Vibrations of Structures

Prof Anirvan DasGupta

Department of Mechanical Engineering

Indian Institute of Technology, Kharagpur

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Vibrations of Rings and Arches

Today, we are going to discuss the vibrations of rings and arches. In the previous lecture, we initiated some discussions on the dynamics of curve beams, and we had discussed about beams with constant curvature which are in your plane. So, before we look into the vibrations of rings and arches. Let us recapitulate briefly, but we discussed in the previous lecture.

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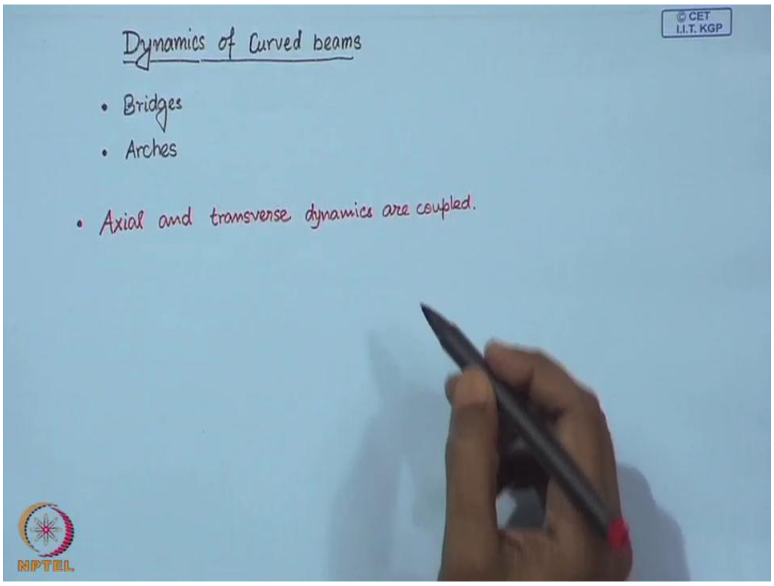

So, we considered… So, curved beams are found in various civil structures like bridges; they are used in arches, so and various other places. Now, we looked at the dynamics in the previous lecture on the dynamics of curved beams, and we observed that the important aspect of the dynamics is that the axial and the transverse are coupled. So, they are coupled because of this curvature of the structure of the beam. So, we had made some simplifying assumptions when we discussed about the, when we formulated the dynamics. We assumed that the beam is still planar though it is curved in a plane. We considered that, we assumed that the deflection is much smaller than the thickness of the beam; and the thickness in turn is smaller, much smaller than the curvature which is assumed to be a constant; and we also assumed that the Euler Bernoulli hypothesis holds, which means that cross section of the beam which was initially perpendicular to the neutral fiber remains perpendicular to the neutral fiber, remains flat and perpendicular to the neutral fiber even after deflection. So, we neglected shear which means that we considered that the beam is infinitely stiff in shear. So, with such considerations in the previous lecture, we have derived the equation of motion using the variation of formulation.

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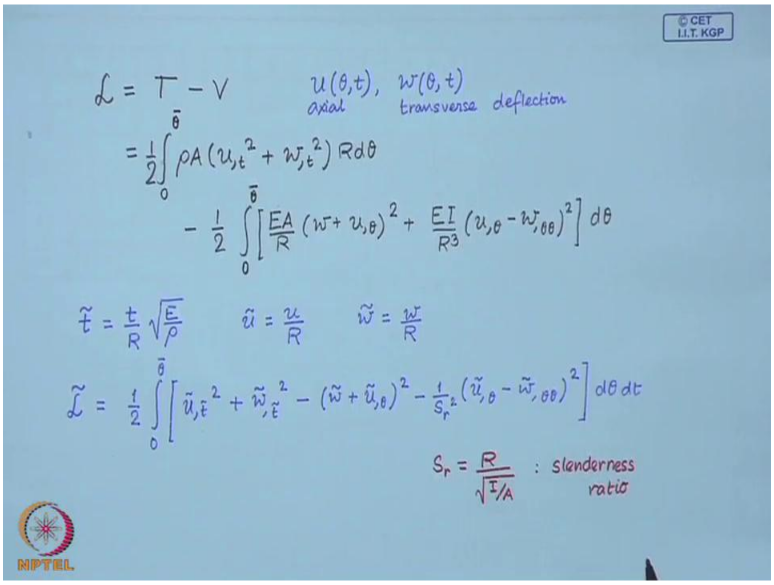

