Vibrations of Structures

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Module No. # 01
Lecture No. # 03
Axial and Torsional Vibrations of Bars

So, today we are going to look at two examples on vibrations of one dimensional elastic structures that we have started with strings. So, today we are going to look at axial vibrations of bars and torsional vibrations of circular bars.

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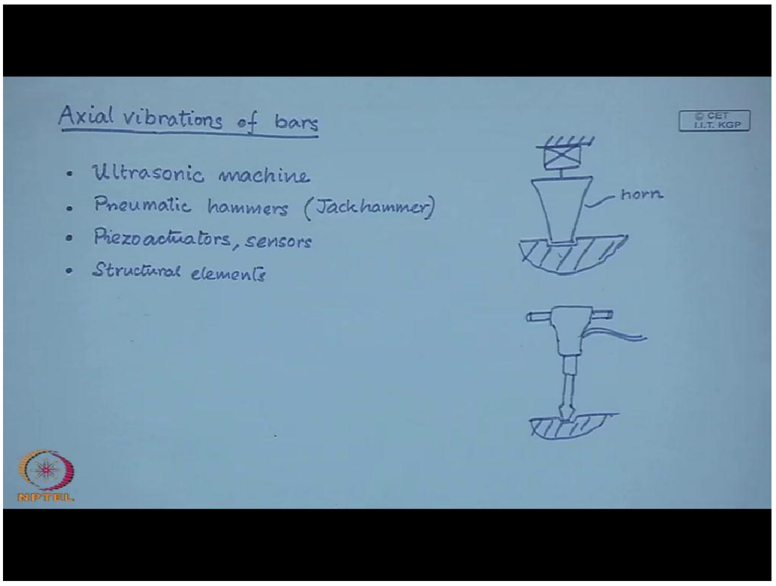

So, let us begin with axial vibrations of bars. So, where do we find bars in axial vibrations? So, some examples are in an ultrasonic machine. So, in an ultrasonic machine what you have is a bar, which is shaped like this, which is connected to an actuator. So, this bar is called a horn at the ultrasonic machine and this in ultrasonic generator which passes ultrasonic waves in this bar and because of the shape you have large amplitude motions at this work piece. So, such a machine is used for machining builder materials for example. Then you find bars in axial vibrations in pneumatic hammers, sometimes also known as jack hammers. So, in a jack hammer this looks roughly like this.

So, these are used for drilling or chipping operations in construction sites. So, here you have a bar which is also in axial vibrations. Then you have piezo-actuators/sensors, in which you find a bar, which is made up of piezo-electric material which is a under axial vibrations. Then in various structural elements, you may find bars in axial vibrations.

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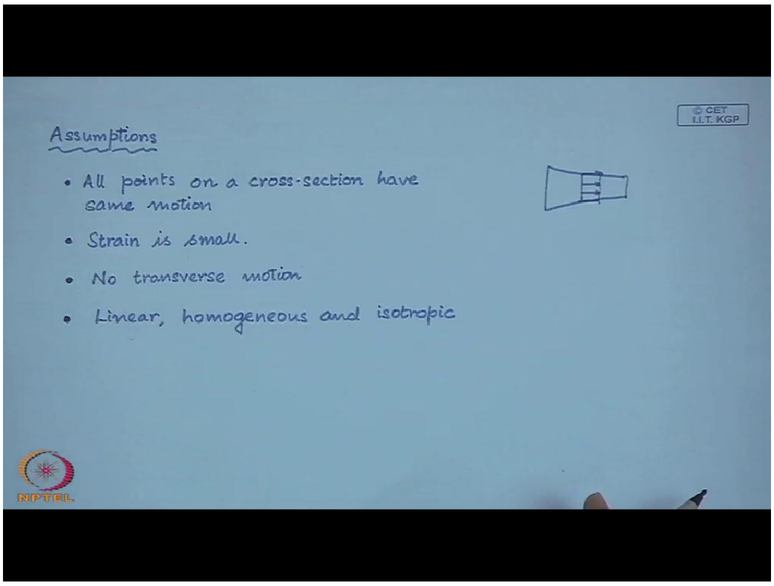

