Vibrations of Structures
Prof. Anirvan DasGupta
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur
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Dynamics of Membranes

Today we are going to initiate some discussions on vibrations of Membranes. So, today first we are going to look at the dynamic modeling of membranes. Now to begin with what are the membranes? A membrane is the two-dimensional elastic continuum which cannot resist or transmit bending moment. Now, when we discussed about the strings, the string was a one-dimensional elastic continuum which does not resist bending moment. But now this is, membrane is a two-dimensional elastic continuum; but then there are some fundamental differences between string and membrane, which means that membrane is not just a two-dimensional extension of string.

So, let us first, in order to start our dynamic modeling, let us first enumerate our assumptions that we make further the modeling, mathematical modeling of membranes. First of all where do we find membranes? We find membranes in ear drums, on drum heads and in certain kinds of microphones etc. Now, we study membranes not just to study drum heads, ear drums or microphones but to understand this two-dimensional elastic continuum, which is simplest of its kind, because it does not transmit bending moment. Now in order to model membranes, we will assume that the membrane is flat; and the deflection is purely transverse. So, when you have, let us say an inflated membrane, let us say a balloon, in that case the dynamics is more complicated, because the deflection can no longer be considered as just transverse; there is a coupling between the transverse and the in-plane modes. So, we are going to make this assumption that our membrane we are going to analyze is flat so that only we have to consider the transverse deflection of the membrane. The second assumption that we make is the slopes are small. So, if the deflection is, if you have certain deflection, then by small slope you can imagine that the maximum deflection divided by the characteristic dimension of the membrane; let us say, I have a membrane of this size, then the transverse deflection should be much much smaller than this dimension or this dimension of the membrane. So, slopes should be small. We assume that the tension is uniform and it does not change with the deflection; so it is constant; it does not change with the deflection; and finally we also assume that the thickness does not change. So, under these assumptions, we are now going to look at the modeling of the membrane.

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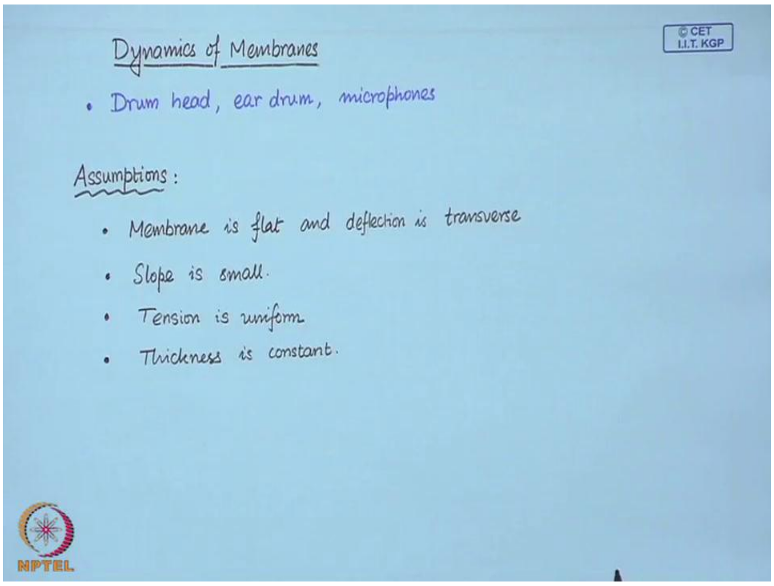

So, let me first draw an infinitesimal element of this membrane. So, here we have, I have considered an element of this membrane, which was initially flat on the x-y plane. So, we have, in order to locate the membrane, we consider this field variable w which is a function of x, y and time. So, here we have these forces. So, I have taken a little element of the membrane; and I have drawn the free body diagram. If this point is (x,y), I will denote this on the force on this edge as T. So, here I have, here this is dx, this is dy; similarly the force on this edge is T(x,y,t). So, these are the forces acting on this little element. Now here this T is the force per unit length. So, you can think in this way that there will be stresses on this cut. So, if you integrate this stress over the thickness, then you obtain this T, which is force per unit length. So, stress integrated over the thickness of the membrane is force per unit length; so, some kind of resultant, stress resultant. Using this, we are now going to write down the equation of motion.