So, we considered the Lagrangian which we wrote as the kinetic energy minus the potential energy; and the kinetic energy was one half of rho A… Here, these are the field variables. So, u… So, this is the field variable for the axial or circumferential motion or deflection and this is the transverse… w is for the transverse deflection. So, this and minus the potential energy we calculated as… So, here the angle varies from say zero to whatever angle you have. So, the angular extend of the beam. Now, in the previous lecture, we also made some simplifications based on certain redefinitions. So, let us consider some non-dimensionalization. So, the time is non-dimentionalized. So, t tilde is the non dimensional time; u is non-dimensionalized using the radius of curvature of the beam; similarly w was non-dimensionalized. Now using this non-dimentionalization, we can rewrite this Lagrangian. So, this is our Lagrangian. Here S_r, we have defined as the slenderness ratio, which… So, this is the slenderness ratio which tells us how slender the beam is. So, higher the value the more slender it is. So, it is the radius of curvature divided by the radius of gyration of the cross section about the neutral axis. Now, so with this Lagrangian, we derive the equation of motion. So, this was the equation corresponding to u, the circumferential motion and corresponding to the transverse motion… We have these two equations. So, today we are going to first discuss the vibrations of rings. So, in the case of… So, the equation, the two equations of motion… So, this is the circumferential motion, and this is the radial or the transverse motion. So, we imagine that we have a uniform ring. So, this is the radial direction and this is the circumferential or tangential direction. This is the angle theta.

Now, the boundary conditions; we discussed the boundary conditions for a complete ring like this. So it turns out to be periodicity conditions on the field variables. So, we have the periodicity conditions on the field variables. Now, we are going to perform the modal analysis of this system. So, we search for solutions with the structure… So, we are interested in solutions with this separable structure.

Now you see that, this is a function of theta and t. Now it must periodic in theta. So, we must have solutions of the form like this… where n can take values 0, 1, 2, etc. So, this is to enforce the periodicity conditions, to satisfy the periodicity conditions that we have written here. So, we search for solutions of this form. Now, here if n is zero, then as you can see this becomes independent of theta, which means then we are talking about axi-symmetric modes; so, modes which are independent of theta. For non-zero values of n, we have non axi-symmetric modes. So, let us see what happens when we consider a solution like this. So, we substitute the solution in the equations of motion.

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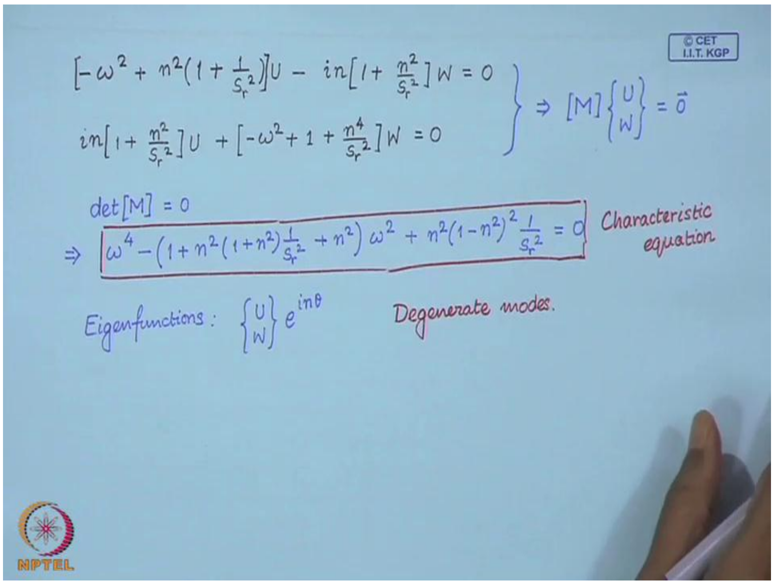

