Vibrations of Structures

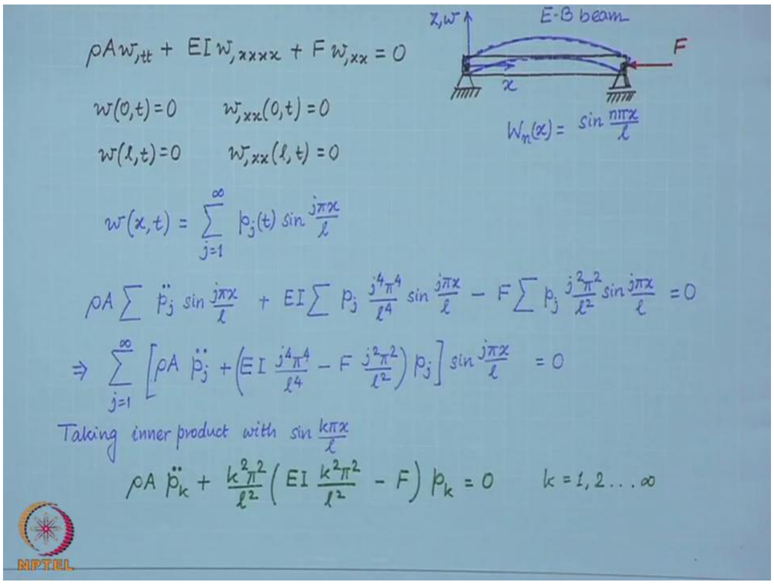
Prof. Anirvan DasGupta
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture No. # 29

Topics in Beams Vibrations- II

Today, we are going to discuss an important problem in beam vibrations or even beam stability you can say, which occurs when beam is subjected to an axial load. So, you find for example, column; in a column it is subjected to an axial force on the column, and we know that, I mean this column cannot with stand any amount of load. There is a limit after which the column will buckle or get destroyed actually, it my get destroyed. This is one kind of axial loading that can happen. There is another kind of axial loading that occurs, in case of let us say missiles. So, in case of missiles you have jet coming out from the rear end. So, there is an axial force from the rear on this, but now the property of this axial force is that, it will always maintain the angle which is there at the rear end of the missile. So, if the missile is flexible it is going to this forces going to maintain the angle. In the case of columns, even if the column deflects the force for small deflection, the force is still going to be vertical. So, there is a difference in these two axial forces, which we are going to now discuss.

(Refer Slide Time: 02:10)

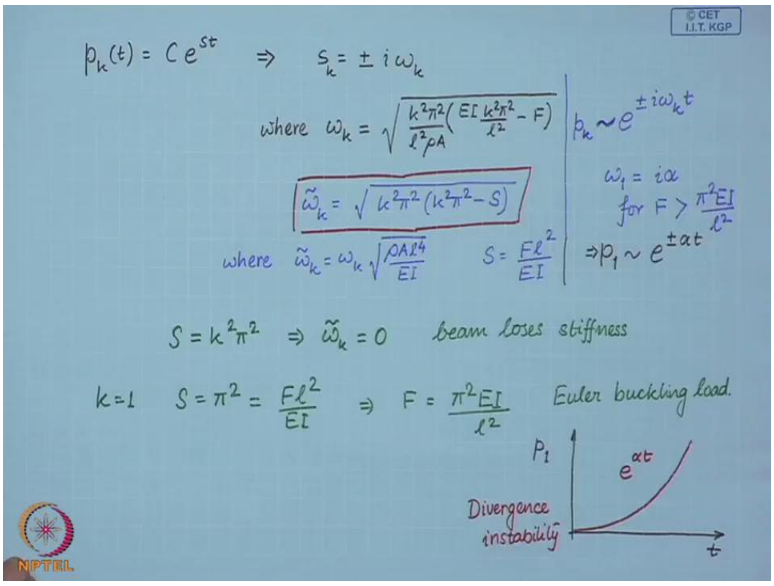


So, when we say beam with axial force, we are going to consider a column like loading. So, let us consider a simply supported Euler Bernoulli beam with axial force F. So, we have an axial force F and the beam can deflect this manner. The force retains its direction in this particular case. So, the equation of motion for this beam with the axial force… So, these are the two terms for the Euler Bernoulli beam and in addition, because of the axial force which is now compressive, you have this additional term. So, the case of strings for example, the force is tensile. So, this is with the negative sign. So, now it is compressive. So, this is the term, because of the axial force. Now, the boundary conditions…

So, the deflection and the bending moment at 0 and l they must vanish. Now, for this problem, we can use our modal expansion, we know that the Eigen functions of a simply supported beam is given by sine n pi x over l. So, the solution of these dynamics with axial force, let us construct the solution as an expansion in this form. Now, if we substitute this in the equation of motion that must be zero now, this can be simplified.

