Vibrations of Structures

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Lecture No. # 34

Vibrations of Rectangular Membrane

In the last lecture, we have discussed the dynamics of membrane, a flat membrane; and today, we are going to carry forward our discussion on this, and look at the vibrations of rectangular membrane. So, just to recapitulate, what we discussed in the last lecture let me write down the general mathematical formulation of a flat membrane.

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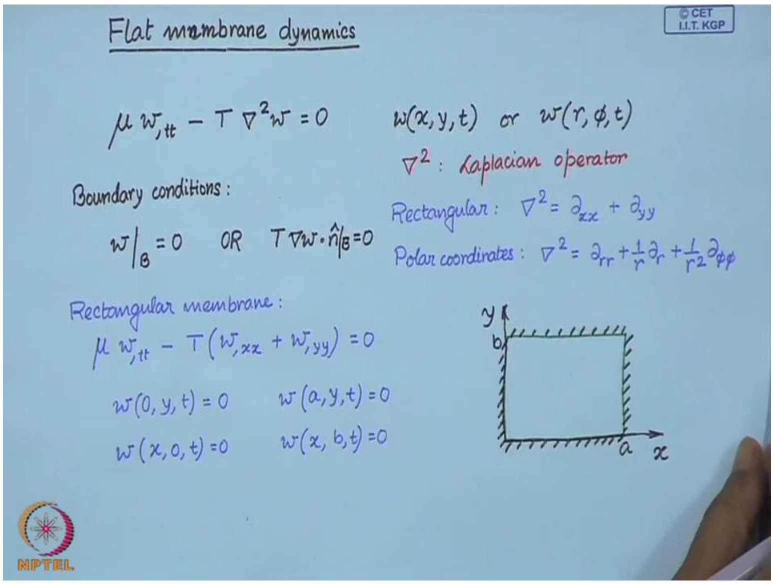

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So, we have derived the equation of motion of a membrane with density per unit area or the arial density mu, undergoing small transverse vibrations where this w is the field variable, which is the function of (x,y,t) or w could be in the polar coordinates; for example, could be function of (r,phi,t); and nabla's square is the Laplacian.

So, in the case of rectangular membrane, so in the Cartesian coordinate system, nabla's square is double derivative with respect to x plus double derivative with respect to y; in the polar coordinates, nabla's square is double derivative with respect to r 1 over r single derivative with respect to r plus 1 over r square double derivative with respect to phi. So, we have this Laplacian operator in two coordinate systems; and along with this we have boundary conditions. So we can have a fixed boundary or a sliding boundary. So, this is the force per unit length T times the gradient of the field variable dot the unit normal at the boundary. So, for example, for the rectangular membrane which we are going to discuss today; and we will assume that the boundaries are fixed. So, let me first… So, we have w at x equal to zero, which is the y axis and for all y and all time… at x equal to a… for all x at zero which is the but y equal to zero, which means the x axis… So, this is our mathematical formulation for this rectangular membrane. So, we are going to look at the modal vibrations of the rectangular membrane.

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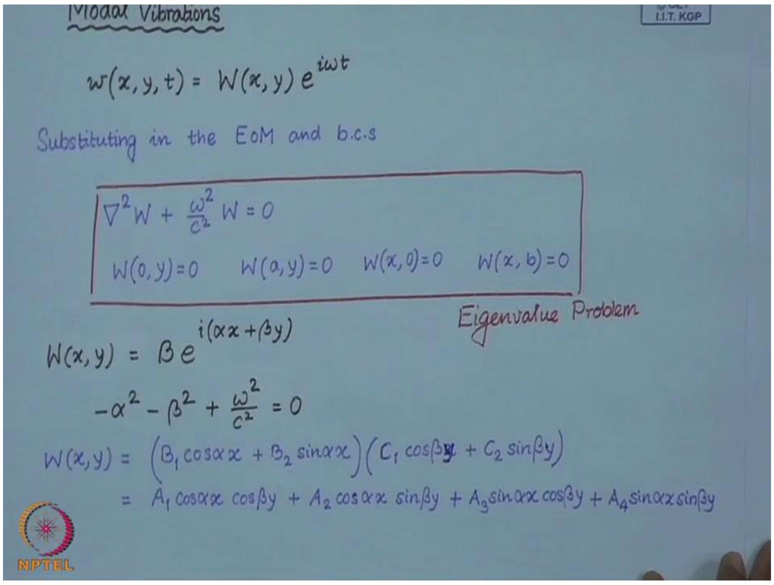