So, if you do that then you can check that upon simplification of these equations… So, this is the first equation. The second equation reads… So, these are the two equations that we obtained by substituting the solution form, modal solution form in the equations of motion. Now for non trivial solutions of U and W, this capital U and capital W, we must have the determinant of this matrix… So, we can write this as matrix. So, determinant of M must vanish. So, for non trivial solutions of U and W… and that leads to the characteristic equation, which can be obtained easily. So, this is our characteristic equation. Now, we have to solve for omega from this equation, substitute in these two equations, and then solve for this Eigen vectors U and W; and then, we will obtain the Eigen functions. So, you see the Eigen functions will be complex like this. So, U and W, themselves, will be, can be complex, because you have this i here in these equations. So, U and W are themselves complex; and hence, this Eigen functions are all complex.

Now, we have discussed already that when we have complex Eigen functions, both the real and the imaginary parts of these can be the Eigen functions and so, what we can conclude is that for a given Eigen frequency, we can have more than one Eigen function. This is called degeneracy. So, we have multiple Eigen functions for a given Eigen frequency. Now, let us see then; solve this equation, characteristic equation, and try to find out the Eigen frequencies and the Eigen functions which will characterize the mode of vibration. So, we start with the value n equal to zero.

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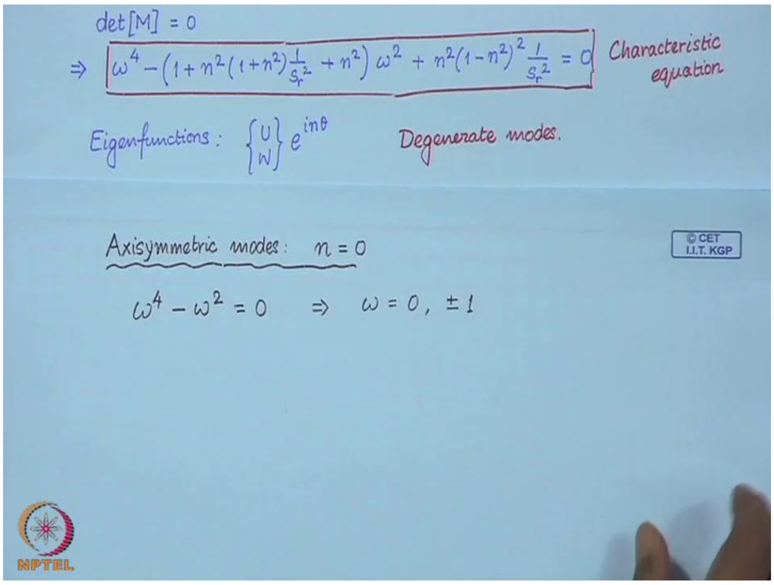

So, let us consider axisymmetry. So, n is equal to zero. So, if n is equal to zero, then you can see straight from here; so from here, so n being zero, this is the characteristic equation for axisymmetric modes. So, that would imply… So, we have omega equal to zero and omega equal to plus or minus one. So, let us first look at omega equal to zero.

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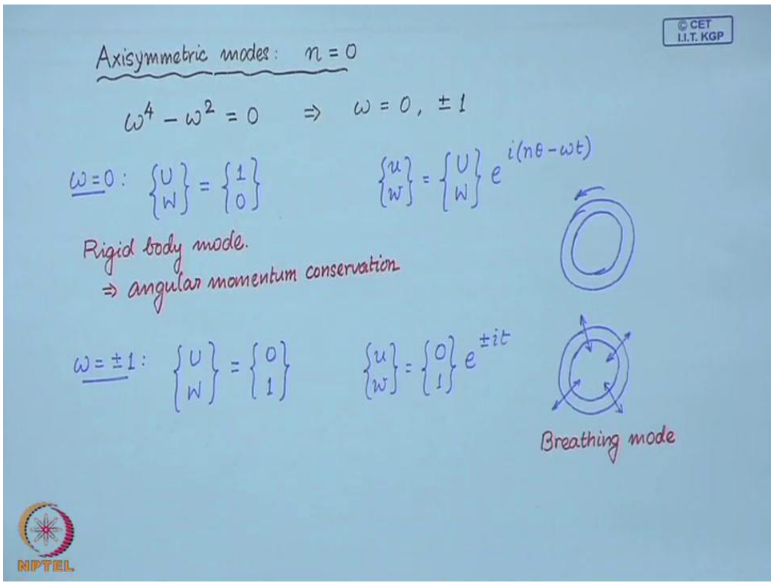