Now, because of the orthogonality of the Eigen functions, we can use the orthogonality property to determine this coordinate p_j the dynamics of the modal coordinates p_j. So, we multiply this for example, with sine j, Pi x over l and integrated over the domain of the beam. So, if we take this inner product… So, you have the dynamics of the modal coordinates given by this equation.

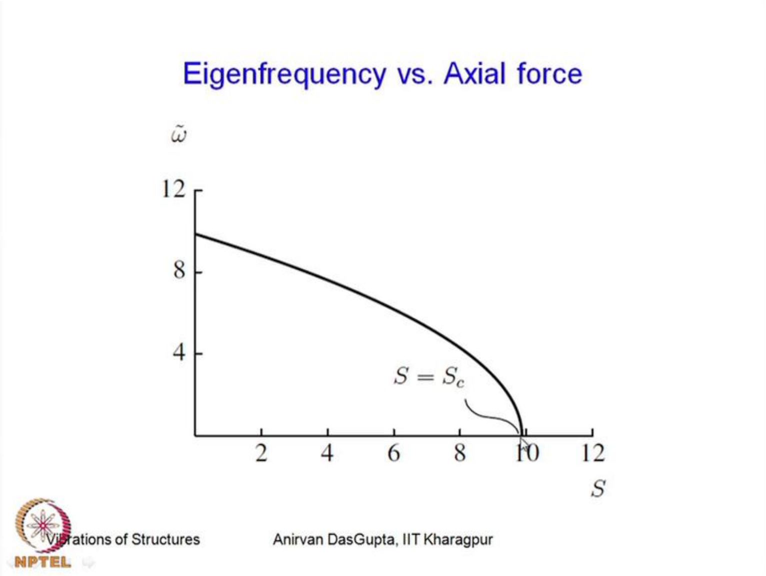
(Refer Slide Time: 10:41)



Now, suppose you look for solutions of the form c times exponential s t. So, if you substitute in here, so, we will obtain solutions of S in terms of this omega k, where omega k is given by this expression. Now, we can make a little bit of simplification, here by some redefinition, we can write this… where… So, using these redefinitions, we can recast this in this non dimensional form the Eigen frequency in the non dimensional form now, here for. So, this S represents the load, the axial force. So, there is a value of this axial force, when omega k goes to zero. So, if S is k square pi square, when omega k till the goes to zero, which means that the system looses stiffness. Now, so this if you now substitute here for the lowest mode, so, when k is 1 then S is pi square… and from here we know that F l square… Now this expression we know as the Euler buckling load. So, this is the Euler buckling load now. So, we have write at this Euler buckling load is through the dynamics. So, let us first see this. So, when we are increasing S from 0 then this omega k is a real number. So, you have solutions in the form exponential. So, when omega k is real you have solutions.