Now in order to model the dynamics of bars in axial vibrations we begin with some assumptions that we make on this modeling. So, the first assumption that we make is that all points of the bar, or in all points on a cross section, have same motion. So, what I mean by this is suppose you have a bar; at a certain cross section all points will have the same motion. The second assumption that we make is that the strain in the string is small, so that, we do not have non-linear effects. We are going to discuss only linear vibrations of bars. The third assumption is that there is no transverse motion of the bar. So, this bar the material the material points are vibrating only along the access of the bar. There is no transverse motion; and the fourth assumption or that we make is that the material of this bar is linear homogeneous and isotropic. So, with these four assumptions we are going to now look into the equation of motion of a bar in axial vibrations.

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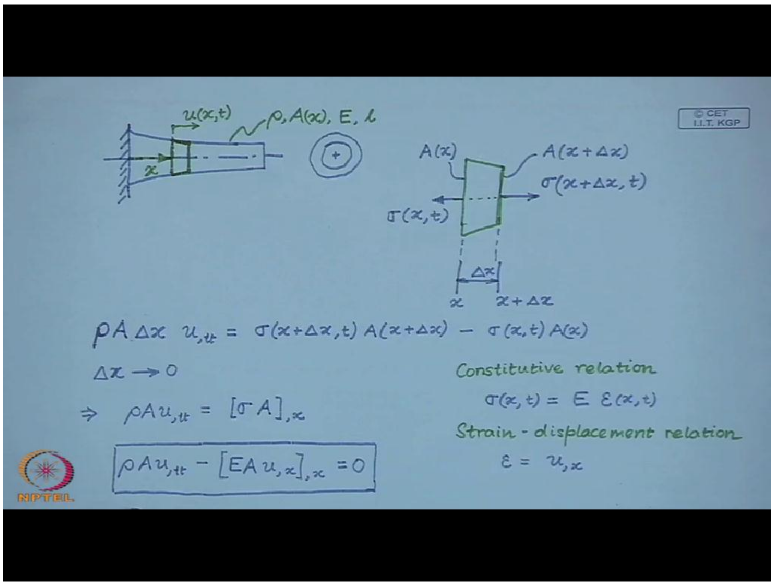

So, let me draw bar. So, this bar is made of material of density say rho, and an area of cross section A, which may be a function of the spatial coordinate x, has Young's modulus E and has a length l. Now at any location x of the bar, the displacement of the cross section in the axial direction is measured by this field variable u as the function of x and time t. Now we are going to derive the equation of motion of this bar using the Newtonian approach. So, what we will do is, we will consider and the infinity symbol a small section of this bar as I shown here and draw its free body diagram. So, we will consider the stress.

So, this element is of length delta x, lies between x and (x+delta x). So, the stress on the right face I will write the sigma (x+delta x,t) and on the left face the sigma(x,t). Let this area be A(x+delta x) and this is A(x). Now we are going to write the equation of motion using Newton's second law for this infinitesimal element. So, the mass of this little piece may be written as. rho times A is mass per unit length and the length of this little element is delta x. So, this is the mass of this element. This mass times the acceleration in the longitudinal direction, that is the double derivate of the field variable u with respect to time, that must be equal to the forces in the longitudinal direction. So, the force on the right face is given by sigma times A on the right face minus the force on the left face which is given by sigma times A on the left face.

Now, if I divide this whole equation by delta x and take the limit delta x tends to 0, then that could imply rho A times the acceleration is equal to the partial derivative of sigma into A. Now we require to represent this stress in terms of the displacement, the field variable. So, in order to do that, we will need two things. The first is the material constitutive relation, which will relate the stress with the string. So, we know that the axial stress is proportional to the strain and the proportionality constant is the Young's modulus. So, this is the Hooke's law in one dimensional. Along with this we need the strain displacement relation. So, which means epsilon the strain is del u/ del x. So, if I substitute this expression in the constitutive relation and put this back in the equation of motion, what I obtain on rearrangement of terms. So, this is therefore, the equation of motion for axial vibrations of a bar. You remember that, this equation has been derived by considering a small element of the bar that in no way, tells us or describes to us the full physical picture of the bar.

