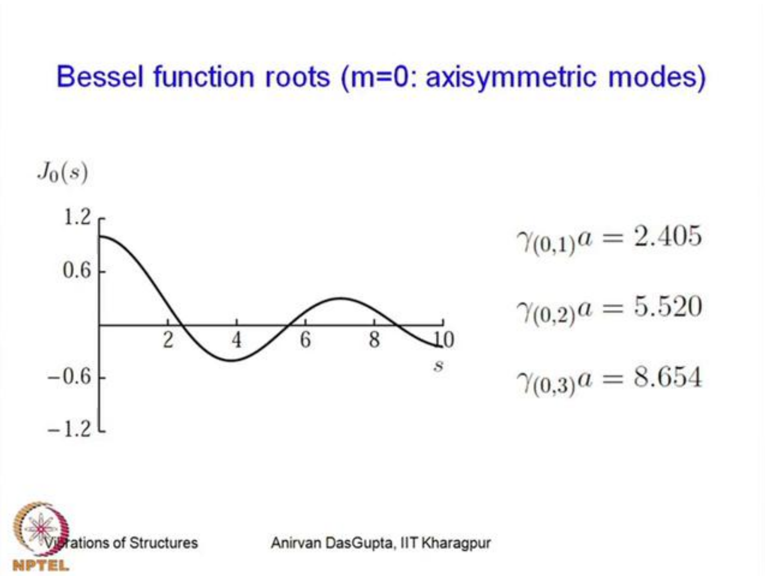

Now let us look once again at the Bessel function of the first kind, let us say at with n equal to zero, so zeroth order. So, this function goes to zero at these points; so, in this figure, there are only a three visible, but then this function has infinitely many roots, so this goes on and on. So, the first three values of gamma times a, which are now indexed with m and the root numbers; so this takes a number one; this is the second root; this is the third root. So, 1 is 2.405; so, this is approximately 2.405; this is 5.520, and the third root is 8.654. So, the first three solutions of this condition therefore, are obtained here. So, therefore our frequency is also now get indexed. Let me first write this in this form.

So, this is gamma m n times c. Now gamma m n, let us say omega 0 1, as we have seen is 2.405 over a times c; so that is the circular Eigen frequency corresponding to the mode 0 1 and in this way, you can determine the higher modes.

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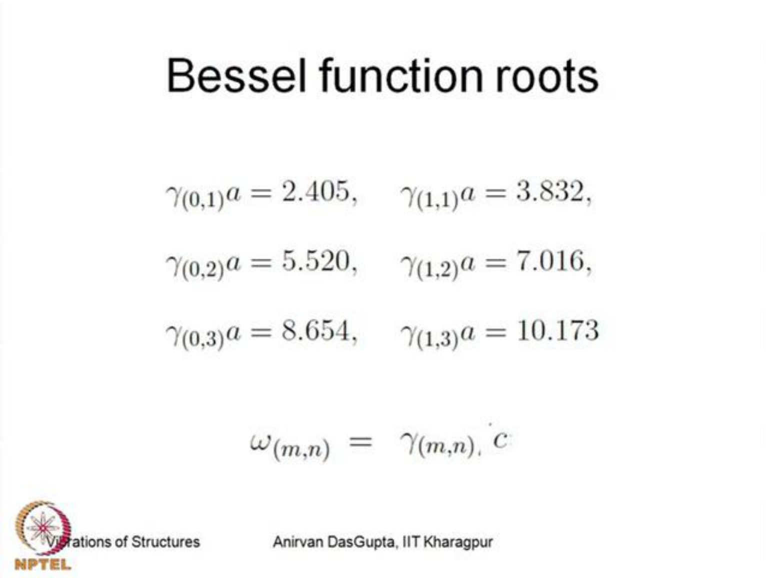

So here, I have listed out the some of the higher modes as well. So, so this is these are with m equal to zero, these three, what we have seen and this is at m equal to 1. So, one the first root with one, the second root etc.; and therefore the circular Eigen frequency or natural frequency of the circular membrane is given in this form. So, let us once again have a look therefore, so our solution was in this form. Now when m equal to zero, so when m is equal to zero, we have what are known as the axisymmetric modes and for m not equal to zero, we have the unsymmetric modes. Now let us look at and so finding these roots of the Bessel function, so this is the essentially finding roots of the Bessel functions of the first kind; now finding these roots is actually little cumbersome, though in various softwares or numerical analysis programs these are coded, they are available; but let us look at an interesting approximation of the Bessel function.