So, let me write down… So, we are interested in a solution as we have discussed before; so, this is separated in space and time. So, we have this spatial function W which is the function of x and y and this complex time function. Now, we substitute this solution form in the equation of motion and also the boundary conditions. We obtain so upon simplification… So, this is our Eigen value problem, where of course, this Laplacian is once again the double derivative with respect to x plus the double derivative with respect to y. So, in order to solve this, let us consider the solution of this differential equation of the Eigen value problem as some constant exponential i times alpha x plus beta y, where alpha and beta are as yet unknown constants. So, if you substitute this solution form in here… Since, we have double derivative with respect to x, so, that gives as alpha square, minus of alpha square, minus of beta square from the double derivative with respect to y plus… So, this is what we obtained. So, alpha and beta have to be chosen such as that this equation is satisfied; but that is not all, as you know that we have to also satisfy these boundary conditions.

So, we expect the solution in the form, let us say, since we have considered separable solution even in x and y, so, I can write… So, this may be simplified and written out like… So, I multiplying out and redefining this product as A_1, A_2 etc. So, we have this as the general solution of the spatial function W(x,y). Now, this general solution must also satisfy the boundary conditions which are going to give us further conditions.

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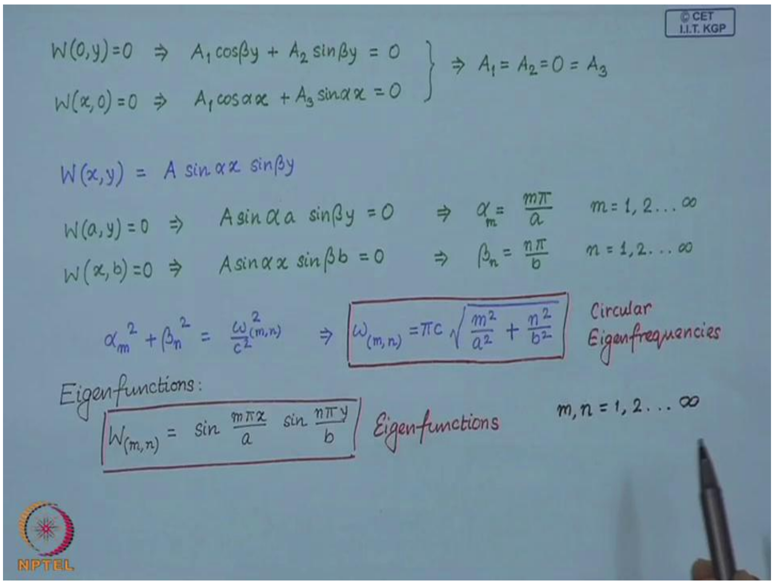

So, let me substitute this solution in the boundary conditions. So, you can see that this boundary condition would imply… So, x is put at zero… So, this follows from this term and here its sine alpha x, sine alpha again sine alpha x here; so at x equal to zero these two terms vanish; so we are left with this. Then, let me take this boundary condition. So at y equal to zero, so, this term will vanish, and this term will vanish. So these are the two conditions I obtained from these two boundary conditions. Now, it is very easy to see that these two boundary conditions imply… So for all y and all x, so for all y this condition should be true which immediately tells as that A_1 and A_2 must be zero; and this also tells as that A_1 and A_3 must be zero; so, A_1, A_2, A3 are all of them are zero; so we are left with only A_4. So, which means in order to satisfy the boundary conditions… let me call this only A since, there is only one term in this function, and we are left with two more boundary conditions. So, these are… So, when I substitute replace x by a, and the fourth one…. Now again, this condition should be valid for all y and this condition should be valid for all x. So, this should imply that sine of alpha x should be zero, since A cannot be zero. So, therefore, alpha must be m times pi over a, so this gets indexed; and there are infinitely many values of alpha m; and similarly, from here this gets indexed by n and there are infinitely many values of beta n. So then finally, we recollect that alpha m square plus beta n square must be equal to omega square over c square. Now this should also therefore get indexed. I will write as (m,n) within brackets. So, I can take this phi as well outside. So, it is square root of m square over a square plus n square over b square. So, that is, these are the Eigen frequency of the rectangular membrane; and then we obtain from this function, the Eigen functions which should also now get indexed. So these are the Eigen functions. Of course, here we have these infinitely many Eigen functions. Now here, you can notice that the higher, so the fundamental circular Eigen frequency is obtained when m and n are one and you can see that higher circular Eigen frequencies are no longer just integral multiples of the fundamental what we have seen in strings. So, in the case of strings, suppose I simply block this, then you obtain actually the Eigen frequencies of the string. So, there you have seen that the higher Eigen frequencies are just integral multiples of the fundamental frequency. Now, in the case of the membrane, this is no longer true. There is another possibility that we are going to discuss shortly and that is when m square over a square plus n square over b square is some r square over a square plus s square over b square. So, that is a possibility that means, there are two different modes which have the same circular Eigen frequency. So, that we are going to discuss shortly.