So, the Eigen vector corresponding to omega equal to zero turns out to be… If you solve the matrix equation, then U and W turn out to be 1 and 0. Now, this means that, see the solution was… So, n is 0, omega is also 0. So, this term is absent and so the motion is, this is 1, and this is 0. So, which means there is a motion along the circumferential direction of the ring with zero frequency.

So, this is nothing but rigid body mode. So, this is the rigid body mode which implies that angular momentum is conserved. So, this is not vibratory mode. So, next let us look at the other solution which is omega equal to plus or minus 1. So, in this case if you solve the Eigen vector that turns out to be 0 and 1. So now, there is no motion in the circumferential direction, the motion is… So, here n is equal to 0, but omega is plus or minus one, which means it is an oscillatory mode. So, it will be an oscillatory mode which is only in the radial direction. So, you have something known as breathing mode. This is sometimes known as a breathing mode. So, the ring expands and contracts axisymmetrically. So, this is a breathing mode. So, this shows the breathing mode.

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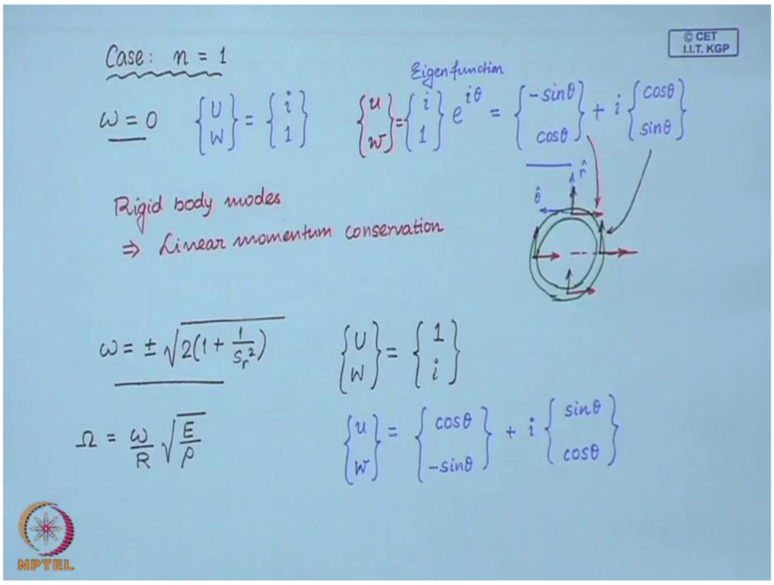

Now, next let us consider the case n equal to one. So, now, we are talking about non axisymmetric modes of course, because there is theta dependents; as soon as n is non zero, we have theta dependents. So, in that case if you substitute n equal to 1 in the characteristic equation, once again you will find that the first solution is omega equal to zero. So, this is one solution for n equal to one, omega equal to zero is again a solution.

So, that you can see directly from here once again; so, if n is equal to 1, this term drops out in the characteristic equations. So, once again omega equal to zero is solution and then there is another solution which can be determined from here. So, the first solution is omega equal to zero; again we suspect that this is a rigid body mode which it is. So, if you calculate once again the Eigen vectors, that turn out to be i and one. So, let us see what this means. So, this gets multiplied; the Eigen functions are nothing but… So, these are the Eigen functions and that can be written as… So, if you consider this solution then let us see how is the deformation. So, this is the ring. So, now, this is the deflection of the center the neutral fiber. Now, sine theta with the negative, so this implies that… So, the theta equal to zero is the datum. So, this is u, this is the u motion, and this is the w motion. So, u is zero here, whereas w is plus one; and then you will find that at this point theta equal to pi by 2, so, this is minus one; now minus one would mean, because the axis here is like this, so minus one would mean this; and similarly you can find out that this represents the motion like this, and similarly this Eigen function vector will represent a motion like this.

