Vibrations of Structures

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Lecture No # 28

Topics in Beam Vibrations-I

Today, we are going to take up some special topics in vibrations of beams. So, the first topic that we are going to discuss is that of a tensioned beam. So, what happens when uniform beam is subject to tensile loading? So, this is important, because when we discuss strings, taut strings are in tension; but as we have discussed before, that there are examples, for example, the guitar string which looks like more like a beam under very low tension and as you make it taut, it becomes string. So, we are going to look at, in this transition of behaviour from a beam like behaviour to a string like behaviour, when you put beam under tensile loading.

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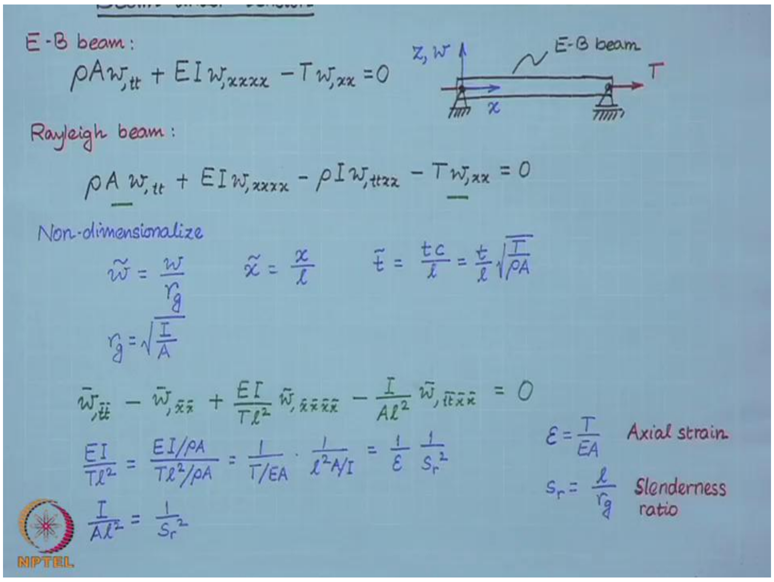

So, let us look at this example. Let us consider this Euler Bernoulli beam which is subjected to a tensile loading say T. Then the equation of motion... So, this is in Euler Bernoulli beam. So, these two terms are from the beam equation. Now, when you have the tensile loading and then additionally you have… So, this is the equation of motion for an Euler Bernoulli beam. In case of a, so, this is for an Euler Bernoulli beam. In the case of a Rayleigh beam, you have this additional term, because of rotary inertia.

So, this is for a Rayleigh beam which has this additional term, because of the rotary inertia. Now since, this can always fall back to the Euler Bernoulli beam, if you drop this rotary inertia term. So, we can discuss the Rayleigh beam. So, let us first non dimensionalize the equation of motion. So, if you do this non dimensionalization using this scheme, so, the spacial coordinate is non dimensionalize with the length whereas, the field variable is non dimensionalize with this r_g which is the radius of zyration, this is defined as square root of the second moment of the area divided by the area of cross section. Further, we non dimensionalize time using the speed of propagation of transverse waves.

So, if you use this non dimensionalization scheme, then the non dimensional equation of motion turns out to be in this form. Now, here we make a; so, now you see that these two terms are from the string equation of motion whereas, these two terms are from the flexural stiffness and the rotary inertia. Now, if you look at these coefficients, so, these coefficients can be written, if you divide the numerator and denominator by rho A, then I can recast this as… where epsilon is always the axial strain in the beam. So, T divided by EA, so, that is going to give us the strain, because of this tensile loading and S_r is defined as, the length over the radius of gyration; this is known as the slenderness ratio. So, this is the slenderness ratio and this is the axial strain.

Now, so, this coefficient is nothing, but inverse of the axial strain and inverse of the square of the slenderness ratio. Similarly, this coefficient is 1 over slenderness ratio square. Now, using these definitions then we can rewrite the equation of motion. So, this is our equation in non dimensional form and the parameters and the coefficients defined in a very special way. So, here I have of course dropped the tilde symbol for simplicity. Now, let us look at these terms. So, this is the slenderness ratio which indicates how slender the beam is, so its length over the radius of gyration. So, if the beam is very slender, then this slenderness ratio is very large, because its length is much larger than the area property.

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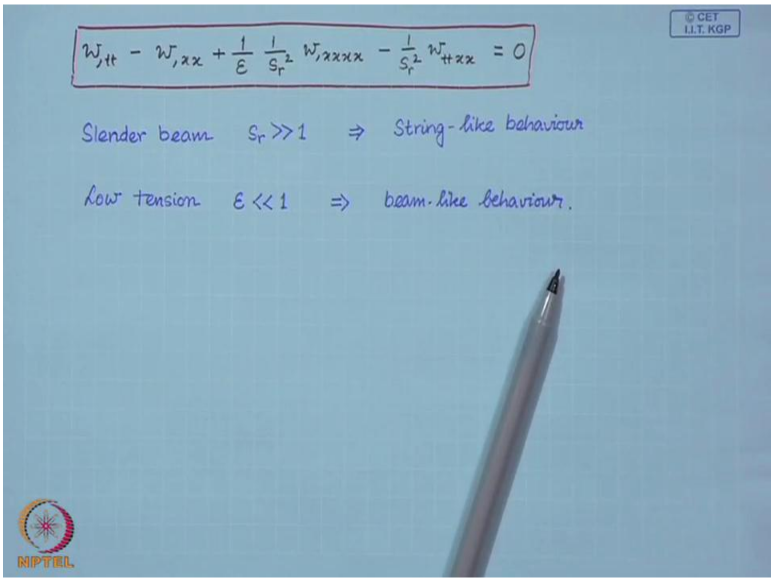