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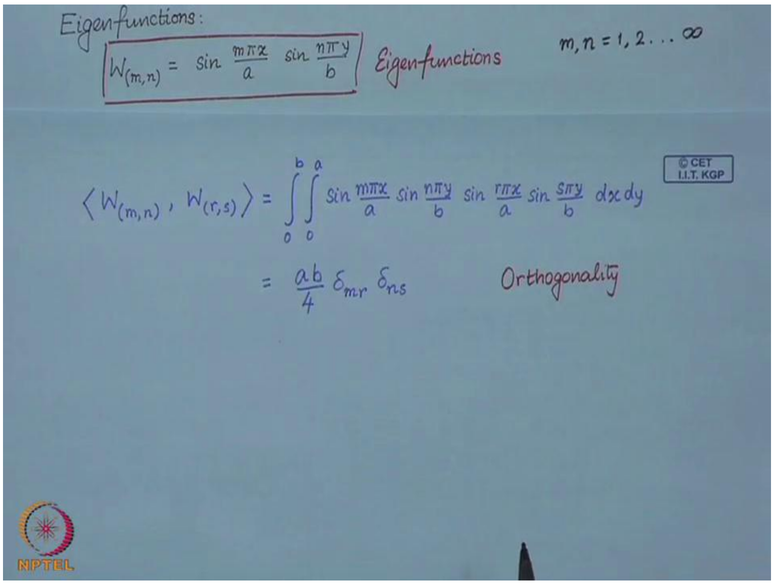

So, before we discuss, so here let me just tell you about the orthogonality of these Eigen functions; so which is very clear from these expressions. So, this inner product which is defined as the integral over the domain of the membrane… and we know from this trigonometric functions, properties of this trigonometric functions that this is nothing but… So for example, the integral of this over x from zero to a is a by 2; similarly this is b by 2; so this is a b by 4, but then that happens when m is equal to r and n is equal to s. So, these are Kronecker’s delta functions. So, this takes a value one, only when m equals r and similarly, this takes a value one when n equals s; otherwise they are zero. So, this is the orthogonality property of the Eigen functions.

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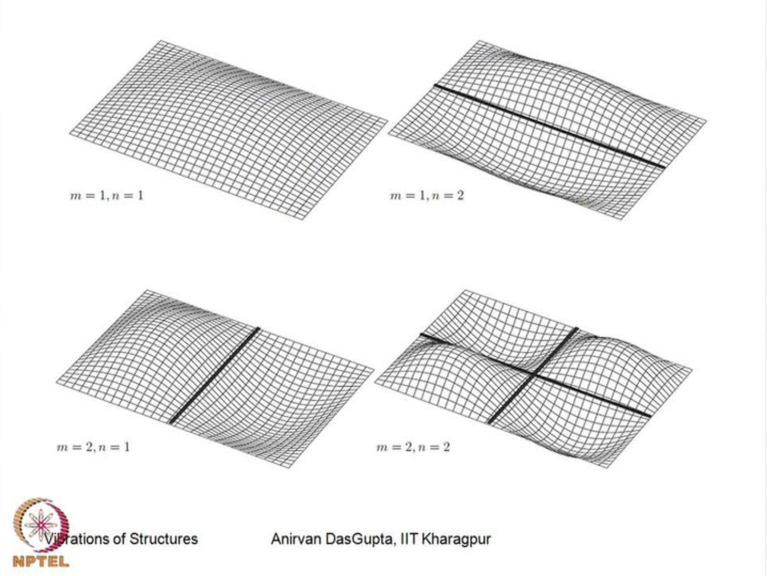

