Vibrations of Structures

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Lecture No. # 31

Dynamics of Curved Beams

Discussions on vibrations of beams, we have till now looked at only straight beams. However, there are various places, you find curve beams; for example, in arches, in bridges, also we have rings, which can be treated as beams, which are curved. Today, we are going to initiate some discussions on the dynamics of curved beams. Today, we are going to look at essentially the modelling aspects of a curved beam. Now, the first and the fundamental difference between a straight beam and a curved beam that we will see is, because of this curvature, the axial motion or the circumferential motion is coupled to the transverse or the radial motion of the beam. There is a coupling between these two directions.

We can no longer treat using one field variable for our deflection. So, we must use two field variables; one for tracking the circumference of the axial motion, and the other is for the radial motion. As with any beam theory, we are going to make some assumptions to simplify our modelling process.

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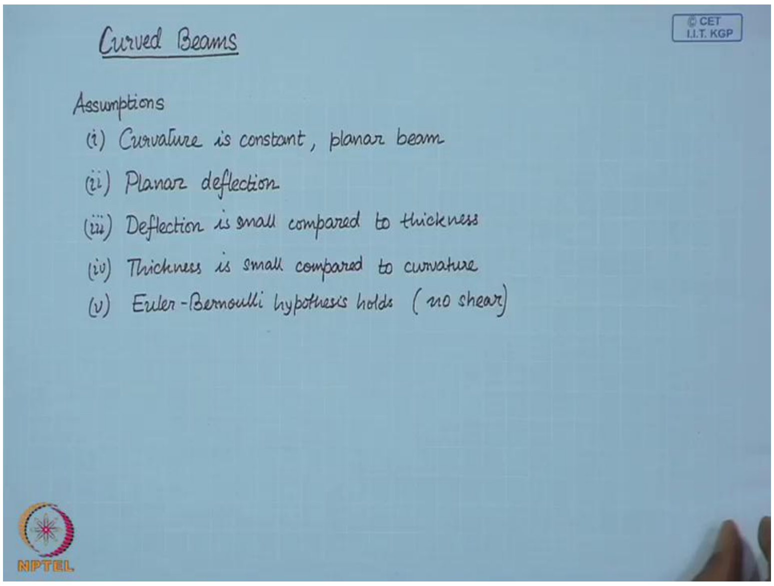

The first assumption; so, the first assumption we make is that this whole the curvature of the beam is; we will look at a very special situation where the curvature of the beam is constant and the beam is planar. The second assumption we make is the deflection is also planar. Thirdly, we will assume that the deflection is small compared to the thickness; compared to the thickness of the beam, the deflection is small. We also assume that the thickness itself is small compare to the curvature.

Finally, we assume that the Euler Bernoulli hypothesis holds and along with that we say that there is no shear. So, under these assumptions we are going to model a curved beam. We are going to restrict ourselves to the case of a beam with constant thickness.

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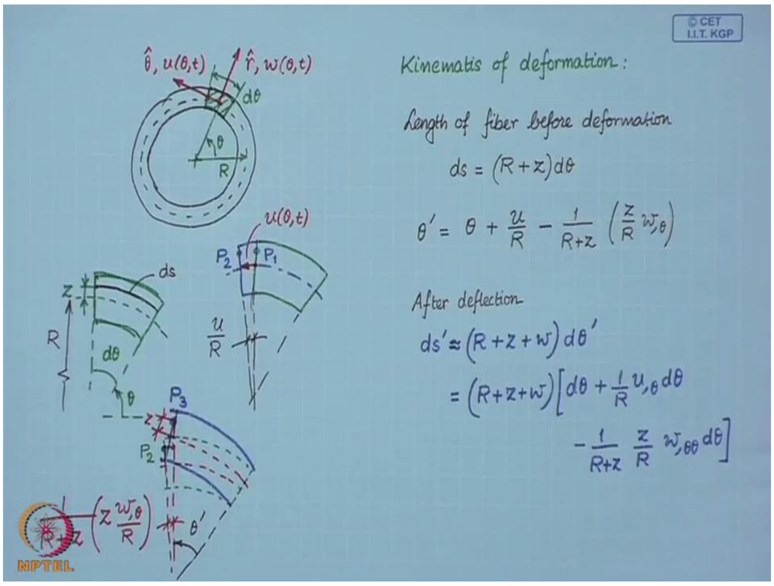

