Vibrations of Structures

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Lecture No. # 30

Wave Propagation in Beams

In a previous lecture, we had studied wave propagation in one-dimensional continuous media governed by the wave equation; and we have looked at propagation of general wave forms, as well as harmonic waves. Today, we are going to discuss wave propagation in beams.

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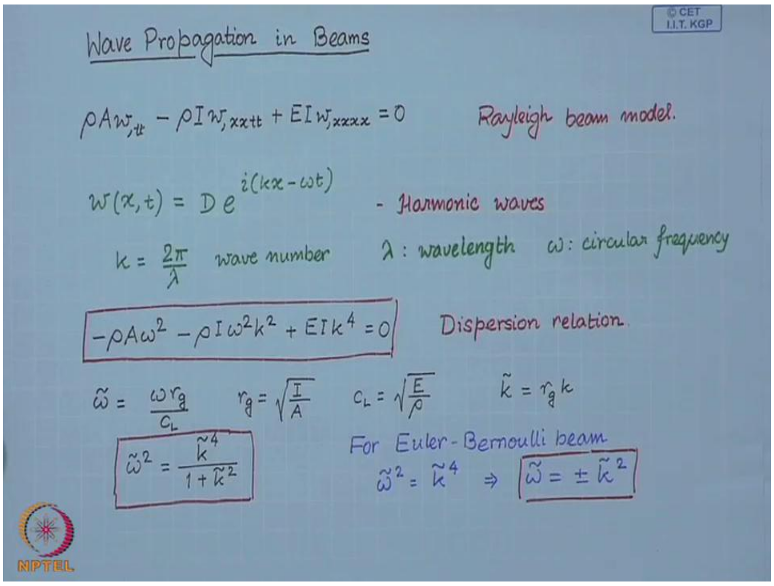

So, let us first look at the equation of motion of a Rayleigh beam. So, this is the Rayleigh beam model. Now, when we talk of wave propagation, we would be interested in propagation in an unbounded medium. So, we do not consider any boundary conditions for the moment, assuming that either the boundaries are too far off, or the time interval in which we are looking at this phenomenon is very short. So, we are interested in a looking at propagation of harmonic waves, which we will represent in the complex form. So, we would like to look at the propagation of waves of this form, where this k is the wave number, lambda is the wave length, and omega is the circular frequency.

Now, we have discussed previously that this is the form of harmonic waves. So, it has got two parameters k and omega and the amplitude D. Now when the harmonic wave propagates in a medium for example, string or here a beam, then this k and omega can no longer be independent variables. So, they get coupled. So, let us see that coupling. So, when we substitute this in the equation of motion, so, we obtain this relation between omega and k. So, this here I have simplified as… I substituted in here and simplify by removing the exponential term. This is known as the dispersion relation.

So, this relates the frequency and the wave number of the harmonic wave that can propagate in the Rayleigh beam. Now we will simplify this little bit. So, if we define omega tilde as omega times r_g over C_L, where r_g is the radius of generation of the cross section and C_L is the speed of actual waves in the beam; and we also define k tilde as r_g time's k. So, with these definitions of omega tilde and k tilde the dispersion relation can be written as… So, this is our dispersion relation represented in terms of omega tilde and k tilde which are non-dimensional frequency and wave numbers. Now in the case of an Euler Bernoulli beam, for an Euler Bernoulli beam this relation… which implies that… So, this is the dispersion relation for the Euler Bernoulli beam. Now we have seen before that the speed of propagation of the harmonic wave is which is known as the phase speed.

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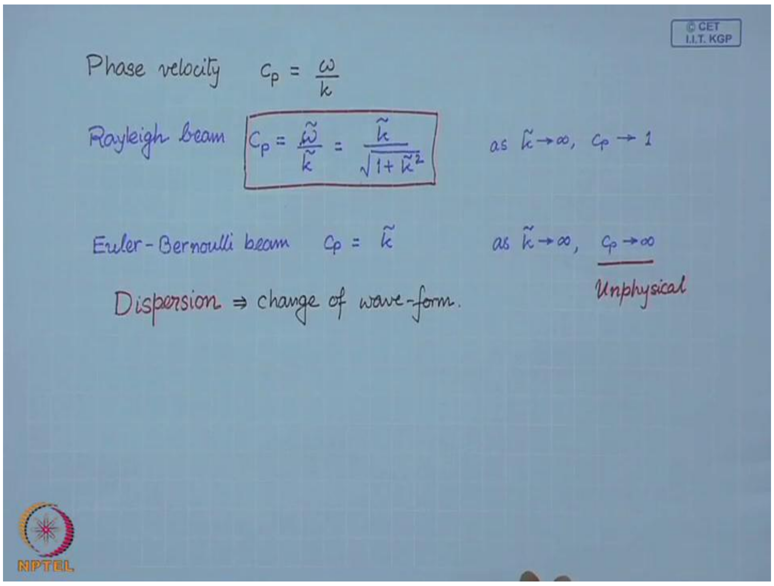