So, let me consider this little element; so the transverse dynamics… So, let mu be the mass per unit area. So, mu is the mass per unit area. So, the area of this element may be written as mu times dx into dy; so, that is the mass of this little element times the acceleration in the transverse direction that must be equal to the net force in the transverse direction. So, I can write… So, this is… So this T is the force per unit length of the cut now, because it has been integrated over the thickness; so, it is equal to the force per unit length of the cut. So, for example, the force because of this on this face is t times dy, because the length of this face is dy. So, that is the force, but it must have a… So, I must take a component which is the in the transverse direction; so that component, so if this angle let us say is phi, then I must have sin. So, I must take sine of phi to find out the component of this force in the transverse direction.

Now, sin of phi is approximately equal to tan of phi and that is equal to del w/del x. So, to find out the transverse force because of this, I will use del w/del x at x plus dx and y, and of course at time t; and on this side, it would be del w/del x at (x,y); and similarly from these two edges. So, this T dx is the force. So, since we are assuming that the tension in the membrane is uniform; so it is essentially isotropic, is homogeneous. So, everywhere it is T though I have written it like this to indicate that this vector might be different. So, magnitude wise it is the same. So, here it is del w/del y is in this direction it is del w/del y.

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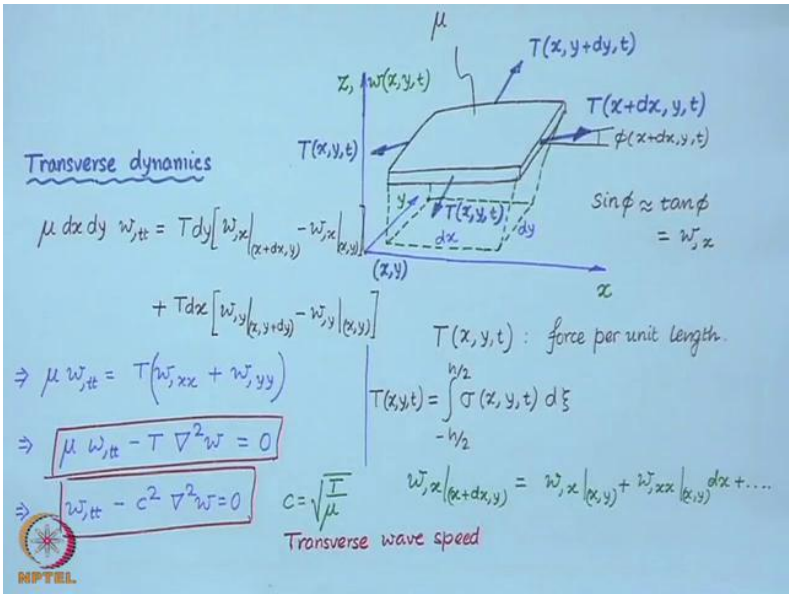

So, this must be the equation of, this must lead us to the equation of transverse dynamics. So, if you divide through out by dx dy and make some simplifications, so this, so del w/del x at x plus dx., so let me write this down. So, del w/del x(x+dx,y) may be written as… and so on. So, I can have an expansion, the Taylor series expansion of del w/del x at (x+dx,y) in terms of del w/del x at (x,y) and del w/del x and its higher derivative at (x,y).

So, using this expansion here I can therefore finally, write this as… So, from this will contribute a term del square w del x square, and this is going to contribute the term del square w del y square. So, that implies now this is the Laplacian of w; and this can also be written as… where this c is… Once again looking at the structure of this equation and the dimension of this quantity, so, this has a dimension of speed. So, this is the speed of transverse waves in the membrane. So, this is the transverse waves speed in the membrane. So, this is the structure of equation for transverse vibration of membrane. Now, you see that here we have double derivative in x as well as y. So, we would require boundary conditions at on these edges. So, let us look at the boundary conditions.