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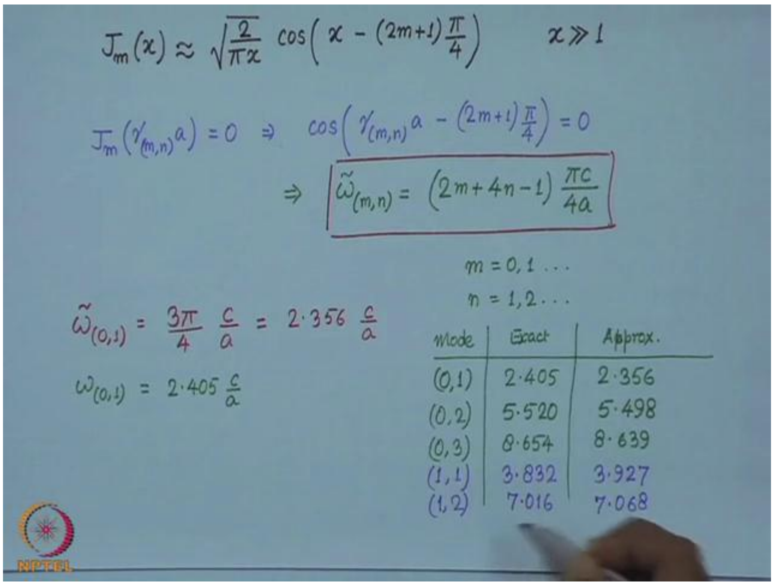

So, you can write this, approximately as… when x is very large. So, for large arguments, we have this approximation, and let us see how good this approximation is?

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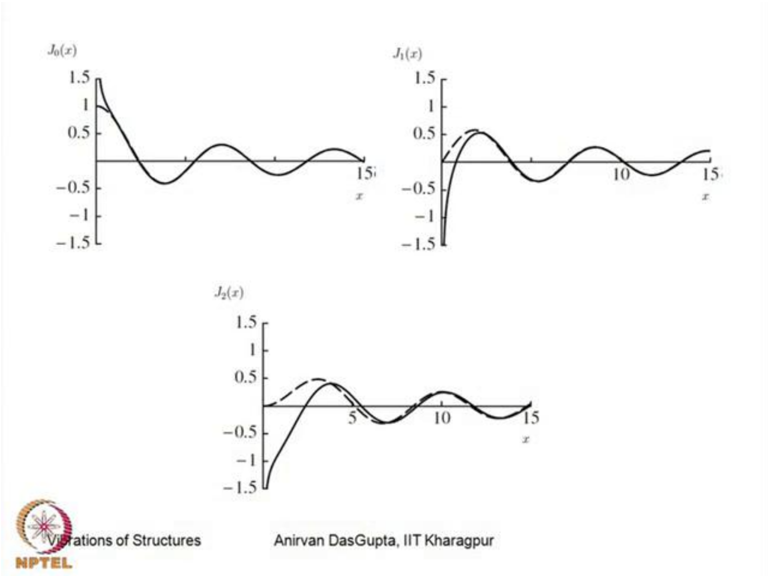

So, in this figure I have plotted out the Bessel function here for the first kind of order 0, order 1 and order 2. So, this dashed line is actually the exact Bessel function and this solid line is the approximation. So, we can see here for example, the solid line and the dashed line are almost indistinguishable except for this; but we are interested in this approximation only to find out the roots of the Bessel function. So, you see the, we expect for example, for J_0, the approximation to work very nicely, even starting from the first root; whereas for J_1, there is slight error in the first root; but then from the second and third and fourth etc., these are very good. Now for J_2 this approximation is not so good for the first root; well there is some error in the second root; third root is close; and from the fourth root, it is quite good. So, you can expect that this is going to, this approximation is going to give us the roots… Now so for large arguments we say, so, let us see what happens, we want to have… So, this must be zero. Now this gives us, so if I write this as omega m n over c and write out… So, this is an approximation. Let me put a tilde to denote that this is an approximation. So, we can… So, m equal to zero as I said is the axisymmetric mode. So, we can have for each of these modes, we have infinitely many roots, so we have this circular Eigen frequencies corresponding to these modes. Now let us sees how good is this approximation?