So, what I am trying to get at is we need to complete this description of this bar in axial vibrations we need the boundary conditions.

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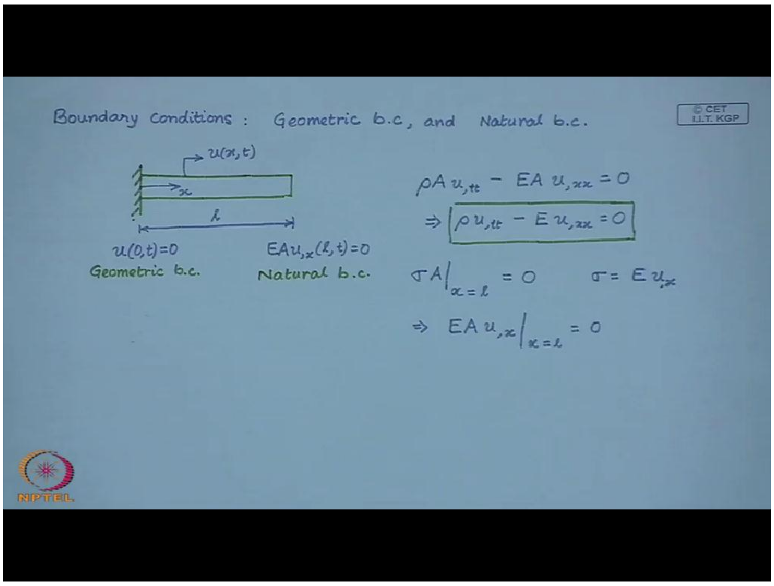

Now as we have discussed before, this boundary conditions are of two types: geometric and natural. Now let us look at some examples and identify the boundary conditions. So, we begin with uniform bar. So, the equation of motion in this case simplifies to this equation, because the area is now constant; it is no longer function of x. So, it can come out of the partial derivative and this can be simplified further to obtain this equation of motion for a uniform bar. Now the boundary condition of this bar, as you can see, this end of the bar is completely fixed at the wall. So, at x equal to 0, there cannot be any axial motion of the bar. Now this boundary condition is fixed by the geometry of the problem.

So, this is a geometric boundary condition. Now this end is free; free would mean that there is no axial force on this face of the bar. The force, as you know, force hear on any cross section is given by sigma times A. So, at x equal to l, the axial force must be 0. Now we have used the Hooke's law and the strain displacement relation previously to obtain this relation between the displacement and the stress. So, therefore these two will imply E times A del u/del x, this computed or evaluated at x equal to l must be 0. Since A is uniform, so, the boundary condition turns out be this. Now this comes from a force condition. The such a boundary condition is an natural boundary condition. So, we finally, have the equation of motion of a uniform bar and the boundary conditions which complete the description of these fixed free bar in axial vibrations.

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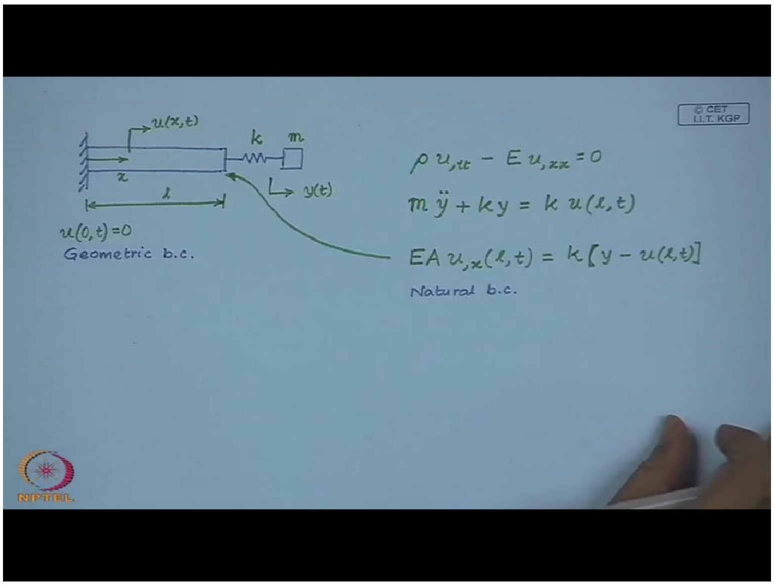