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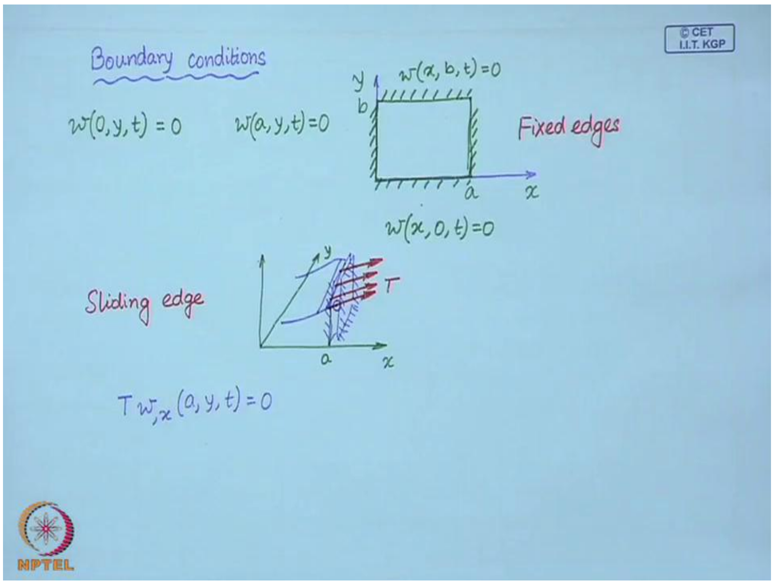

So, if you have a, let us say a rectangular membrane; please note one more thing that we have derived the equation of motion considering the Cartesian coordinate system. We will look at the general situation in a moment. So, if you have this x-y plane and a flat rectangular membrane, then if this is fixed on these edges, then we can easily write the boundary conditions. So, at x equal to zero, that is on this face and for all of y and for all time, this is fixed. Similarly, on this face we would have w at x equal to a, for all of y at all times must be zero. On this edge, for all of x at y equal to zero, this must be zero; and on this edge… So, when the edges are fixed, you have zero displacements. There is another possibility, where you can have sliding edge. So, in a sliding edge, the membrane can, so, it is very similar to the sliding edge of a string. So, if you have an edge of a membrane which can slide, in that case the condition at this boundary, let us say at x equal to a… It is very similar to that of the string. So tension times the force per unit length times the slope that must be zero. So, this is the force per unit length at this edge. So, the whole thing is… So, this is that the tension in the in the membrane, the force per unit length. So, we must take the transverse component of this force. So, that is determined by taking the, multiplying it with the derivative of w with respect to x as we have seen. So, that must be zero for a sliding boundary.

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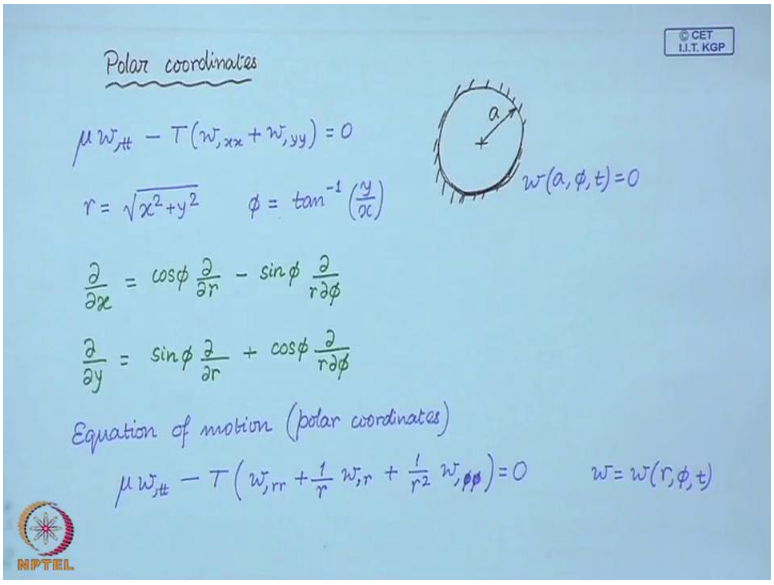