So, the phase speed or phase velocity is given by omega over k. So, that is the phase speed or phase velocity of a harmonic wave in a medium. Now if we calculate the phase speed in Rayleigh beam for example, so we have will write as… So if you use the expression of omega tilde and find out this ratio, then we have the phase speed. It is a plus or minus, but then it represents the speed in the positive and negative directions. So, the phase speed is k tilde over under root one plus k tilde square; and similarly for the Euler Bernoulli beam is given by just k tilde, because omega is plus minus k tilde square. So, omega tilde over k tilde is just k tilde. So, this indicates that as k increases the phase speed in an Euler Bernoulli beam increases linearly with the wave number.

So, waves with higher wave number which means, shorter wavelengths will propagate at higher speeds in an Euler Bernoulli beam, and this increases as k tends to infinity c_p tends to infinity; but if you look at for the Rayleigh beam then, as k becomes very large, much larger compare to one in that case this becomes negligible. So, it is under root k tilde square. So, this becomes one. So, c_p is bounded whereas, in an Euler Bernoulli beam it becomes unbounded. Now this is unphysical. So, we can understand that the Euler Bernoulli beam represents an unphysical model when the wave lines become very small which means, wave number goes to infinity.

So, for very short wave lengths they can travel at infinite speed. So, as the wavelength goes to zero, the speed goes to infinity which is quite unphysical, whereas, in the Rayleigh beam this is bounded and the reason it is bounded is that is the presence of the rotary inertia term. So, you can see the important role played by the rotary inertia term in a beam modal. Now, because of this infinite phase speeds something very strange peculiar can happen in an Euler Bernoulli beam model.

So, suppose you disturb the beam at a particular point. Then the effect of the disturbance can be felt at infinite distances from the source of disturbance, because short and shorter wave lengths, because if you, for example, give an impulse then you are giving disturbance of all possible wavelengths and impulse is represented by all possible wavelengths; it is a combination of all possible wavelengths, because it is Fourier transform is one. So, all possible wavelengths have equal contribution in the impulse. Now since for very high wave numbers, so very short wave lengths, the speed is infinity. So, you can feel the effect of that impulse at large or infinite distances in an Euler

Bernoulli beam which is quite unusual. So, it is quite unphysical that wave. So, Rayleigh beam is a in that wave better model for the beam.

So, this is, because of this rotary inertia term. The other thing that you must note here you see the wave speed is a function of in both cases the wave speed the phase speed is a function of the wave number. So, what this means, is that if you have collection of harmonic waves constructing a wave pulse or a wave form and when this wave form propagates since different wavelengths are propagating at different speeds so obviously, we expect that see the initial wave pulse of the wave form is constructed using these individual harmonic waves in a certain manner. Now suppose some waves travel at some speed and some other waves travel at some other speeds.

So, different waves are travelling different wavelengths are travelling at different wave speeds. Then this combination the combination of all these harmonics is going to change the wave form. So, the wave form change as it propagates this is known as dispersion. So, we have dispersion when a wave pulse propagates in a beam. So, this is an effect of dispersion is change of waveform. So, when you have dispersion, the waveform changes. This is an effect of dispersion. Now here, let us look at this dispersion relation of the Euler Bernoulli and the Rayleigh beam models.