So, these are nothing but the rigid body modes in the x direction, and in the y direction. So, these are again rigid body modes which imply the linear momentum conservation. Now, let us consider the other solution for n equal to one. We saw that there are two solutions; one was zero; the other one turns out to be this and the Eigen… So, for this, the Eigen vectors happen to be given by these complex notations. So, this turns out to be… following these notations. So, if you multiply this with exponential i theta, so, here we have… So, these are the Eigen function vectors. Now, here these are actually the non-dimensional frequencies. Now, if you want to find out the dimensional frequencies, they are given by… So, these are the dimensional frequencies. So, these are the Eigen functions that we have for this mode. Now, let a have a look at this mode. This shows, because of some aspect ratio problem, it might seem like an ellipse; but actually this dashed one is a circle which is the undeformed position of the ring. Now here, this black one is the mode corresponding to the imaginary part as you can see here. So, let us understand this motion before we look at this picture again. So let us consider this imaginary part. Once again, let me draw the ring. So, if you consider the imaginary part of the solution, so this is the Eigen function, the motion can be written as… where omega is of course given by the circular Eigen frequency. Now, in the circumferential direction, you have this as sine theta, which means here there is no circumferential motion, but there is radial motion. Here, you have circumferential motion as you increase; here the circumferential motion is maximum. This is positive plus one in this direction. When you come here, the motion is purely radial, no circumferential motion at this point. Similarly, we have circumferential motion in the negative direction here at this point and no radial motion here. So, which means you expect the ring to… So, the centerline of the ring will look like this. It seems that the centerline, atleast the geometric center of the figure is shifted; this is an exaggerated figure of course. So, it seems that at least the geometric center is shifted, but actually the center of gravity of the ring is not shifting, because of combination of radial and circumferential motion. So, the particles are actually moving in this direction circumferentially, while radial motion is like this. So, we look at this figure one second now. So, here you can see this is the unreformed ring and I have drawn these empty circles to indicate the particles before deformation, and this filled circles are the particles after are this material points after deformation.

So, you can see that there is a clustering of these materials points here. So, you have compression here and expansion here. So, that finally, it is a momentum conserved mode. So, continuing this way you can then solve for higher modes.

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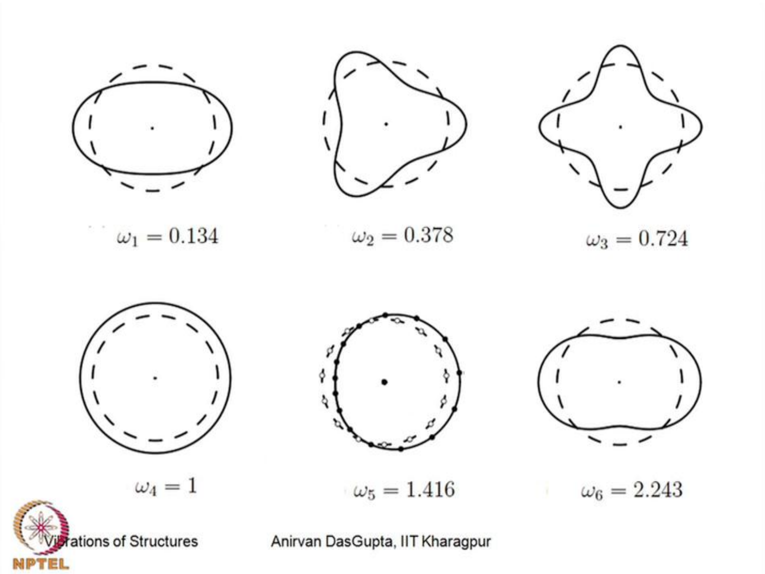

But then let us see something interesting. Now, we have discussed as yet, till now this mode which is the breathing mode and this mode; but they are not the lowest mode. The lowest mode appears to be like this. Once again there is some aspect ratio problems. So, this dashed one is the unreformed ring, and this is the deformed configuration. So, you see this, so in this mode, it is moving out here and moving in here and vice-verse. So, this is going to oscillate like this and. So, there is a phase difference between this and this which is pi.