Now, when we go to the polar coordinates, we use polar coordinates when we have a circular membrane. So, suppose you have a circular membrane of radius a; then first of all the equation of motion for this membrane… So, we have the equation of motion for a rectangular membrane written out in this form. So, we consider, we can derive the equation for a circular membrane from this by simple coordinate transformation. So, if you consider that the radius, radial coordinate r is square root of x square plus y square and phi is tan inverse y over x; so using this, you can derive these operators del/del x in terms of derivative with respect to the radial and the angular coordinate. So, if you use these transformed operators, then finally you can get the equation of motion in the polar coordinates. So, this is obtained as… where now w, the field variable is the function of r, phi and t; and for fixed boundary we have… So the boundary condition is quite simple.

Now, this choice of coordinates actually is guided by the boundary. So, for simple boundaries is like, rectangular membrane or circular membrane the choices is very clear; but for other shapes some of which we are going to discuss later on we will see how they are dealt with. So, here we have discussed about, right now about the Newtonian formulation. Let us also look at the variational formulation.

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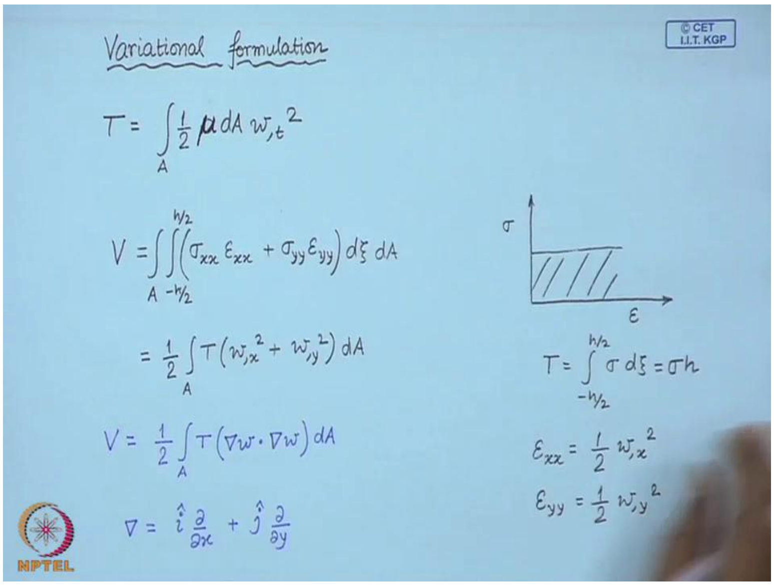

So, the variational formulation for the dynamics of membrane; once again I mean to motivate, so, the variational formulation is quite powerful; not only it gives as the equation motion also the boundary conditions; and even more important than that it gives us methods by which we can solve the or discretizised the dynamics of continuous systems, for example, we have looked at Ritz's method. So, for that reason we must also discuss the variational formulation for the membranes. Now to begin with, we write down the kinetic energy expression of membrane. So, if w is our field variable, then half, so we have to consider mu as the mass per unit area; so, mu times dA is the mass of a little element of the membrane; so, half mass times velocity square, and this when we integrate over the area of the membrane, we are going to get the total kinetic energy of the membrane.

Next, we would like to find out the potential energy, when the membrane deflects. So, to determine the potential energy, we have consider that the force per unit length or the tension in the membrane, the stretch in the membrane, so, that remains almost constant it does not change. So, if you thing about it in this way that the stress and the strain, if the stress in a material, because if stress remains fixed only then the force per unit length, which is determined by the integrating the stress; so that must also be constant, because T is fixed. So, the energy per unit volume is given by the area under this. So, it must be this sigma in the x times epsilon in the x plus sigma in the y times epsilon in the y. So, this is per unit volume; so, you must integrate this over the volume; so first over the thickness, and then over the area of the membrane. So, zi is the along the thickness direction.