Next we look at another example.

So, this a bar at the end of which is fixed free; this is a fixed bar at the end of which an oscillator is attached in this manner. So, this is a point mass attached with a spring of stiffness k. So, the displacement of this mass from the equilibrium position is measured by this coordinate y. So, let us first write down the equations of motion for this system; now here you have, as you can see a bar and this mass. So, you are the field variable which measures the displacement of the bar, the material points of the bar, and we have this coordinate y which measures the displacement of this mass, discrete mass m. So, the equation of motion of this bar remains the same as before. So, we can write…

Now for this oscillator which is connected at this end, we can easily write the equation of motion as m y double dot plus k times y is equal to k times the displacement of the bar at this end. This equation can be easily derived, if you write down, if you take the oscillator separately and write down its equation of motion. Now these are the equations of motion for the system. So, as you can see they are coupled. Now let us look at the conditions at the boundary. So, at this end of the bar there is completely fixed. Therefore, we can easily write the displacement is 0 at this end. On the other hand at this end of the bar where this oscillator is attached, we can expect an interaction force with the oscillator, from the oscillator. So, we are already written the force on any cross section as E times area del u/del x at x equal to l at time t. So, that is the force; had there been no oscillator it would have been 0. Now since there is an oscillator, you can write the force that this oscillator puts on this end which is easily obtained as k times y which is the motion of this point minus the displacement of the bar at this end. So, this is the boundary condition at this end of the bar. So, here we have a geometric boundary condition, whereas on the right end, we have a natural boundary condition. So, with these two examples we will move on to the next example or the next case that we are going to consider which is torsional vibrations. So, we are going to consider the case of torsional vibrations of circular bars only.

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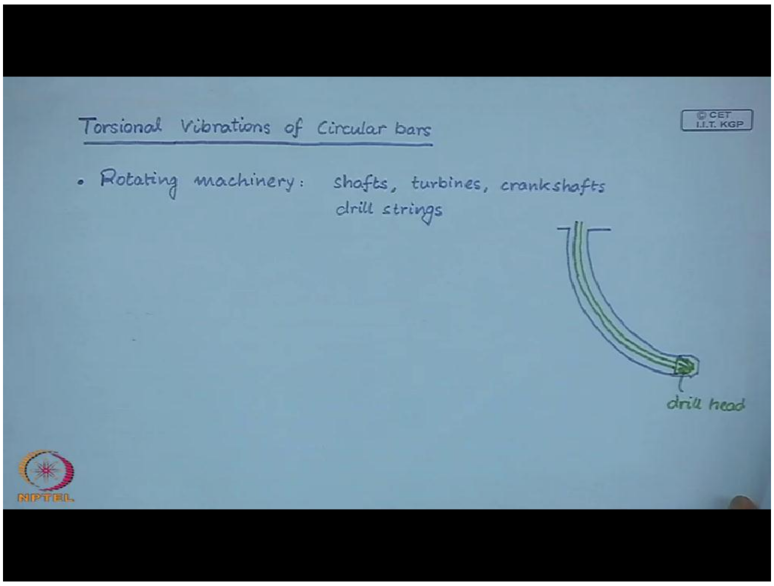