Here, you have the higher modes. You can see the circular Eigen frequency non-dimensional; this is the next higher, and this is the fourth mode, and this is the fifth mode and so on. So, you can calculate all the Eigen frequencies of various modes and also plot the Eigen functions. Now, looking at these figures it might seem that these are nodal points, but this has to be checked properly, because now we have not only radial motion but also circumferential motion. So, if node is considered to be a point at which there is no motion then this might not be nodes.

For example, here this is definitely not node. So, occurrence of actual nodes has to be checked by looking at motion of material points on the ring. So, till now, we have been discussing about vibration of rings.

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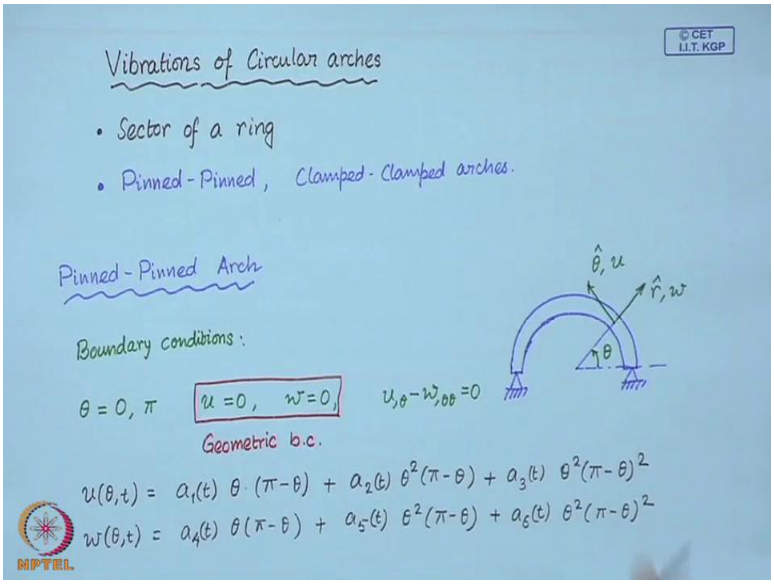

Next, we will discuss vibrations of arches, circular arches. So, circular arch is nothing but a sector of a ring. We will consider two examples of arches; one is the pinned arch and a clamped. Now, as you will realize that the equations of motion for the arches, for the curve beams they are coupled and complicated, so what we are going to do for the case of arches is that we are going to solve this approximately using the Ritz's procedure. So, let us see what are the boundary conditions; because we need to choose admissible functions for applications Ritz's method. So, let us look at first pinned-pinned arch. So, this is a schematic representation of a pinned-pinned semi circular arch. Now the boundary conditions, if you look back the discussions that we have on the boundary conditions of the curved beams, then at theta equal to zero and at theta equal to pi, we must have u equal to zero, which is the circumferential motion and w equal to zero; and we also must have… So, we have six boundary conditions as we have discussed. Here, we have zero displacement at these two points and this is the zero moment conditions at these two ends. Now, we have these as the geometric boundary conditions. So to choose, now we have to choose the admissible functions; we can choose the admissible functions. So, we expand our field variables; so the way I have chosen for this problem; so the first admissible function, this is zero at theta equal to zero and also zero at theta equal to pi. So, I can now construct… So I have taken a three term expansion for u and similarly a three term expansion for w using the same admissible functions. Now these expansions we substitute in the Lagrangian. So, we have a semi circular arch zero to pi. So, this is our Lagrangian; we substitute these expansions here and do the integration over theta; and finally we obtain the discretized equation of motion in terms of this coordinates a. So, we will obtain… where this vector a is a_1 to a_6, and we can perform the modal analysis and obtain the circular Eigen frequencies. So for this arch, pinned-pinned arch, the non dimensional circular, the first two non-dimensional circular Eigen frequencies are obtained as… Now from here, you can calculate the dimensional circular Eigen frequency. Now this figure shows the first two modes of vibration of the pinned-pinned arch. So, you can see that, so, this is the unsymmetric mode and this is the symmetric vibration mode which is having a higher frequency.