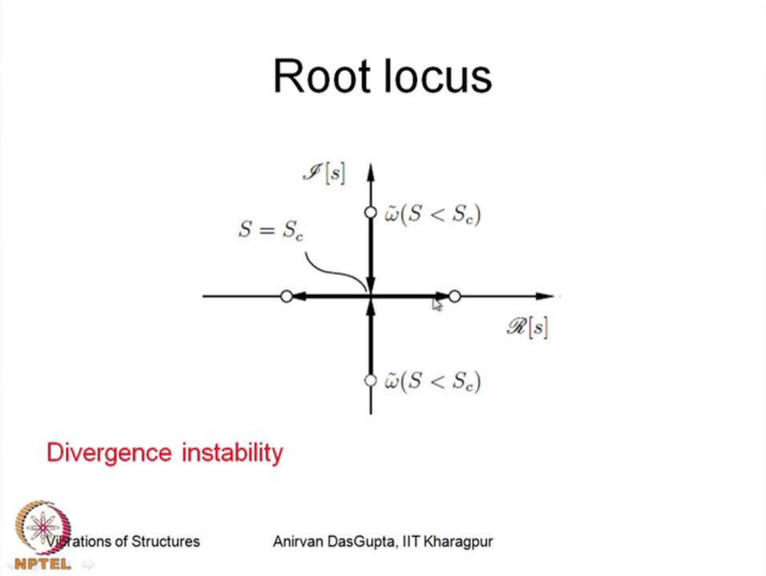
So, solutions have these terms. So, this times; so, the temporal variation of these solutions is like this; so, exponential i omega k t. So, which means these are harmonically waving. Now, when this goes to zero, omega k goes to zero then, it loses the temporal nature. So, this is neutral stability. So, you have reached the Euler buckling load for k equal to 1 for example, and the beam has lost its stiffness. Now, if the load is increased further beyond that points. So, beyond this value, if we increase the load beyond this value then omega 1 will become imaginary. So, let us say it becomes i alpha for F greater than. So, for F greater than this buckling load omega 1 becomes imaginary. In that case… So, you have the solution as exponentially increasing and exponentially decaying terms. Now, at a finite time the exponential increasing solution is going to dominate. So, even for a very small disturbance initial disturbance, the solution is going to have this nature. This is known as divergence instability. So, this kind of behaviour is known as divergence instability. So, when the axial load exceeds the Euler buckling load for the first mode, then you have this divergence instability and the beam actually buckles. So, let us look at the variation of this frequency with increasing load.

(Refer Slide Time: 19:28)



So, here I have plotted this non dimensional frequency, circular frequency with the non dimensional load. So, when the load is zero, you have; this is the first circular natural frequency non dimensional and else the load increases this falls and here, at a certain load it goes to zero, the circular natural frequency goes to zero. So, this is the buckling load and the critical load.

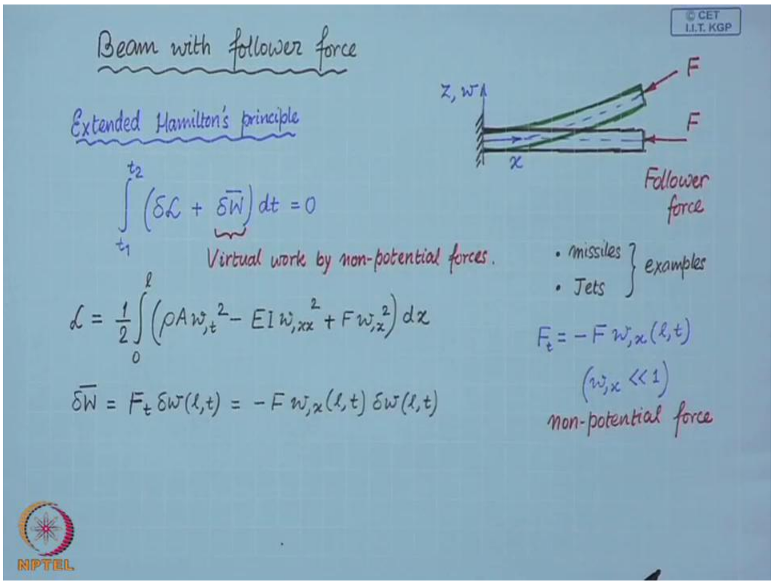
(Refer Slide Time: 20:07)



Now, let us look at this picture in the Root locus diagram. So, S is the coefficient in the exponential term in the solution. So, here I have plotted the real part of S and the imaginary part of S. So, S is plus or minus i omega. So, plus i omega tilda and this is the minus i omega till the, when the load is less than the critical load of the buckling load. As you increase the load, these roots they go to zero once they reach zero then the move on the on this axis. So, here as we are see; so, we have this plus minus alpha. Now, there is this root on the positive side which is going to lead to instability. So, that is what we look here in the solution. So, we have an exponentially increasing solution and which looks like this; even with a very small disturbance it is going to exponentially divergence. So, this is the divergence instability. So, beam with axial force will show divergence instability, if the force exceeds the Euler buckling load. Now, this can be used to estimate the buckling load of a column without destroying the column, if you want to estimate the buckling load of a certain column; so, if you give this axial load and keep increasing it and when it reaches the buckling load this column is going to deflect have large deflections and very likely this column is going to get destroyed or spoiled you cannot use it. But suppose if you want to have, if you want to revise the procedure of non destructive evaluation of the buckling root of the column experimentally, so, you can use this dynamic feature. Now, you increase the load and look at the lowest Eigen frequency and when it reduces and comes to very small value which you can tolerate, you can estimate from there and use the expression of the variation of the Eigen frequency with the axial load to finally find out the intersection without actually doing the experiment without actually destroying the column.