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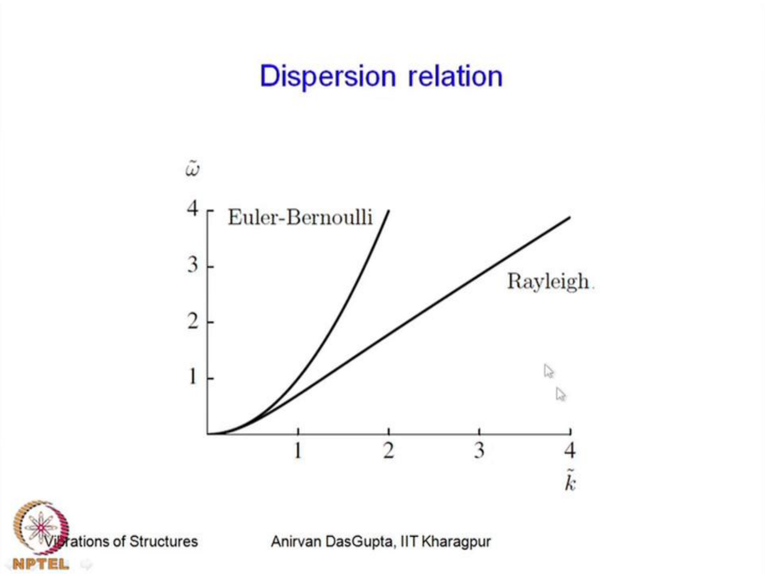

So, you can, so this I have plotted omega tilde versus k tilde. So, you can see that this is this is quadratic in k tilde as we have seen omega tilde is k tilde square, whereas, this tends to a line with a certain slope and omega over k becomes finite. So, this is the Rayleigh beam and this is the Euler Bernoulli beam. Now so, we have looked at this dispersion relation and now let us calculate the wave that can propagate in the beam. So, for that, we are going to once again.... so let me write down the dispersion relation once again.

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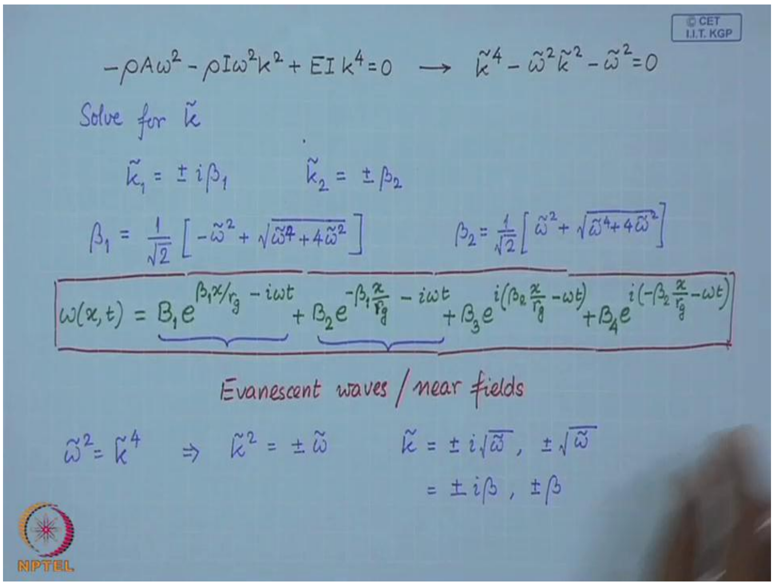

Now, so this dispersion relation can be used to solve for k in terms of omega. So, let me, so if you do that solve, if you solve for k. So, the non dimensional, so we will solve for k tilde the non dimensional dispersion relation was... So, we had converted into non dimensional form. So, if you solve for k tilde, you obtain two values. So, you can see this is a quadratic in k tilde square. So, you have two values for each k_1 and k_2. So, these are represented, I am representing them as i beta 1 and beta 2. So, here beta 1 is… and beta 2… which you can very easily find by solving this and representing…

So, I am representing this k_1 and k_2 in this using this beta 1 and beta 2. So then finally, if you represent the full wave solution, so you have these four solutions of k tilde. So, you have four terms in the solution. So, this represents the waves that can propagate in the beams; so as per the solution of the wave number in terms of the non dimensional frequency omega. Now if you look carefully at these solutions then you find that these two terms, so, this term and this term, they actually do not represent any travelling wave.

They are not actually travelling wave they are in fact, this is decaying in the positive x direction, and this is decaying in the negative x direction.

So, if you actually have an infinite beam then these two solutions, these two terms actually will not exist. So, these terms can exist of course, when you have a boundary or when you have an obstacle or beam interacting with something external element, may be a massive element or a stiff element. So, whenever you have interaction of the beam then these terms can appear, otherwise these terms in an infinite beam these terms cannot be present. These are actually as I mentioned I am not travelling waves they are called evanescent waves they are decaying or evanescent waves. They are I mean, though they are called evanescent waves they are really not travelling waves or they are also called near fields. So, they exist near the boundary or mass point or an interaction point.