Now, before we move on, so let us have a look at some of this Eigen functions. So, here I have plotted four. So, this is the fundamental which is with m equal to 1 and n equal to 1; so, there are no nodal curves in this mode. Here, this is m equals 1 and n equals 2; so there are two lobes here. So, m is for this direction, n is this direction. So, we have one nodal line along, so this is parallel to the x axis. This is for 2 1; so, one nodal line parallel to the y axis; and this is m equals 2 and n equal to 2; so, this is the W (2,2) mode. So, you have two nodal lines like this. Now the general solution, so, this Eigen functions can now be used for constructing general solution for any problem related to rectangular membrane.

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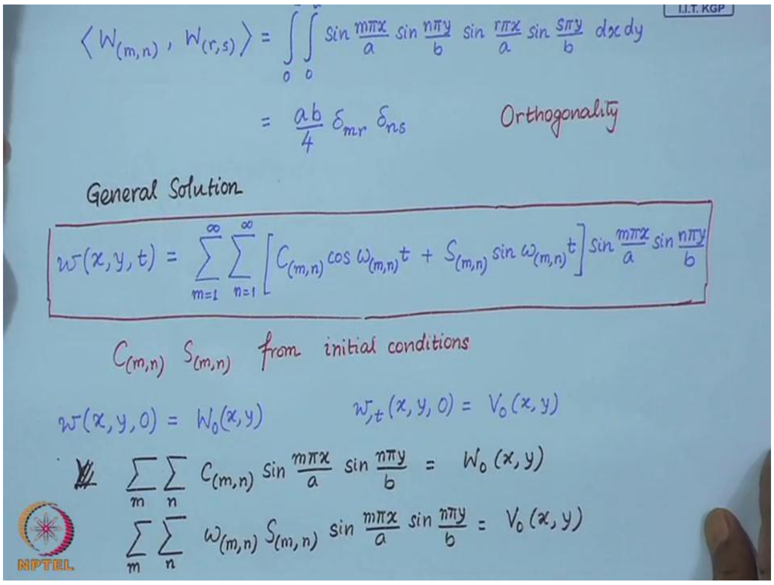

So, let me write out the general solution. So here, so motion of the membrane, the general motion of the membrane may be written as the summation over… Now, all these constants that I introduced, we have these indices (m,n). So, this is the general solution, which can be used for solving the initial value problem; and here these constants, they are determined from the initial conditions. Now in order to do that, we have to use the orthogonality property. So the initial conditions will be in the form… so the initial configuration of the displacement of the membrane and the initial velocity distribution. So, given these two functions capital W_0 and capital V_0, then we can write this… and similarly, for the velocity condition… we have this. So, we can solve now for this infinitely many coefficients C and S using the orthogonality property that we have just now discussed; and we have discussed this in a previous lecture as well. So, we can determine these constants and finally, construct the solution using the summation.

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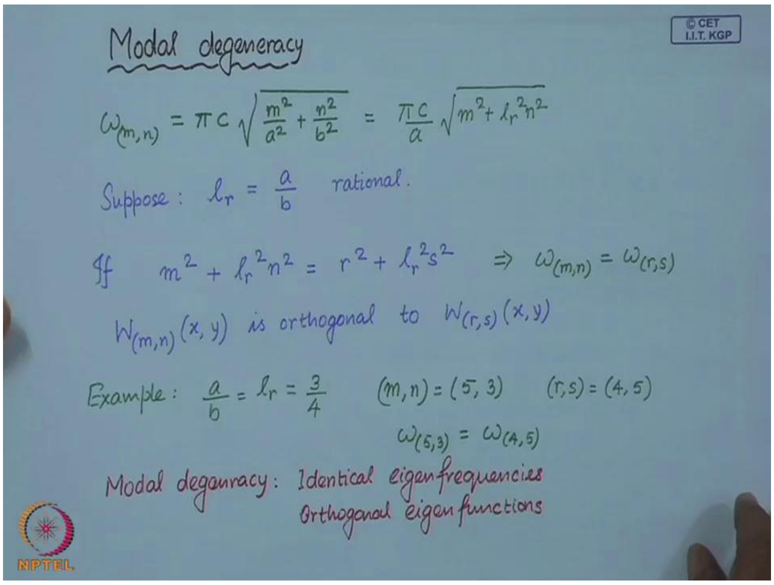