So, let me calculate, let us omega 0 comma 1, now, so with tilde, so, this is, m is zero; so, n is 1; so this gives me 3; and this turns out to be… Now the exact value, exact means solution of the Bessel that we obtained from the by solving the roots of the Bessel function, this turns out to be 2.405 c over a. So, you can see, they are quite close; so let me just… So, 0,1 this factor is 2.405, this factor is 2.356; if I take 0,2; this the exact solution is 5.52, and this one turns up to be 5.498; 0,3, the exact is 8.654, and the approximation obtained from here is 8.639. So, you can see progressively you are approaching the exact solution; similarly if I have 1,1 the exact solution 3.832 and the approximation obtained from here is 3.927; 1,2 the exact is 7.016 and the approximation obtained from here is… So, like this, you can, you see that as you go to higher modes in each stage, you are approaching the exact solution; so that is the argument for large value of the argument. Now if you go for m equal to let say 2, then initial, this approximation will be in some error, but as you go to higher values of n, therefore you will have better approximation. So, that will tend to exact mode. So now, let us… So, this is the calculation for the Eigen frequencies.

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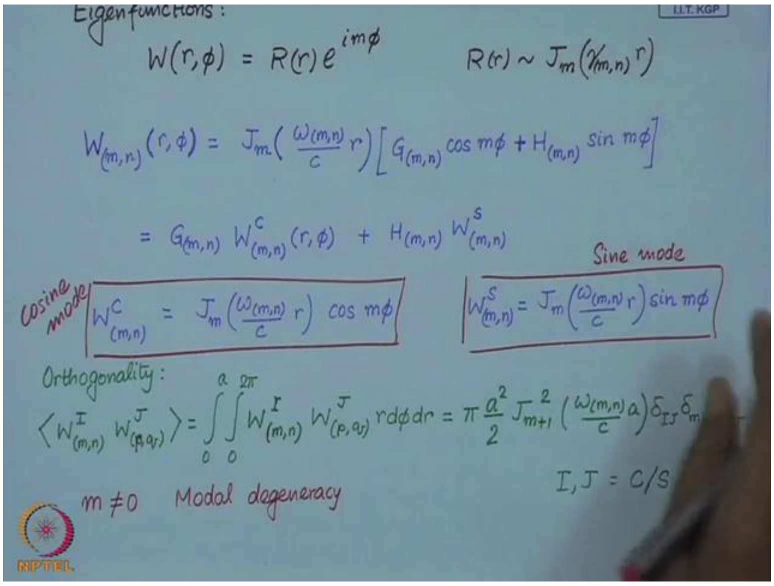

Let us look at the Eigen functions. Now we the solution that we have is… Now as you know that so this R is proportional to this function therefore, and this complex function is cosine m phi plus i times sin m phi; now as we have discussed before, we can take the real part or we can take the imaginary part or we can take a linear combinations of this two parts. So, we can write the Eigen functions as a linear combination of the cosine and sine. Now this I will write this as omega m comma n over so c into r; so that multiplied by… So here I can have a linear combination where the indices also get indexed.

So, it is a linear combination of the cosine m phi and the sin m phi, and that multiplied by the Bessel function of the first kind and this form. So, that is the Eigen function m n. Now let us look at the… So here, I can write this as… So, I am introducing another notation, so in the subscript capital C over on W indicates Bessel function multiplied by the cosine and similarly… So, I am introducing this notation with the subscript C or S to indicate this cosine or the sine. Now so, these are also functions of r and phi and you know that here if you look at the orthogonality, which in this case is defined… So using the property of the Bessel functions, you can write…

So, here this capital I capital J, they can have this values C or S, so, corresponding to whether it is a cos or whether it is sine. So, you see that if it is… So, you can see immediately from these two even for m n, W m n C and W m n S, they are orthogonal; these are orthogonal modes; and they are actually two distinct modes corresponding to a single circular frequency omega m, n. So, this is called cosine mode and this is known as the sine mode; and so, we have two orthogonal modes corresponding to a single Eigen frequency omega m comma n; and this happens only when m is not equal to zero; as you can very easily see, if m is equal to zero, then you have only this mode; you do not have this sine mode. So, for m zero is equal to zero, you have modal degeneracy, so which we have discussed in our previous lecture. So, in the case of circular membrane, all unsymmetrical modes, so this m not equal zero are the unsymmetrical mode. So, all unsymmetrical modes are modally degenerate. Now let us see what how the modes look like.