So, they can exist only close to a boundary and their amplitudes will fall off. So, this will fall off in the positive x direction; this falls off in the negative x direction. Now, let us look at scattering of waves in which we will now have an opportunity to look at these evanescent waves on near fields.

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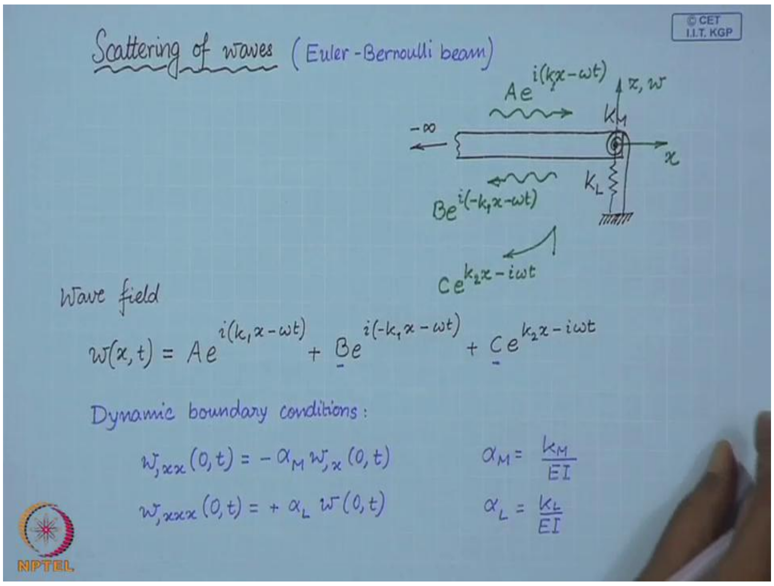

So, now, one more thing before we proceed in the case of the Euler Bernoulli beam, the dispersion relation was omega square is k power 4. Now here therefore, we will have solutions of k. So, k square is plus or minus omega tilde; therefore, k tilde… So, you have 4 solutions. The only thing is here suppose I call them as beta then I have plus or minus i beta and plus or minus beta. So, both are; so, there is no beta 1 beta 2 as in the case of a of a Rayleigh beam. In the Euler Bernoulli beam it is just beta. Now we are going to look at scattering of waves in an Euler Bernoulli beam. So, let us consider this semi infinite beam. So, this end is connected to linear spring of stiffness k_L and a torsional spring of stiffness k_M. So, let us consider that there is an incident harmonic wave represented by…

So, we have an incident harmonic wave in this form and then this is incident on this boundary, which has a stiffness element, there will be a reflected wave harmonic wave which we represent as… So, let me put this; let me call the wave numbers at present k_1 for the reflected incident and reflected and there will be now an evanescent wave, which we will represent in this form; this decaying wave is in the negative x. So, its representation would be… So, we have an evanescent wave, which is decaying in the negative x direction. Now therefore, the total wave field in the beam; so, you will not have a wave with negative x, since, as the beam goes up to minus infinity, such a wave would diverge. So, from physical considerations we have only one evanescent wave in this form. Now the total wave field is given by… Now here we have dynamic boundary conditions. So, at this end of the beam, we have two boundary conditions, because of this linear stiffness we will have shear force condition, and because of this torsional stiffness we will have a bending moment boundary condition. So, these may be represented as… where this alpha m is nothing but k_m the stiffness of the torsional spring, torsional stiffness over E I. So, that is going to give the moment and similarly the shear force condition… So, this is k_L over E I.

So, there is a positive, because the relation between moment and sheer force there is a negative sign. Now here we have finally, the boundary conditions which are dynamic boundary conditions. Now we substitute this wave field, wave form solution in the boundary conditions and we will try to compute this coefficients B and C in terms of A, because A is the incident wave which is, so, A is known. So, in terms of A we will like to know B and C which means, the amtitude of reflected harmonic and the reflected waves.