So, you extrapolate the curve, but I am just showed you with load versus the natural frequency. So, you extrapolate the curve and you obtain the intersection with the x axis, which is going to give you the buckling load without spoiling the column.

(Refer Slide Time: 23:48)

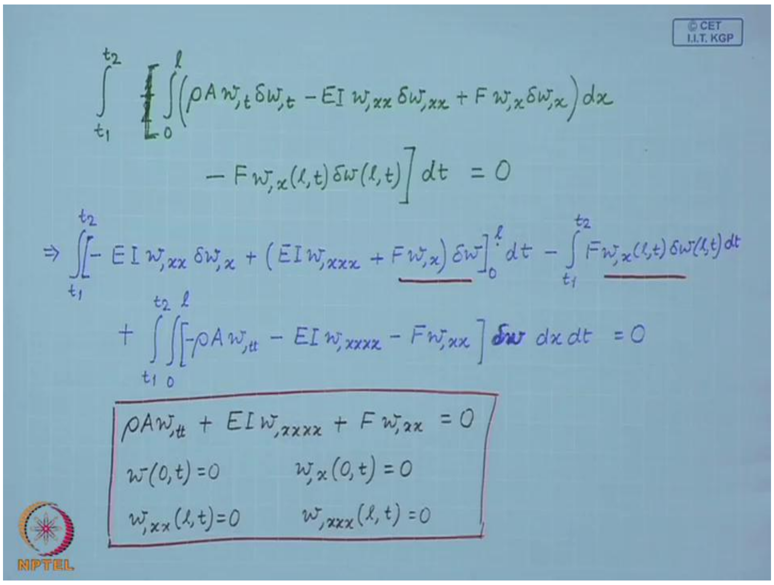


Next, let us look at this example of beam with follower force. Now, what is this follower force? So, let us consider a cantilever beam with force at the free end. This an axial force once again. But then this force has a property that, when the beam reflects, so, this is an exaggerated figure, so, when the beam reflects, the force is going to maintain the same angle as the centre line of; so, it is always going to be tangent to the centre line of the neutral fibber of the beam at the free end. Such a force is known as follower force. So, it follows the tangent to the neutral fibber or the centre line of the beam. Now, we need to; so, such examples are observed in missiles. So, this kind of dynamics is observed in missiles, in flexible missiles, then also in fluid caring jets. So, when you have a pipe or a tube, which has a jet at this end, so the water comes out and that gives force on the free end of this pipe and that is also follower force.

So, these are some examples. Now, we need to derive the equation of motion of this system. One thing you can immediately see that this force has a transverse component which is given by… This is of course assuming that this is small. So, if you make this assumption, then that the transverse component of this force at x equal to l may be written in this form. Now, this force is the non potential force, which means that this does not have this cannot be derived from a potential, it is not radiant of any potential.

So, to derive the equation of this system, if you want to the variational principle to derive this equation, which is safer way for the system, because the dynamics is little tricky as will very soon see. So, let us. So, since we have a non potential force, we have to use the extended Hamilton's principle, to derive the equation of motion for this system. So, extended Hamilton's principle it says is this must be zero. Now, here this is the virtual work done by non potential force. So, in our case we have this as the non potential force and we must include the virtual work done by this force. Now, the Lagrangian, let us first write down the Lagrangian. So, this is one half. So, we have the kinetic minus potential energy. So, that comes from the kinetic energy of beam element minus the potential energy due to flexure and then there is a potential energy, because of this axial force. So, because of axial strain, I have taken the sine of this F approximately. So, that this indicates a compressive force. So, if F here is positive, then it is a compressive force now. So, this is the Lagrangian of the full beam. Now, the virtual work done by this non potential force is given by this. So, that turns out to be… So, you have to note that this virtual work is done at the point x equal to l. So, now we put these things back in our extended Hamilton's principle now.