Now, let us consider this situation we have an interesting property called… So, here what we have, let us look at the expression of the circular Eigen frequency which reads like this. Now, suppose you have this ratio a over b, let me call it the length ratio; suppose this is a rational number. So, if this is a rational number, in that case, we have, we can have two integers a, b such that that equals the length ratio. Now when this happens, then we can have an interesting possibility that so, we have this possibility m square plus l_r square, this ratio's square, for different number r and s, different integers, if this equals this, so once we have this, we have defined this ratio, so you can write this as… So, I can express this in this form in terms of l_r. So, now, if for two, so for a given (m,n), if there exists another mode with r and s such that these are equal, in that case, we have two identical Eigen frequencies, circular Eigen frequencies. So, there are two modes with identical circular Eigen frequencies, but then remember that the Eigen functions they will be orthogonal, they still will be orthogonal. So, we have two modes with identical circular Eigen frequencies, but orthogonal Eigen functions. Now, this is an interesting occurrence. Now, let us see an example. So, for example, if a over b which is l_r is 3 over 4; in that case, if you take (m,n) as (5,3) and (r,s) as (4,5), then you can easily check that for l_r 3 over 4, these two give identical circular natural frequencies; but their Eigen functions are orthogonal. Now, this is an interesting situation. So, let us understand, so this is known as modal degeneracy. So, you have identical Eigen frequencies, but distinct Eigen functions or orthogonal Eigen functions.

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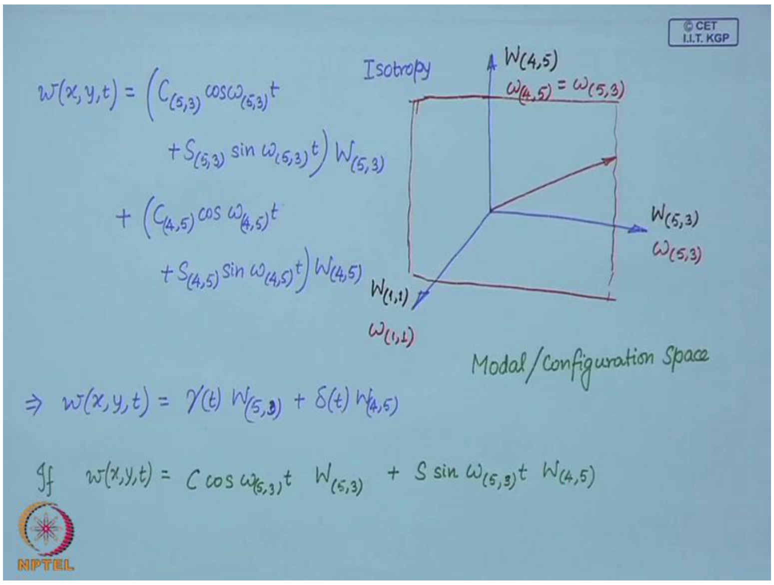