Now this stress, so this is all taken almost uniform over the thickness. So, this turns out to be sigma into h; so, sigma is T over h and in both directions it is the same. Now, the strain, we have seen from the case of the string, the strain is half of del w/del x whole square. So, in the other direction as well we must have… Now, so therefore finally, when you substitute all these expressions in here, so this is what you will get. Now this can be written as… in a coordinate independent manner; we can write this… where is delta is for the Cartesian… so, this is the gradient of this field variable, the magnitude square of that vector, the gradient of the field variable.

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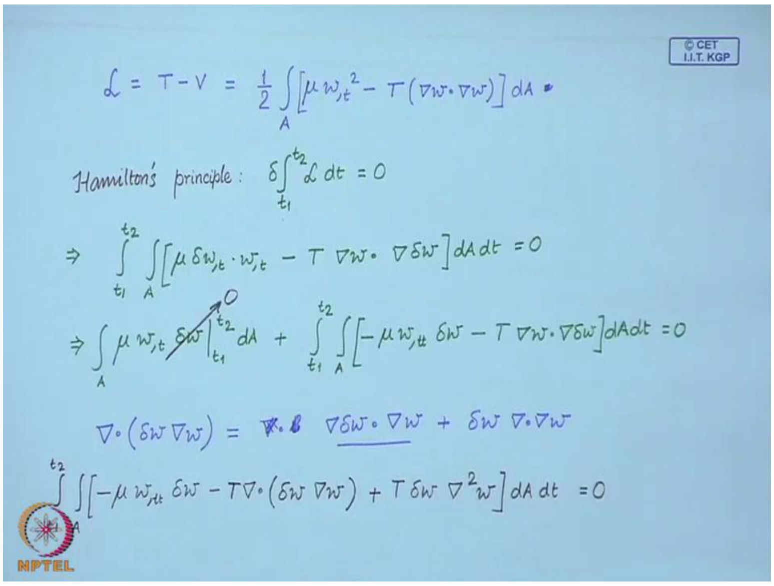

Now, so this is the potential energy, so finally, our Lagrangian… So that is the Lagrangian; and from Hamilton’s principle we can write… so this must very vanish. So, the variation of the action integral, which is the integral over time of the Lagrangian between two time points; so that variation must vanish. So, if you consider this Lagrangian and take the variation; now here I have used this property that the variational operator commutes with the gradient operator. Now this term, I will integrate by parts with respect to time ones. Now, for this term let me… So, this term, in order to simplify this term, let us look at this identity. So, I am taking the divergence of vector. So, this is the vector and this is the scalar. So, this scalar multiplied by a vector and we know that this turns out to be… So, this is the gradient of the scalar dot the vector and so this is the divergence of this vector. Now, so therefore, I can use this here to rewrite this left hand side.

Now, this term goes to zero, because at t_1 and t_2 that there is cannot be any variation. So, what I am left with is this. So, this goes on this side; so this comes with positive and this is the Laplacian. So, this must vanish. So then, now here I have the divergence of vector quantity.

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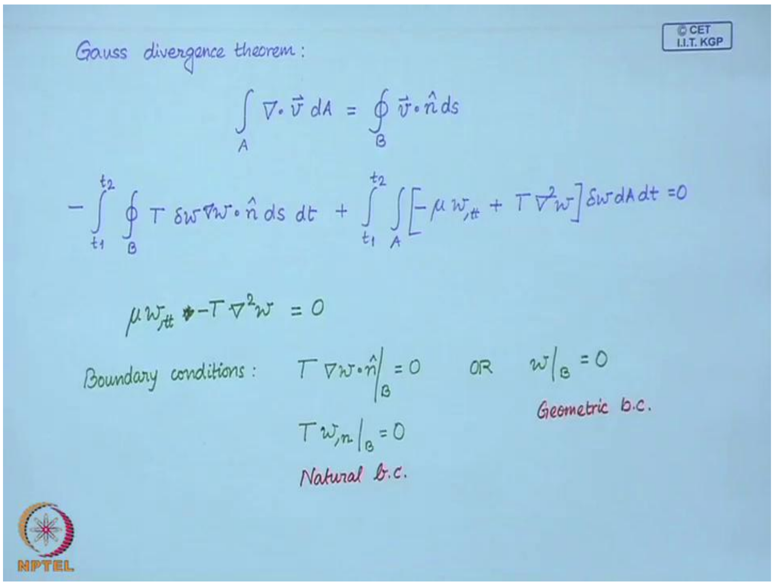