(Refer Slide Time: 33:35)

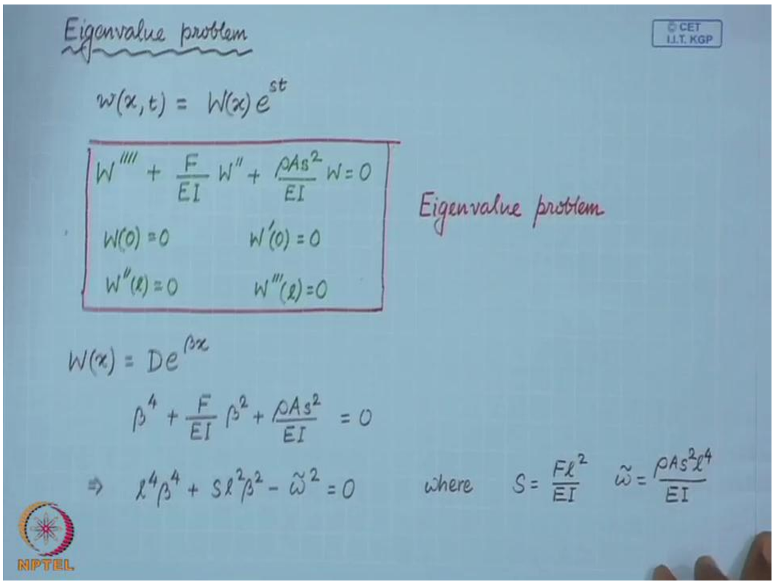


So, if you do that so, we have… So, let me write out the variation directly. So, that is extended Hamilton's principle. So, we note this virtual work already gives a boundary term in this equation. So, now, we integrate by parts this term with respect to time and if you use the arguments of variational calculus; so, at t_1 and t_2 the variation must vanish.

So, I am not going to write the term the boundary term generated, because of from this term at t_1, t_2, then this has to be integrated by parts with respect to space and similarly this. So, if you carry that out… So, once, when you integrate by part this term two times; so, this is one term, this is the other term with delta w and this term also gives you delta w and then we have this already as the boundary term and then we have the integrant in this.

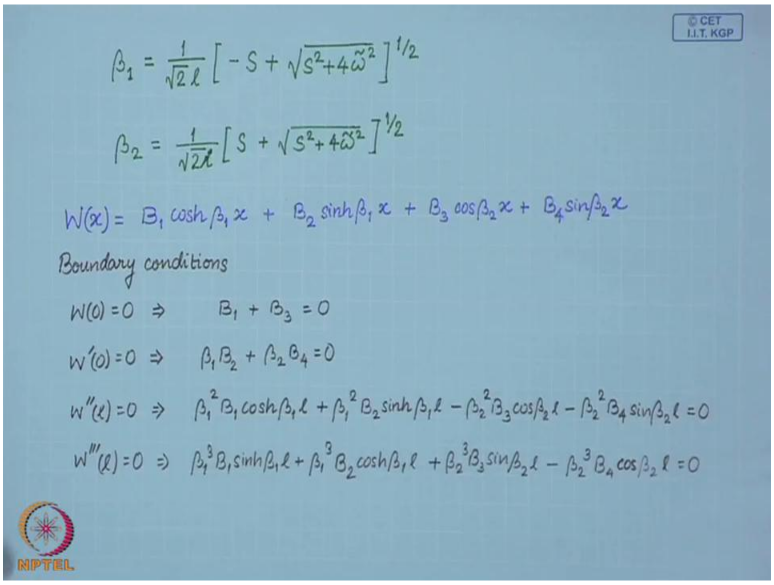
So, delta w taken out and that must vanish. Now, using the standard arguments that the variation over the domain and over the boundary must vanish independently, so, we can easily see that, from here we obtain the equation of motion and then the boundary conditions considering that the beam is the cantilever beam and here if you look at these two terms now here, so, this term and, so, this term evaluated at l and this term is going to cancel and finally, what we have left with at x equal to l… So, then the natural boundary conditions at x equal to l… So, at x equal to 0 this is going to vanish, at x equal to l this must be zero and the other term comes from here which is the moment, the bending moment. So, one term at x equal to l comes from here the other term comes from here; this term gets cancelled at x equal to l. So, this is the equation of motion and the boundary conditions for the beam, for the cantilever beam with the follower force F. Now, it is interesting to note here, you see the, first of all the equation of motion is same as that, with that of the beam with a normal axial load. The other thing is there is this boundary conditions are all homogeneous. Now, that is very surprising, because there was a transverse force at this free end of the cantilever which is, but then this term cancels the boundary contribution from the from this dynamics. So, finally, all the boundary conditions are homogeneous and this is a very surprising thing, in the case of the follower force beam with the follower force.