So, let me graphically try to depict this situation. Now, we have been discussing this visualization in various contexts. So, let us consider the modal space of the membrane. So, this is the modal space or configuration space of the membrane. So, the axis, so which are have drawn perpendicular to depict orthogonality. So, for example, this is let us say W(1,1), and then this is… Let us take the example that we have just now considered; this is W(5,3) and again there are many such axis which are orthogonal to one another; I am only drawing three, and you have to stretch your imagination little bit. So, this mode, in this modal space, in which I have drawn only three of these orthogonal Eigen functions. You see, this is an Eigen function corresponding to omega (5,3); similarly, this (1,1) and this is… but this is same as omega (5,3). Now, if this is an Eigen functions corresponding to omega (5,3), then this is also the Eigen functions corresponding to omega (5,3). So, therefore, any linear combination of these two Eigen functions is also an Eigen function of omega (5,3). So, anything that lies in this plane; so, this is the W(4,5) W(5,3) plane; so anything that lies in this plane any vector, any function that lies in this plane is also an Eigen function of omega (5,3). So, we have some kind of an isotropy in this modal space. So, this plane is isotropic; any function on this is a Eigen function. So, this property is completely missing in the case of strings. So, in strings, for Example, we do not have modal degeneracy; we do not have this property at all. So, this is one distinction of the membrane from the string. So, once again this so this space which is spanned this function space which is spanned by these two Eigen functions the membrane dynamics is isotropic in this. So, imagine that your exciting this membrane close to omega (5,3). So, you can expect the solution… So, the solution would be something like this. So, this can be written as… Now you can imagine that these two, so these two frequencies are actually one and the same. Now, it depends on how the two modes have been excited. So, what will happen is since the combination of, so now the actual solution is the combination of these two Eigen functions; but this is the mode corresponding to single frequency omega (5,3), but these two modes are distinct. So, what is going to happen is you are going to see a combination of these two modes appearing in the vibrations of membrane and it may so appear that the membrane is vibrating in an unsteady manner, because it might be switching from possibly this mode to this mode or through a combination of these two modes; so, that you can understand from this solution form. So, because of this isotropy of the modal space, we have this kind of a phenomenon.

So, for example, if you excite this in a certain manner such that… Since (4,5) is same as (5,3), suppose the excitation initial conditions have been set such that this is the solution, let us say. Now, in that case since one is cos and other is sine, so at time is zero, you have this mode; you can see the nodal lines corresponding to this mode; and at time two pi by omega, you have this as zero; and this is active; so you will see the number of nodal lines corresponding to W(4,5). So, it is going to continuously make transitions from W(5,3) to (4,5) and back; but this is the Modal solution. So, you will actually see changing profile of the membrane; there is no fixed nodal line. So, they keep varying according to this. But suppose the excitation is cos; in that case, there will be a fixed combination of W(5,3) and (4,5), which is the will be C times W(4,5) plus S times W(4,5), because this will be cosine, this is also cosine; in that case, that will come out and therefore, you have fixed kind of a shape of the membrane vibrations. So, this is an interesting feature of modal degeneracy, which is present in membranes and not in strings.

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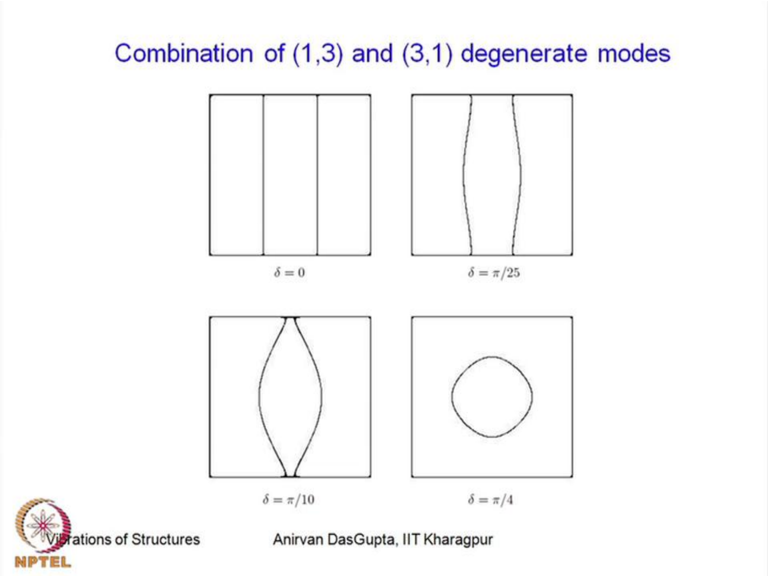