Let us consider… Here I have drawn a ring really, but it could be a part of a ring. We consider the small element of this ring. So this is the radial direction and the field variable will be indicated by w(theta, t); and this is the theta direction; where the radius of this circle, the dashed circle, which we will consider as the neutral fibre that has a radius r, which is constant as we have assumed. Now, let us look at this little element. Now we are going to look at the deformation kinematics of this little element. So, let me draw. When you consider the deformation kinematics, you can understand that, this is at an angle theta and this is a small angle d theta. Now, initially the length of any fibre in this element at a height z; the length of this fibre before deformation; let me indicate this by ds. We are looking at a fibre, which is at a at a height z from this neutral fibre; ds, the length of this fibre before any deformation is R plus z times d theta. Now when this element deforms, now you can imagine that you can consider this deformation in two steps; one is its axial elongation. This moves from any point which was here, moves here. So, this angle is nothing but u over R; u is the deflection of this point in the circumferential direction as we have mentioned that this is u. The second deformation that can take place is that it deflects out radially. So, let me draw that first. Here is P_2 and now here is P_3. So, this point P_2 moves to this point P_3.

Now here again, there is this angle, which can now be written like this that this is the slope of the central line of this element when there is deflection in the radial direction or the transverse direction and which is captured by this field variable w. So, del w/del theta one over R that is nothing but that slope of the central line, because w is the deflection of the of the neutral axes. So, that multiplied by z, z is the distance of this point from this neutral fibre, this time z is the deflection. So, this is nothing but the angle; for small deflections we know that this is the angle times the radius gives the linear deflection of point P. So, it goes from P_2 to P_3; essentially this linear distance is what is being measured by this quantity and when you divide this by R plus z that gives us this small angle.

Therefore: when you combine these two deflections, then the total angle. So, this when this line went from here to here and now this line has travelled back, so, this angle let us say is theta prime. Then theta prime is given as theta plus u over R, because of this motion circumferential motion and minus this angle. So, that is theta prime. So, therefore, after deflection ds prime, if we call it ds prime, then that is approximately, this is R plus z plus w is the transverse or the radial deflection. So, that is the new radius times d theta prime. We have to calculate d theta prime. So, that turns out to be d theta plus, since R is constant, del u/del theta times d theta minus…

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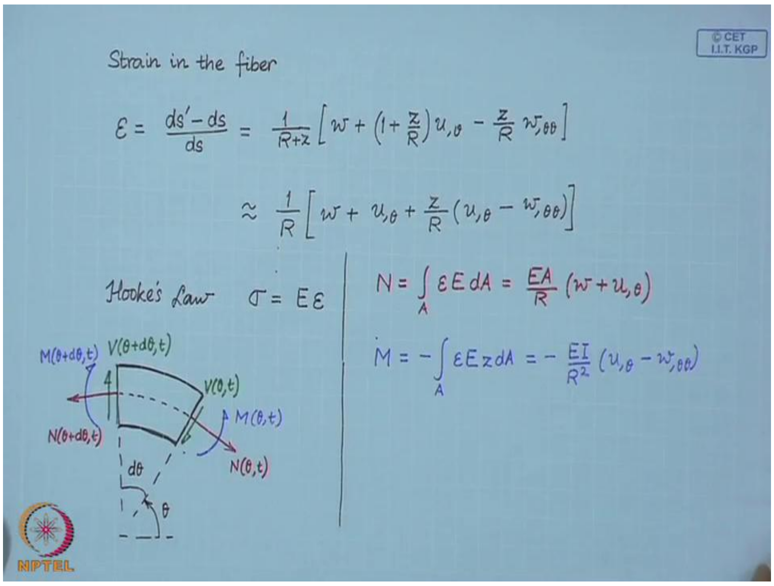