So, where do we find torsional vibrations of bars. So, mostly in rotating machinery. So, you have shafts in rotating machinery which transmits torque. Such shafts are to torsional vibrations, for example, in turbines, in rotors, in crank shafts, in turbines, crank shafts of engines, in the dentist’s drill. So, in the dentist’s drill you have a wire which is under torsion. These are called drill strings; they are also found in petroleum excavation and mining industries. So, these shafts, for example, in a mine like this. So, this the drilling head and here you have a shaft which transmits the torque to this drill head; this shaft is under torsional vibration. It also has some amount of transverse vibrations, but the torsional vibrations are quite dominant in such situations; these are called drill strings. So, in such situations you find shafts or circular bars in torsional vibrations.

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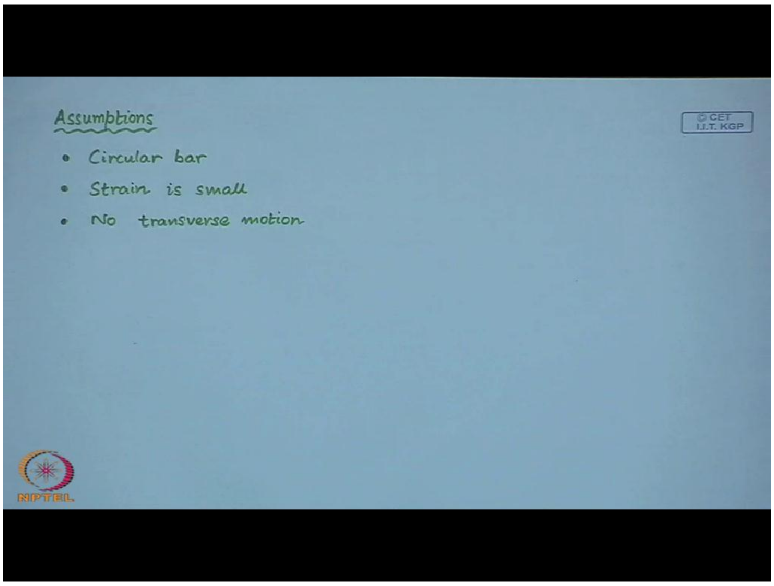

So once again, so the mathematical model of such a bar, we make some assumptions.

So, first assumption that we make is that, we have already said that the bar is circular. So, we are going to study only torsional vibrations of circular bars; that is to ensure that there is no wrapping of the cross section. We will assume that the strains are small, so that the dynamics can be adequately described by a linear model; and we will also assume that there is no transverse motion, There is no transverse motion of the bar. So, with these assumptions let us look at circular bar. So, here I have a circular bar made of a material of a density, say rho, has an area of cross section which may be a function of the spatial coordinate x; the shear modulus of the modulus of rigidity is represented by G and this has a length l. Now at any location x, the field variable that measures the torsion in the bar or the local torsional displacement of the bar is represented by this phi(x,t) at any location x at any time.

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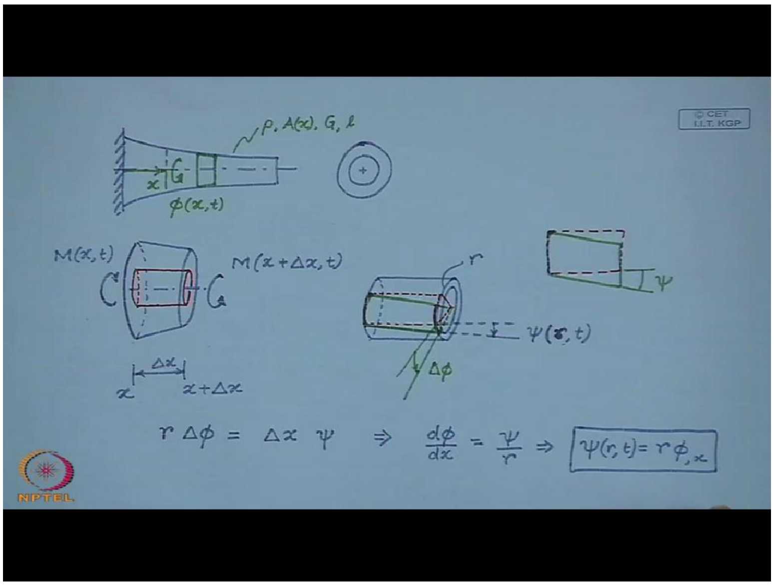