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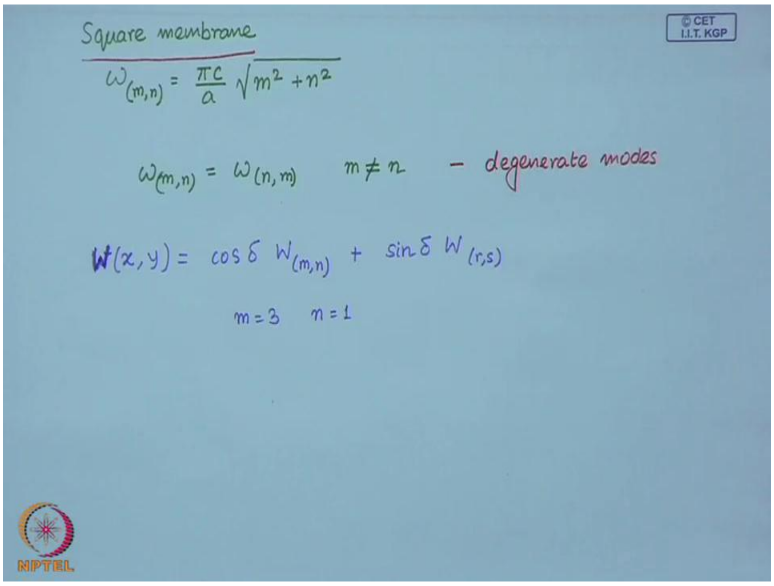

So, in this picture, I have shown this for a square membrane. Interestingly, for the square membrane you see the… So, with side equal to a, in that case this is m Square over a square n square over a square, that comes out; so square root of m square plus n square. So, you will find that omega (m,n) is equal to omega (n,m), where m not equal to n; and then these are degenerate modes. So, all modes with m not equal to n are degenerate modes for the square membrane. So, for the square membrane for m not equal to n these are all degenerate modes. So, once again, let us have a look at this picture. So, here we have this combination; so the combination I have written like this… So, I have written the combination like this for the (3,1) mode.

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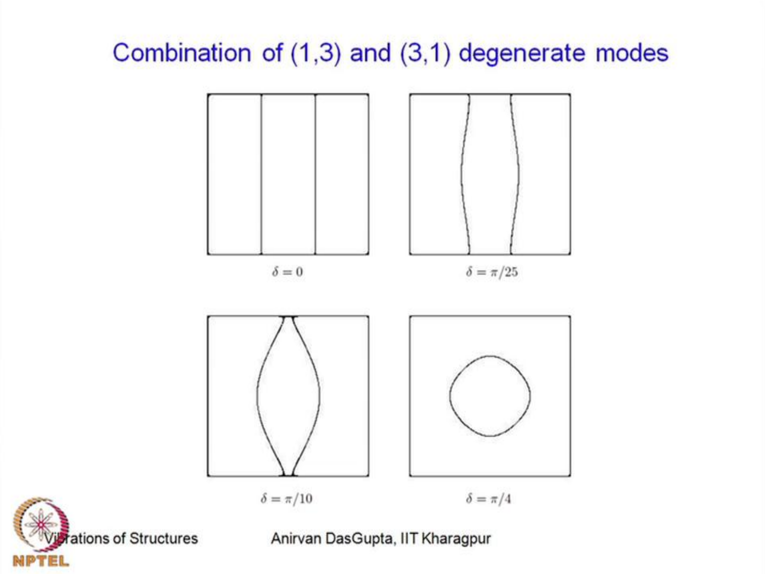

So, let us look once again at this picture. So, here when delta equal to zero, so you will find it is only W(3,1); and here, its value pi over 25; so, you see a combination of (3,1) and (1,3) this is still higher value of delta. So, this combination changes; at pi over 4 this combination of (3,1) and (1,3) gives as a nodal curve which is like this; and this changes for if you increase further to this but rotated, then this rotated, and then this rotated which means the nodal lines will be horizontal for delta equal to pi over 2. So, you see when a Square membrane vibrates in this combination then you will see the nodal pattern, nodal line patterns changing continuously through these shapes. So, you will not see a steady nodal line picture on the membrane. So the nodal lines will be kind of unsteady. So, this is something very interesting that you can observe in an experiment. So, to summarize we have looked at the Modal vibrations of rectangular membrane; and we have observed very interesting property which is the modal degeneracy. Some of these aspects, we will discuss further in our next lecture. So, with that, I conclude this lecture.

Keywords: rectangular membrane vibrations, modal analysis, modal degeneracy.