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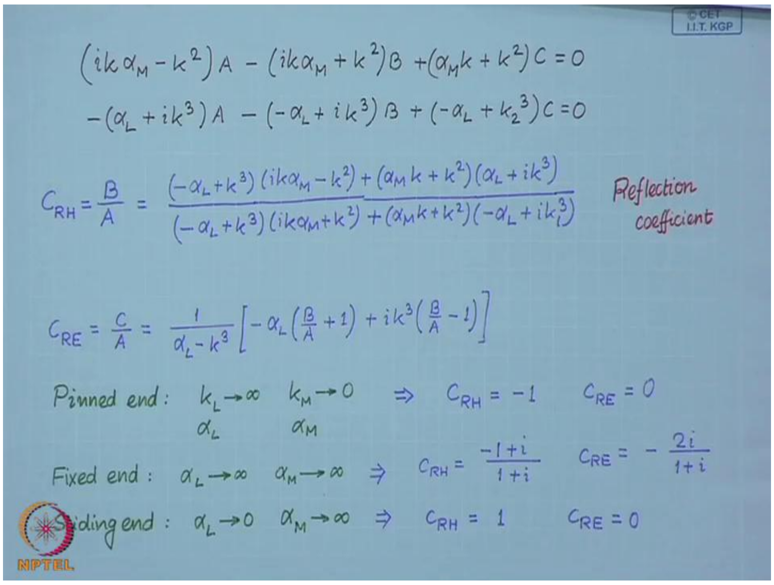

So, if you do that and consider that there is Euler Bernoulli beam… See you have two equations. So, these are the two equations obtained from the bending moment and the shear force conditions. Now these are two simultaneous equations in B and C. So, you can solve for B and C in terms of A. So, we will solve this as a ratio. So, B over A which we will call the coefficient of reflection for the harmonic wave and that is obtained as… So, this is the expression for the reflection coefficient as it is known as; and similarly, you can find out the reflection coefficient of the evanescent wave which is C over A as… So, here you need to substitute this expression of D over A, or the C, reflection of the harmonic wave. So, once you do that you can solve for the reflection coefficient for the evanescent wave. Now we can consider various special cases, for example, you can consider pinned end. So, when the end is pinned, we can consider that, so here if this end is pinned. So, we have infinite stiffness for the linear spring and zero stiffness for the torsional spring. So, in other words this is alpha L; this is alpha M. So, in that case if you substitute this; take this limit in these expressions then you obtain… So, if you, which means, that the harmonic wave is reflected with a phase inversion. So, phase change of pi; whereas, there is no evanescent wave reflection when the harmonic wave is incident on a pinned boundary. The next case could be a fixed end in which case both of them tend to infinity, and in this case the reflection coefficient for the harmonic wave is this and for the evanescent wave is given by this. So, now you have, this introduces a phase in the reflected harmonic wave and also you have additional another phase in the evanescent wave. You can have further conditions, for example, if you consider sliding end which means, that alpha L is zero; so, linear spring is zero, whereas, the torsional spring is infinite. So, in that case the reflection coefficient for the harmonic wave is one and the reflection coefficient of the evanescent wave is again zero. So, we see that now the there is no phase inversion the phase change is zero. So, the wave will get reflected without any phase change and there is no evanescent wave generated, because of this. So, similarly you can study a free end where both the stiffnesses vanish; so, there is no linear or stiffness then again finds out these coefficients the reflection coefficients for the harmonic and the evanescent waves. Next we look at then the situation when the harmonic wave is propagating in a beam and we want to look at the motion of the material points, let us say on the top surface or the bottom surface of the beam. So, we are interested in the motion of the material points on the surface of the beam, when it carries a harmonic wave, a propagating harmonic wave. This problem is interesting, because it has leads to a number of applications, specially in the ultrasonic motors. So, this is used in ultrasonic motors in a certain way.

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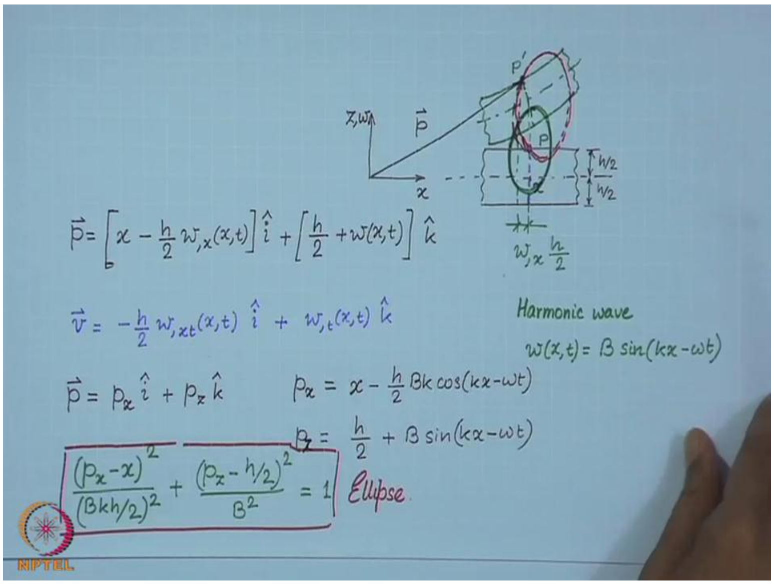