So, this is the expression of ds prime and therefore, if we calculate the strain now, in that fibre at height z from the neutral plane, that can be calculated as ds prime minus ds over ds; and if you do this calculation it turns out to be and which can be approximated by considering that z over R is much smaller than 1. So, you can take this R and write this as 1 plus z over R and take it in the numerator and you would leave out terms which are quadratic in z over R, then it can be simplified. So, that is the strain in the fibre. Now, using this strain, we can use Hooke's law to write the stress as Young's modulus times the fiber strain. So, that is going to give us the fibre stress. Now these stresses can be integrated over the cross-sectional area of the beam to obtain the force resultants. So let us calculate the various force and moment resultants. Let me first draw out the free body diagram. So, this is a little element and we have the shear force on this face, the bending moment and the circumferential forces. Now, let me calculate these stress resultants. So, N is nothing but the integral over the area, stress times the area; and if you substitute these expressions and calculate this, it turns out to be… Since there is a z term here, when you integrate over area, since it is already measured from the neutral fibre, so this term will vanish. So, you are left with this. So, this is the expression for the normal force on the face. Then we have this bending moment, which once again we calculate as we did for the beam.

Now, because of this additional z, this becomes z square and that makes it even function and whereas, this becomes an odd function that cancels off. So, you are left with contributions only from this term. So, that is the bending moment. Now using and the shear force, as I have mentioned that we do not consider shear, it is infinitely rigid in shear. So, that will come out from the equations of equilibrium. So, let us now start writing the equations of equilibrium.

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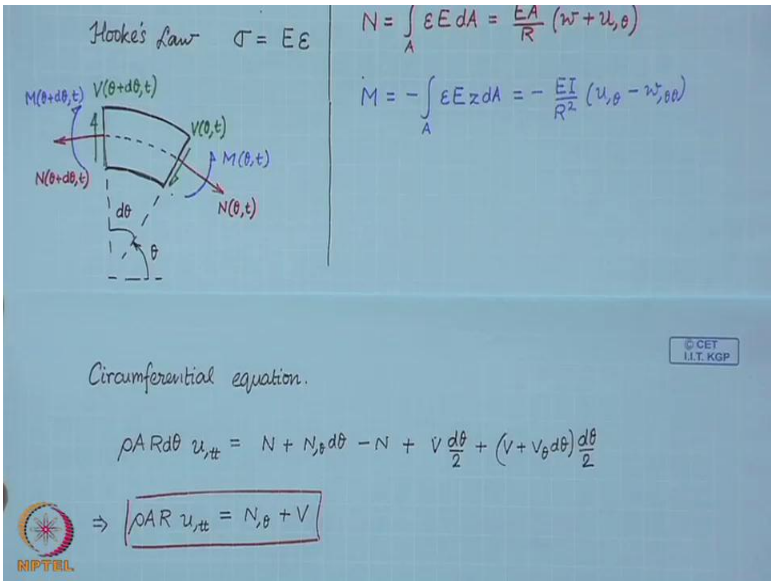

First, we will write the circumferential equation. So, that reads rho A is mass per unit length; R d theta is the small length of that element; this is mass of the little element times u is the circumferential motion or the field variable; the acceleration and that must be equal to the forces in the circumferential direction. Let us look at, refer to this figure once again. So, in the circumferential direction we have, N theta plus d theta can be written as N plus del N/del theta into d theta and this minus this into cosine of this small angle that is small; that is taken as 1 minus N, because of this plus; because of the shear force, for example here it is this into sine of this small angle which is d theta over 2. So, that is almost d theta over 2 and plus; we have this again therefore, if you divide by d theta and drop terms smaller than the first order then, you can easily write this equation. This is the equation of motion in the circumferential direction. Next we look at the radial direction.

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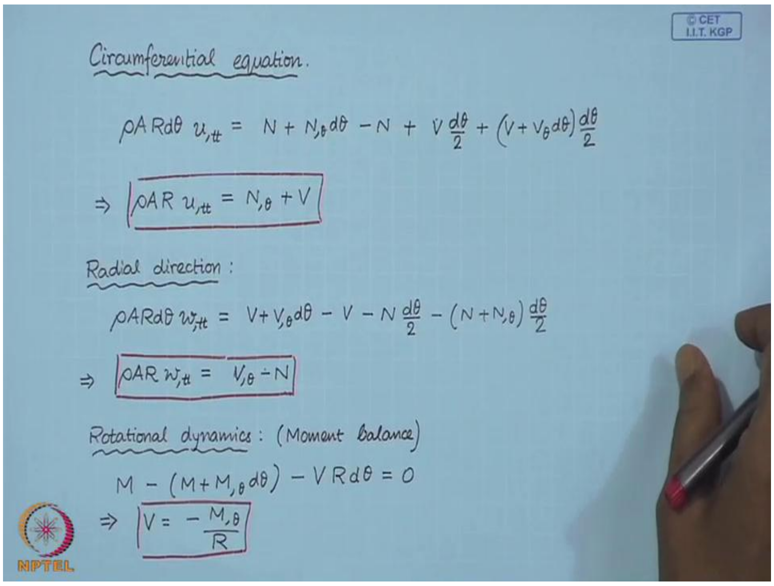