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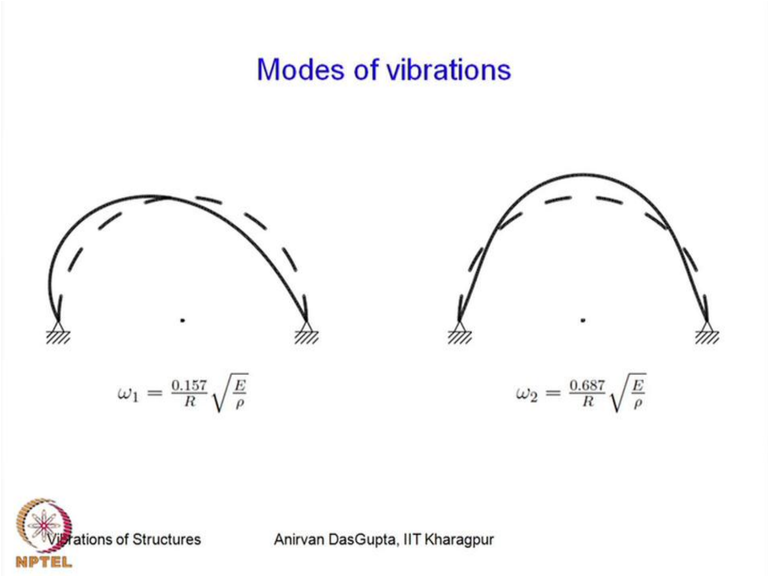

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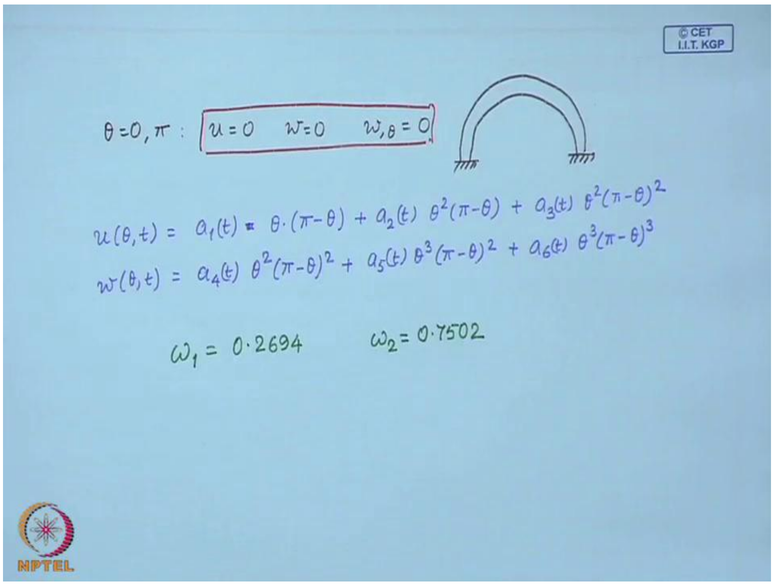

Now, in a similar manner you can perform for the clamped-clamped arch. So, this is the clamped-clamped arch. Here, the boundary conditions at zero and pi happen to be like this. Now, here all the boundary conditions are geometric boundary conditions.

So, in view of these, we have this expansion. Now here, because you have this slope conditions as well, so, del w/del theta condition; so, the admissible functions for w must be taken like this, theta square pi minus theta whole square. So, if you once again substitute this in the expansion, and calculate the discretized equation of motion and further calculate the Eigen frequencies, they turn out to be… So, the non-dimensional circular Eigen frequencies appeared as... So, these are higher than the pinned-pinned case as we expect. So, here this figure shows the modes of vibration. Again this first mode is the unsymmetric mode, and this is the symmetric mode with higher frequency.

So, let us recapitulate what we have discussed today. We discussed the vibration of rings, and arches. So, we have looked at some interesting results in the vibrations of rings; and we have considered the two kinds of semi circular arches, and using Ritz's method we have determined the Eigen frequencies and modes of vibrations. So, with that I conclude this lecture.

Keyword: circular rings, arches, modal analysis, Ritz method.