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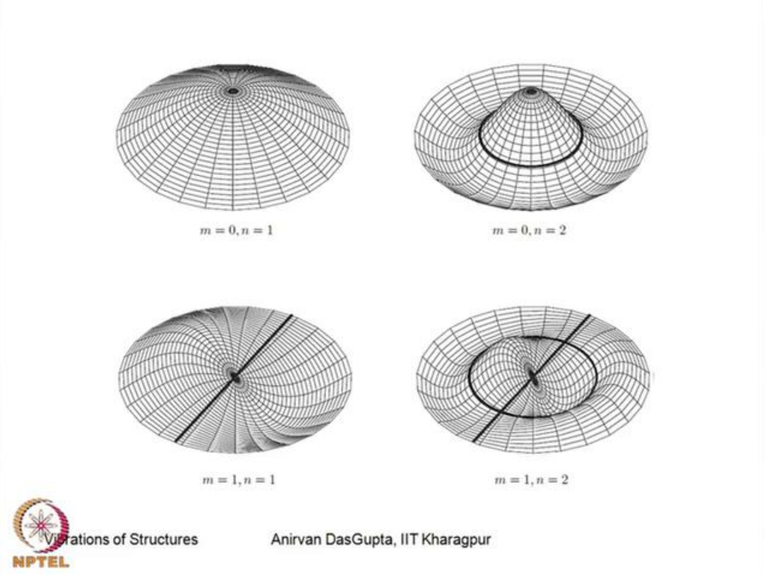

So, in this figure I have plotted out some of these modes. So this is, these two are the symmetric modes m equal to 0 and this is the first, corresponding the first root. So, n equal to 1 and this is n equal to 2. So, as n increase you have these nodal circles; so this is a nodal circle; and these are axisymmetric modes does you can see. So, n equal 2 has one nodal circle; n equal to 3 will have two nodal circles. Now as you increase m, you generate a nodal line or nodal diameter as you can see here. So, n equal to 1 you do not have any nodal circle; whereas, n equal to 2 you have one nodal circle, and for m equal to 1, you have this nodal diameter.

Now you see you, we have m not equal to zero for which means, say for example, n equal to 1, these two modes they are degenerate modes. So, corresponding to omega 1, 1 there are two Eigen functions. So, how are these two Eigen functions, so how do these two Eigen functions look like. So, one is the cosine, the other is the sine; that is the only difference; so that sets the orientation of this nodal diameter, because when it is the cosine mode, so, when m equals 1 and it is the cosine mode. So, it is 0 at phi equal to pi by 2. So, this is suppose pi by 2, then the sine mode will be at phi equal to 0; so, which means the nodal diameter just gets rotate at 90 degree. So, that would be the sine mode. So, why do we have degeneracy? Now it is very clear from this figure, what we had discussed for the rectangular membrane. We had discussed about the isotropy in the modal space; but now we can see this isotropy also in the physical space. You see the reference line from where phi is measured that is arbitrary. So, you can put that reference line here; then this is the cosine mode or if you take this as the reference line, then this becomes the sine mode. So, since there is arbitrariness, there is isotropy in the rotational direction of the circular membrane, so, we have modal degeneracy. So, here in this case, we can look at this nodal degeneracy or the isotropy also in the physical domain; and this definitely is there in the modal space. So, in order to understand this modal degeneracy, we have understood this concept in our previous lecture in terms of isotropy of the modal space. So, there are two Eigen functions, which are Eigen functions of a single Eigen frequency. So for a single Eigen frequency, there are two Eigen functions, which are independent, which are orthogonal. So, this leads to the degeneracy. So, any combination of these functions is also an Eigen function, any arbitrary combination is an Eigen function. So, we call this as an isotropy in the modal space or the configuration space of the membrane. Now in this circular membrane, this is also visible in the physical space.

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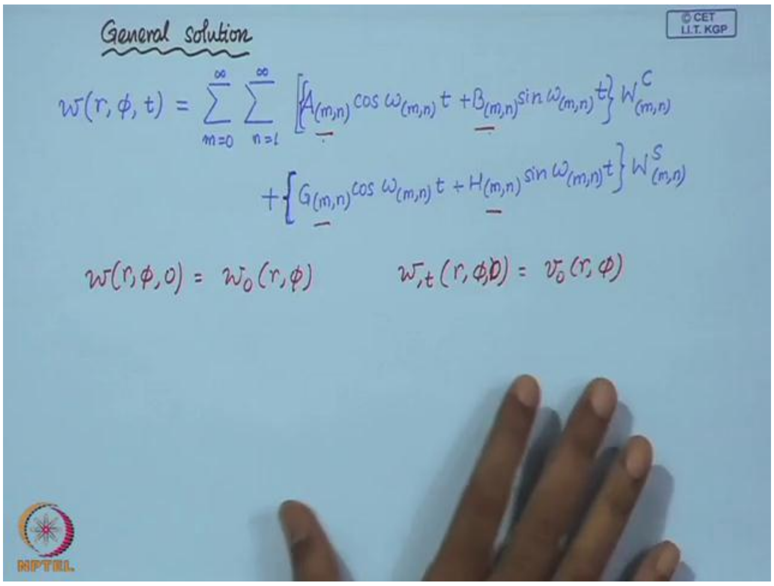