So, here I can use the Gauss divergence theorem, so which says that the divergence of any vector over this area integrated must be equal to this vector dot the normal to the boundary integrated over the boundary. So, if you use this here, then we have boundary term arising out of this which… so this integral over the area of this term, that gives us this boundary integral… dot the normal to the boundary. Now, once again we apply the argument that the variation over the boundary and over the domain are independent; so therefore, we had obtained directly from here the equation of transverse dynamics that we have obtained before.

Now, we have, from this boundary term we obtained the boundary conditions. So, we must have… so this calculated at the boundary of course. This is sometimes written as… so this is the normal derivatives; so, gradient dot the unit normal at the boundary is the derivative along the normal to the boundary. So, this is a, this boundary condition is a natural boundary condition whereas, this is the geometric boundary condition.

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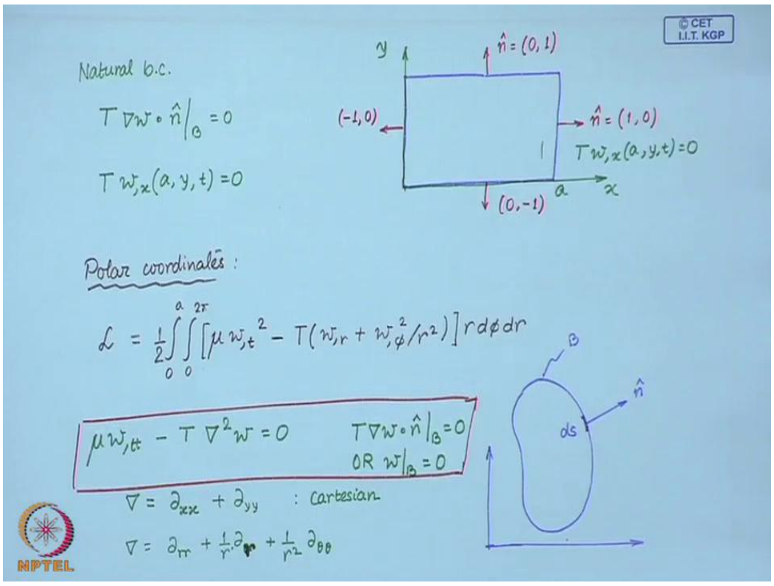

Now, let us consider this example. So, if you have for example, rectangular membrane, so, the unit normal at the boundaries… Let us says this, so let us understand this natural boundary condition which says… So for example, at this boundary at the right boundary at x equal to a, so n is (1,0). So, therefore… so the right boundary, you have this condition if it is dynamic or natural boundary condition. If the boundary is fixed then of course, you have the geometric boundary condition, which says that the displacement is zero.

Now, just briefly let us look at the polar coordinates. So, the Lagrangian in the polar coordinates may be written as… So, in the case of let us say a circular membrane, you have the kinetic energy and the potential energy and of course, there is a one half. Now so in the general case, suppose you have an arbitrary boundary; so here if you have an arbitrary membrane, then you can… So, this is the boundary; you can represent the unit normal ds; and then the boundary conditions for an arbitrary boundary can be written in terms of more general coordinate free notation.

So finally, you have in the coordinate free notation, the equation of motion. So corresponding to, for example, the Cartesian coordinates, this is nothing but… and for the polar coordinates… So, this is the operator and the boundary conditions can be written like this. So, this is the general representation of the equation of motion of the flat membrane, which is undergoing transverse vibrations. So, to conclude we looked at the dynamics of membranes both from the Newtonian as well as from the variational perspective. In the following lectures, we are going to study the vibrations of membranes with circular and rectangular shapes. So with that, we conclude this lecture.

Keywords: membrane vibrations, Newtonian formulation, variational formulation, boundary conditions.