So, once again rho A R d theta is the mass of the little element times the acceleration in the transverse or the radial direction must be equal to the summation of all forces in the radial direction; this is what we have. You can write this as V, then there is this normal force which is towards the centre; and this is negative and there is the projection. These are the forces in the radial directions. So, that implies… so this is equation of motion in the radial direction. Next, we will look at the rotational dynamics of this element. We will neglect the rotational inertia of this element.

In that case, this equation actually falls down to only moment balance about the centre of mass of this element, which can be written as M is this moment and we have two contributions from these shear forces about the centre of mass. So, that can be combined and written as, this is R theta over 2 and this is also R theta over 2 and they produce moment in the same directions. So, it is V time R d theta. That must be equal to zero. That implies… that is what we obtain from moment balance. Now we are going to combine these equations. Essentially, we are going to eliminate this V and also replace this N. If you do that, then you get the equations of motion. So, these are the equations of motion for the beam with constant curvature R.

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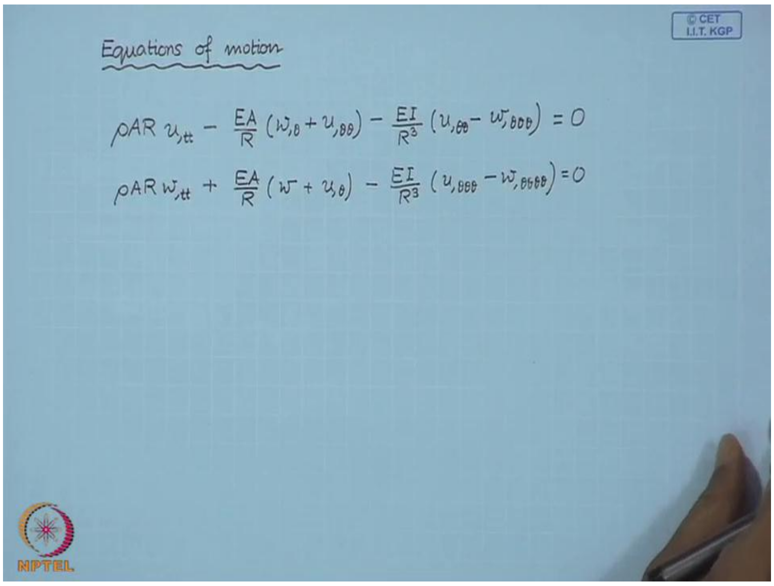

Now, next we will also look at the variational formulation for this. This we will need for the approximate calculation, for example, Ritz method. So, let us look at the variational formulation. We start with writing the kinetic energy; so, one half the mass of the little element times the velocity's square and this integrated over the whole beam. So that is the kinetic energy. Similarly, now we derive the potential energy, which is nothing but half; now we know from the theory of elasticity that the energy per unit volume is the stress times the strain; so one half stress times the strain for the linear theory; the stress is Young's modulus times the strain times the strain; so this is per unit volume. So first we integrate over the area and then over the length of the beam. Now we will substitute the expression of this epsilon; and that turns out to be… This bracketed term is the strain epsilon; so, that square dA and there was an one over R in the strain expression. So, that becomes one over R square. Now, if you square this these terms, so, this will give w plus del u/del theta whole square and that when integrated over the area. So, these terms will have nothing to do with the area; it is the area itself. So, we have… that is the contribution from the square of these terms; and then there is a square of this term. So, you will have z square and you have R power 4 and here this is an R. So, that will give one over R cube and z square integrated over the area that is going to be the second movement of the area.