Now let us write down the general solution then. So, the general solution for the circular membrane may be written like this. So, here we note that m starts from zero, which indicates the axisymmetric modes; and there are these constants. So this is the coefficient of the cosine mode plus we have also for the sine mode. So, that is the general solution. Here these coefficients, the constants, they are to be determined from the initial conditions, so from these initial conditions. Then we can determine these constants, using the orthogonality property of the Eigen functions.

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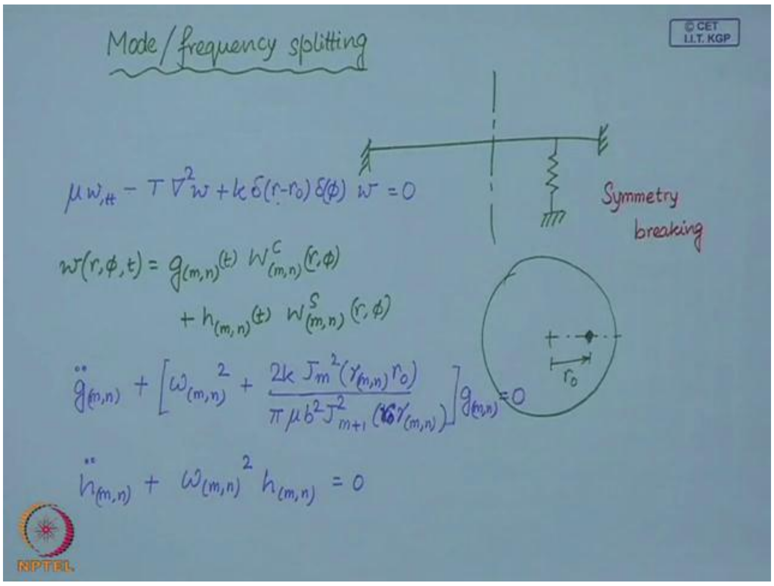

Now, let us look at an interesting property, which comes because of this modal degeneracy, which is called mode split splitting or frequency splitting. So, if you considered a circular membrane and you put an external interaction; in this case I have put spring. So, let us consider that we have, this is the top view. I have put the spring here; so at phi equal to zero; so this is the reference line phi equal to zero and r equal to r_0. So, in that case the equation of motion, I can write the equation of motion. So, here additionally, I have this stiffness at r equal to r_0 and phi equal to zero. Now, to understand what happens in this case, let us just look at a single mode expansion. So, this is the modal coordinate; this is the Eigen function, the cosine one. So, I have taken a single mode, actually corresponding to a single frequency; so there are two degenerate modes; so I have taken this two. If you substitute that and take the inner product and discretize the equation so that you get the dynamics of these modal coordinates, then it looks like this. So you see that this correspond to the sine modes and the sine mode has the nodes here. So, the sine mode remains unaffected; because of the spring the sine remains unaffected whereas the cosine mode has to get affected and this is the additional term that comes with omega m n square. So, which means that the frequency of these two modes now gets separated; this is called mode splitting or the frequency splitting. So, this takes place because of external interaction which now, as you can understand, breaks the symmetry. So, there is a symmetry breaking, as you can understand, I told you that a normal circular membrane has a geometric symmetry; but that is now broken, because of this external interaction and that immediately what it does is, it splits this frequency. There is no modal degeneracy now for corresponding to omega m, n. So, in this situation this is going to split the natural frequencies of the membrane.

So to summarize, what we have looked at, we discussed the vibrations of a circular membrane; we looked at modal degeneracy; and we have looked at this interesting consequence that comes, because there was degeneracy and when you break the isotropy or symmetry of the system, then you have mode splitting or the frequency splitting. So, with that, I conclude this lecture.

Keywords: circular membrane vibrations, modal analysis, Bessel functions, modal degeneracy, frequency splitting.