(Refer Slide Time: 43:05)



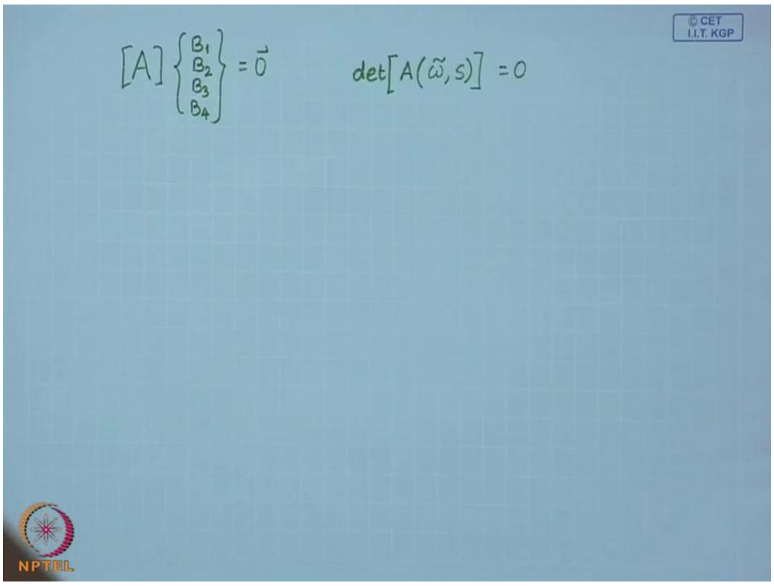
So, let us now look at the Eigen value problem. So, we look for a circular wave solution, complex solution of the form, let us say W(x) exponential S t. Now, when we substitute this in the equation of motion and we do the simplifications, so, that is the differential equation for the Eigen value problem and along with this we have… So, this describes the Eigen value problem. Now, let us make some redefinitions here. So, first we, let us search for a solution of this W. Suppose we look for solutions of W in this form, then substituting here we obtain… Now, we can write this as… where we have used these definitions. So, using these definitions, we restructure our characteristic equation in this form. So, here we are going to solve for beta.

(Refer Slide Time: 47:52)



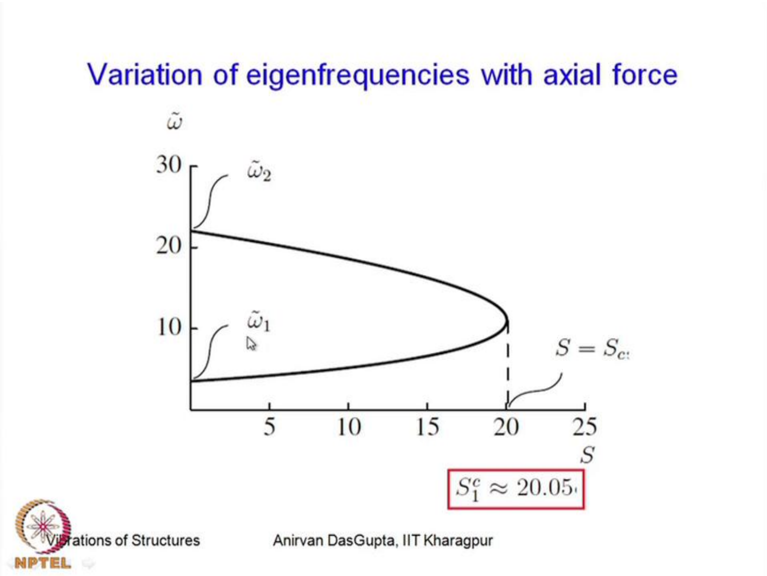
So, we can see that there will be two solutions of beta. So, this is one solution of beta; the second solution reads… So, I have written this beta 1 and beta 2, in such a way that I can write my solution of W as… So, that is the solution for W, general solution. Now, we must use the boundary conditions. So, the boundary conditions as I had written down, so, the displacement slopes being zero, so, this implies… and fourth condition… So, these are the boundary conditions.