This is the torsional displacement of any cross section. Once again we are going to derive the equation of motion of such a bar in torsional vibrations using the Newtonian approach. So, we are going to consider a small portion of this bar between the spatial coordinates x and (x+delta x). Now on the right end of this bar let us consider the moment represented by M(x+delta x) at time t; on the left end, we have the moment on the cross section as M(x,t). Now inside this little element, we are going to consider a ring of certain radius r. So, we will draw this portion, this ring separately. Now consider, in the undeformed ring, two axial lines like this and the corresponding radial lines here. Now this little element, when the bar is under torsion, there will be a differential rotation between the left face and the right face.

So, because of this differential rotation, this red element, which is somewhat like a rectangular element, is going to take up this configuration. So, if you look at this angle, that is the small rotation, the differential rotation between this face, the left face and right face. On the other hand if you look at this angle, this is nothing but the shear strain experienced by this initially undeformed red element. Now this shear strain is a function of the radial collision of this element. So, if I draw these two elements once again. So, this was the red element; this was before the deformation; this is the green element after the deformation. So, this is the shear strain in this element. So, with this kinematics we can write r the radius of. So, this is the radial location. So, we can write r times delta phi, which is this small length, r times delta phi must be equal to this length, which is delta x times the shear strain.

So, that would imply, upon taking delta x tends to 0, which we can also write as… So, this is equation that we obtain from the kinematics of this torsional deformation of the bar. We can now relate the shear strain at any radius r at any time t in terms of our field variable which is phi. Of course, this is also a function of x because phi is a function of x. Nnow we are going to use this kinematic relation further.

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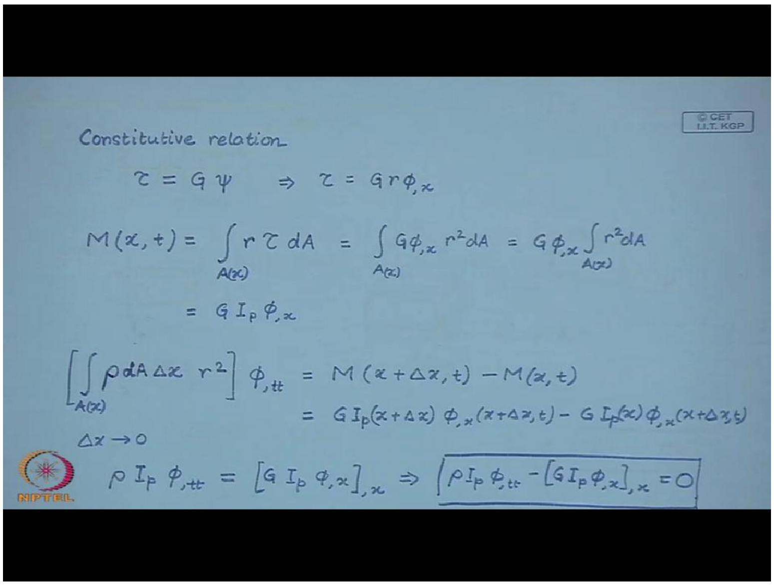

So, let us look at the constitutive relation of the material. So, we know from Hooke's law that the shear stress is proportional to the shear strain. So, the shear stress is G times the shear strain. So, if I use the kinematic relation in this expression, I have the expression of the shear stress at any radius r at any location x at any time t. So, once I have the shear stress, I can integrate this, multiplies with the area and integrate, multiply the shear stress with this area and integrate over the whole face to get the moment, the torque in the part.