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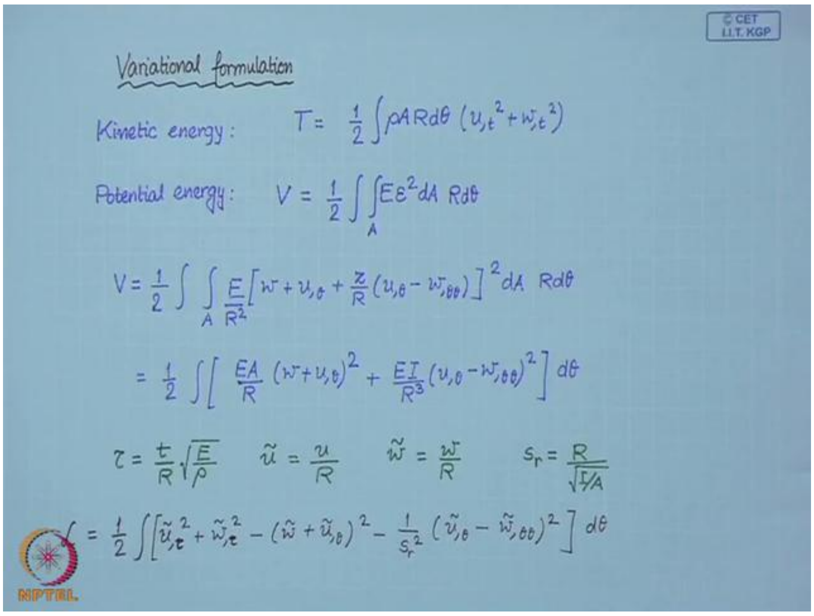

The square of this second term is going to give us… Then there is this third term two times this into this and that is linear in z and when you integrate over the area since z is measured already from the neutral axis or neutral plane; so that term vanishes. So, these are the terms in the potential energy expression. Now to move on further, before we move further we let us make a little bit of simplifications using certain redefinitions. Let me redefine time like this. So, this is non-dimensional time tau and non-dimensional circumferential displacement and non-dimensional transverse or radial displacement w tilde and I will also define the slenderness ratio which will be… I will define like this; the radius of curvature of the beam divided by the radius of gyration of the cross section. So, that reflects the slenderness of the beam. Now, if you use these expressions then the Lagrangian can be written as… so integrated over the length of the beam. So this is tau. So, that is the Lagrangian of the system. Now we use Hamilton’s principle to derive the equation of motion, which says that this must vanish, the variation of this Lagrangian of this action integral must be zero.

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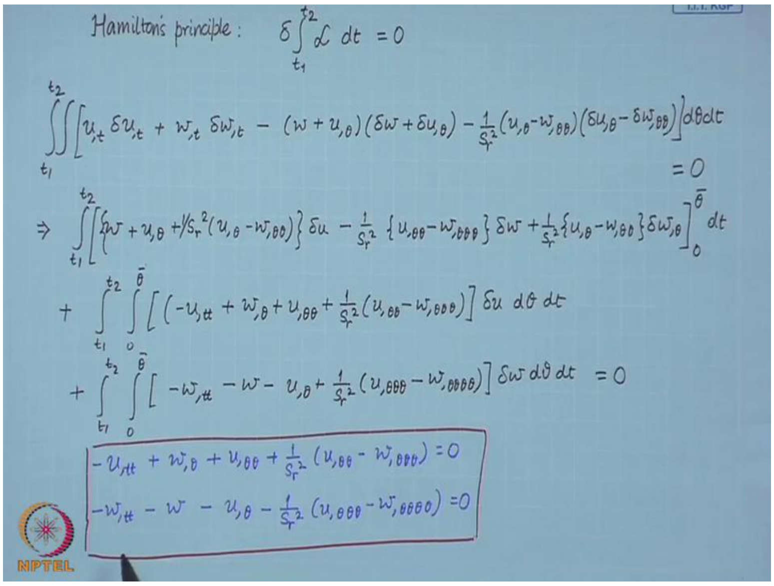

If you do that using the expression of the Lagrangian that we have just now derived, I will drop the tilde in this calculations now. So this must vanish. Now, this has to be integrated by parts with respect to time; these two terms and here we have to integrate by parts with respect to theta and these three terms. If you do that then finally, when you get the boundary terms and the variation over the domain; so, this is. So, this integral, so here you have the limits over the domain of the beam; so, it is zero to some angles that is the theta bar and plus… So, this is what you will obtain. Now from here, it is easy to see that using standard arguments, we can see that this integrant must vanish and similarly this integrant must vanish, delta u and delta w being independent. So, we have the equations of motion.