(Refer Slide Time: 53:01)



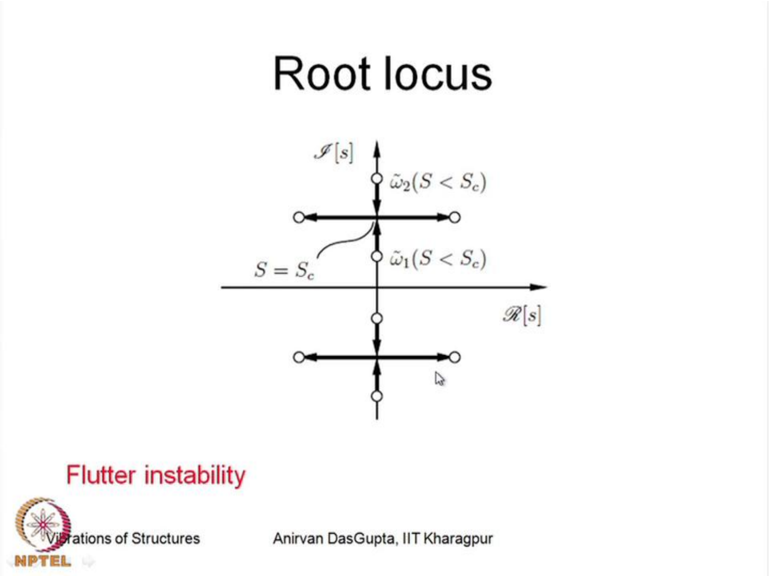
Now so, they can recast as a matrix A times… this is 0. So, for non trivial solutions of these coefficients, the determinant of A and remember this A is the function of omega tilda and S the loading, is must vanishing. So, if you so, you can solve for a given load S, you can solve for the circular natural frequency of the beam and this can be solved numerically.

(Refer Slide Time: 53:51)

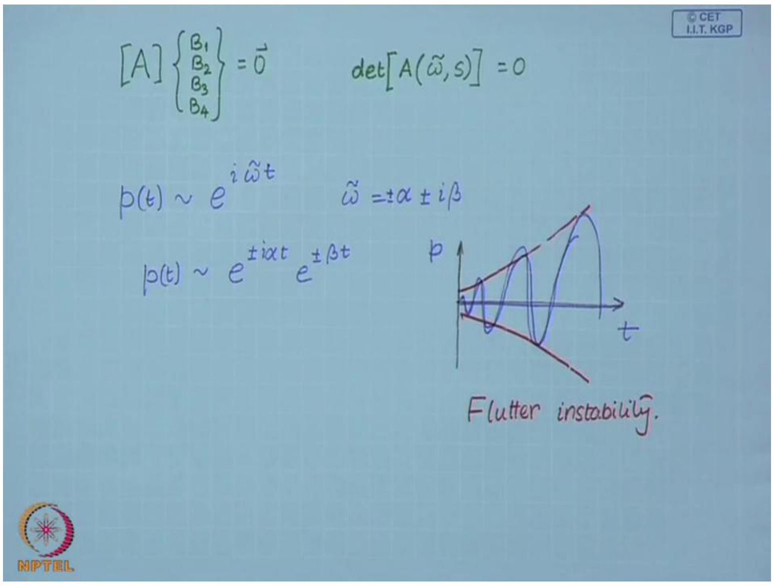


So, here this shows the results of such solutions. So, when the load is zero, the first two circular natural frequency non dimensional are these two points and as the load is increased as you can see that omega 2 decreases while omega 1 increases and they collies at the certain value of load. Now, this is the value which is critical, because after this they vanish. Now, to see what happens to these Eigen circular, Eigen frequencies we look at the root locus, which again we plot S the real part and the imaginary part of S. So, remember that S is plus or minus i omega t. So, when you have the load less than the follower force magnitude less than the critical value, then these are the distribution of the circular natural frequencies plus or minus i omega 1 and plus and minus i omega 2. As the load is increased these two come together and collies at a certain value, which is the critical value of the load and then they move into the, they have they have real part as well. So, this branch, this root has positive real part. Similarly this root has a positive real part.

(Refer Slide Time: 54:30)



(Refer Slide Time: 55:37)



Now, therefore, you have solutions. So, these are the solutions. Now, if omega becomes as you have seen in the root locus, you have both plus minus alpha plus minus i beta. So, if you have, if you consider this solution, then you have solution like this.

So, you have exponential term along with a fluctuating component. So, which means the solution will look like… So, this has an exponentially increasing envelope and so, this is t versus the displacement. This kind of a behaviour is known as flutter and we observe flutter instability in beams with follower force. So, there is an exponentially there is the fluctuation with exponentially varying amplitude. So, to summarize what we are discussed today, we have looked at the dynamics of beams with normal axial force and follower force. So, with that we conclude this lecture.

Keywords: axial force, follower force, non-conservative force, extended Hamilton's principle, divergence instability, flutter instability.