So, M at any location x at any time t, I can write from the shear stress times the small area of this ring and the arm. So, arm times these shear force. So, that will give us the moment the torque. Now if you integrate this over the full face then you get the total torque on this face of the bar. So, if I substitute this expression I have this; and since these two terms the G and phi they have nothing to do this integral over the area, so I can safely bring them out and this, therefore, is the torque; and we know we can easily identify this integral as the polar moment of the area. So, now, I have related the torque at any cross section in terms of my displacement; the torsional displacement. Now I can write down the equation of motion.

So, first I will write the moment of inertia of this ring that we have considering. So, this moment of inertia of the ring is mass of the ring times the radius square. So, mass of this ring can be written as, rho times dA is the mass per unit length into delta x, that is the mass of this ring times r square. So, this if I integrate over the full area, then I get the moment of inertia of this element that we have considered. So, this the moment of inertia times the angular acceleration which is the double time derivative of our field variable phi and that must be equal to the balance of torques; and we have, already have, this expression of moment on the torque an any face here. So, I can write this as… Now if I divide this whole equation by delta x and take the limit delta x tends to 0 and if I identify this r square dA integral; rho can come out of this integral. So, r square dA integral over the face is once again the polar moment of the area. So, you can write rho Ip, the angular acceleration, must be equal to…; and that gives us, upon rearrangement, the equation of motion of torsional dynamics of a bar a circular bar.

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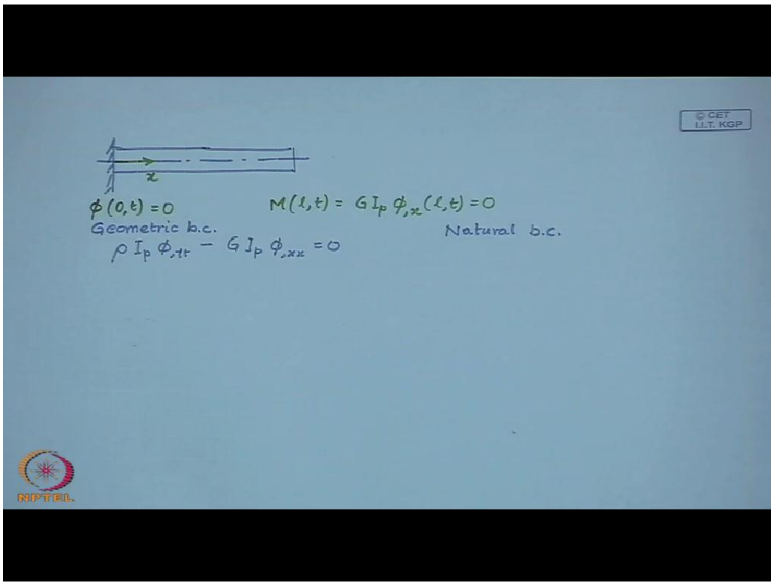

Now let us look at some examples. So, if you have say a uniform bar, then the equation of motion once again simplifies to rho Ip phi,tt minus g Ip phi,xx equal to 0. Now the boundary conditions for this problem which will complete that description of the physical situation. So, here this bar is connected to the wall and therefore, this end of the bar cannot have any rotation. So, this again is the geometric boundary condition; and this end of the bar the torque is 0 which is therefore, given by... So, this is a geometric boundary condition where as this is at right end of the bar we have a natural boundary condition.

So, with this we complete our discussions on axial and torsional vibrations of bars. Now to summarize we have considered axial vibrations of bars; we have derived the equations of motion; we have considered we have seen the boundary conditions of two types, namely the geometric boundary condition and the natural boundary condition. Then we have also derived the equation for a bar interacting with a discrete system; and then finally, we have looked at the dynamics of torsional vibrations of circular bars and we have derived the equations of motion equation of motion and the boundary conditions. So, with that we end this lecture.

Keywords: axial dynamics of bar, bars interacting with discrete system, torsional dynamics of circular bar.