So, these are the two equations of motion which we have derived early as well in a slightly different form. Now you can once again see the coupling between the circumferential and the radial directions. Now here, since this is non-dimensional these equations are non-dimensionalized, you can look at the contribution of these terms in the equation. Now if the beam is very slender, which means slenderness ratio is very high, then these terms will become insignificant in that case, the equations will get simplified.

So, you have only three terms in each equation which you can then try to solve. But if the beam is not slow slender, in that case these terms will also contribute to the in the dynamics.

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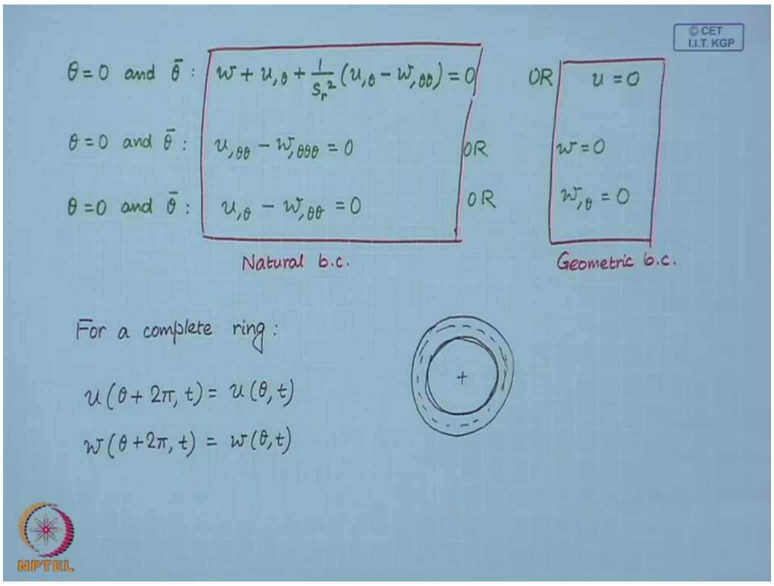

Now, let us look at the boundary conditions. They are obtained from boundary terms. We have at theta equal to zero and theta bar, we must have this equal to zero or u was equal to zero; either the circumferential motion is restricted or which is the geometric condition and this is the natural boundary condition. Similarly, at again at theta equal to zero and theta bar, so this or… So, these two boundary conditions follow from these boundary terms; this is the first boundary term and this is the second boundary term corresponding to displacement, so delta u and delta w. There is this third boundary term which is in terms of the slope. Once again at theta equal to zero and theta bar, del u/del theta, so, this must be equal to zero or the angle must be equal to zero. So, you have these boundary conditions. So, these are the geometric boundary conditions whereas, these boundary conditions are the natural boundary conditions.

Now, in case of a complete ring, so when there is a complete ring, then in that case, you do not have boundaries like this. So, what you have is you have these periodicity conditions, which means that u at theta plus 2 pi must be equal to u at theta for all time and similarly… and all that follows from these periodicity conditions. So, everything is going to be periodic, displacement, slope, bending moment, shear force etc. So, they are going to satisfy the periodicity conditions. So, for a complete ring we have these two conditions.

Let us briefly recapitulate, what we have discuss today. We have today discussed, initiated some discussions on the dynamics of curve beams. We have considered in particular beams of constant curvature; and we have derived the equation equations of motion using both Newtonian as well as the variational formulation. This is very interesting and peculiar about this curve beams that the transverse and circumferential motions are coupled. So, in the straight beams, we have only the transverse motion; we can treat the transverse and the axial motion separately; they are decoupled. But this curvature in the case of curved beams couples the circumferential and the transverse dynamics; that is what we have seen through the equations of motion. So, with that we conclude this lecture.

Keywords: curved beams, constant curvature, rings, variational formulation.