So, let us look at motion of material points in the beam. So, consider this, so this beam element; so, we consider… and when this gets deflected… So, this distance… So, P was a point on top surface of the beam in the undeflected position and when this gets deflected this point moves to P prime. So, the axial motion of this point, because of the deflection may be written as, for small deflections it may be written as del w/del x is the small angle and times h by 2; so, the small angle times this height will be the axial motion of this point P. Now then I can write this vector this position vector P; so, this location was x. So, this location was x, now this is x minus, so, this is the x coordinate of the point P prime and the z coordinate, so, this was the initial z coordinate. So, that is the position vector. Once we have the position vector, we can write down the velocity by differentiating the position vector. So, that is the velocity vector. Now suppose you have harmonic wave propagating in this beam. So, this is the representation of the harmonic wave. Now if you substitute in the position equation, then you obtain… so, if you write this as p_x i cap plus p_z k cap, then you obtain p_x as and p_z as… So, this is the, so one is p_x and other is p_z, these two coordinates. So, one here, we can very easily see that this satisfies… So, this is nothing but the equation of an ellipse; so, which means that this point P actually cases out and ellipse. So, its location, the centre of the ellipse is at x and h by 2. So, actually it is in this form; and you can find out its semi major and semi minor axes. So, the point traces an elliptic path centred x and h by 2.

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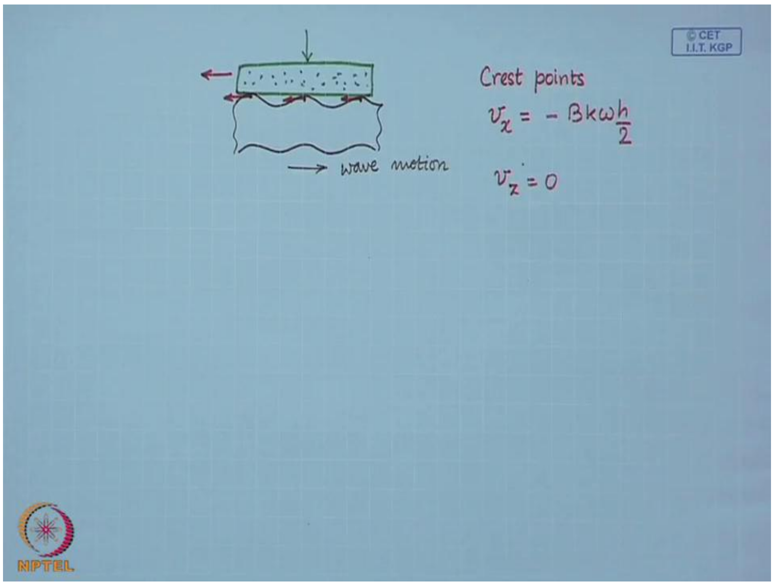

Now, if you look at the beam, let us say a rigid body on top of it, which is pressed down upon on this beam; and this beam, if it carries the wave in this direction, then from the velocity expression, we can easily find out that velocity of these points on the top of at the, at this crest; the velocity is the negative direction. This can be find out by looking at the velocity expression; and this velocity set on the, at the crest. Say if you have a rigid plate or a body in contact these points then it is going to be transported with this velocity.

So, to just that we have discussed today, we have looked at wave propagation in beams, we have looked at the dispersion relation and the wave propagation characteristics of the beam, in a beam; and we have seen how the Rayleigh beam is a better model than Euler Bernoulli beam. Finally, we have looked at the scattering of waves in a beam and the motion of material points of the beam. So with that, we conclude this lecture.

Keywords: travelling waves, harmonic waves, dispersion relation, evanescent waves, cut-off frequency, reflection of waves, material point motion.