So, when the beam is very slender, then this is very large; and therefore, these terms can be dropped in comparison to this term. So, in that case, we expect the string like behaviour. So, in other words, we can treat beam under tension, very slender beam under tension like a string. On the other hand, if this stream is very small, so, the axial strain is very small; so, if the tension is small and the axial strain is very small, in that case this term might become significant. In that case you may have to include this flexural term in the equation of motion. So, this rotary inertia term therefore, is negligible in the case of slender beams, whereas, this term is negligible if the beam is again slender in the tension is very high.

So, the tension is very high and beam is slender then these two terms can be drop therefore, it will have a string like behaviour or the system can be modelled as a string, whereas, if tension is very small then this flexural term may be significant. So, this must be considered. So, if you tend to more like a beam like model; so, this is precisely what we observed from this equation and elements like the guitar string, which are extremely slender, when they are taut even though they have sufficient bending stiffness; but when they are subjected to a large tense tension, in that case the behaviour is like a string. So, you can treat guitar string under tremendous amount of tension as the string. So, this is what we have observed. So, another way to look at it is that the restoring force in a guitar string is more because of tension than because of its flexural rigidity or flexural stiffness.

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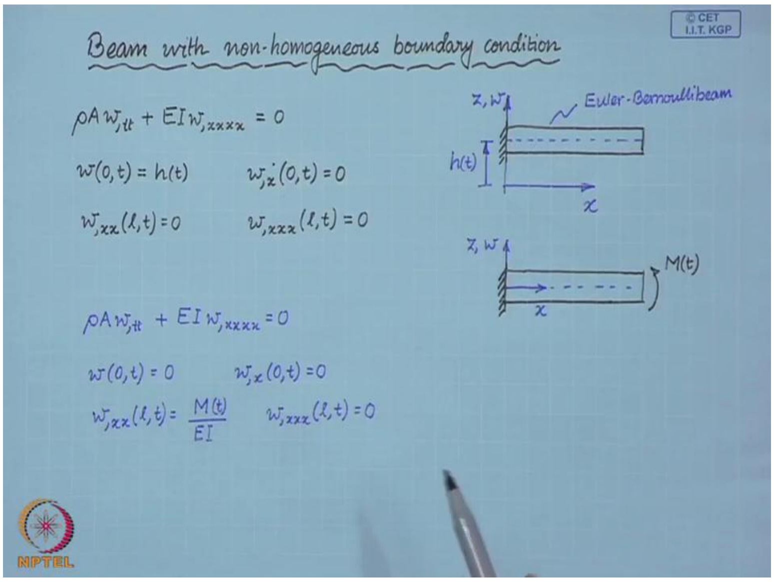

Next, let us consider a beam with non homogenous boundary condition. So, in various situations, you find beams which are excited at the boundary. So, for example, when you have an absorber attached to a vibrating structure; an absorber made of a continuous system like a beam which may be attached to a vibrating surface or structure, for the purpose of vibration absorption. So, usually these beams are like cantilever beams. So, they are fixed on the structure and they vibrate, because of base excitation. So, let us consider this example of a cantilever beam. So, this is the cantilever beam, which is subjected to base excitation. So, we can consider that, this motion of the base is given by h of t.

So, this is the time varying function, which is specified let us say, and we will consider that, this is an Euler Bernoulli beam. So, the equation of motion is given by this and the boundary conditions... So, the displacement of this boundary suppose, given by this function h of t at x equal to 0. We also have this condition that the slope is zero, whereas, for the free end, we have the bending moment as zero and the shear force is zero. Now, if you have, so, this is one kind of non homogenous boundary condition. You can have other kinds for example you can have the cantilever beam once again, but let us say with the time varying moment at the free end.

So, in this case, we can write the equation of motion and boundary conditions. So, once again at the free end, now they are both zero the displacement and the slope, whereas, the bending moment condition at this free end may be written like this and the shear force at the free end is zero. So, this is another kind of non homogenous boundary condition, where you have the force. Here it was the displacement, here it was the force. Now, when you have force, it is possible to include it in the equation of motion. So, we are going to first discuss this kind of system where we have excitation, boundary excitation in terms of displacement.

When you have forces again it is possible very easily to include this force in the equation of motion and then we can treat it has system with homogenous boundary condition. Now, why you we need this, because whenever we have solved the Eigen value problem, we have considered a homogenous boundary conditions usually. So, if you want to use this modal expansion theorem for solving the system, in such cases you will require that the boundary conditions be homogenized first. Otherwise, the modal expansion, the Eigen functions do not satisfy the boundary conditions with time varying functions.

So, we would like to have a method of homogenize the boundary conditions for the problem. So, let us look at this problem first. So, we have this cantilever beam with base excitation.

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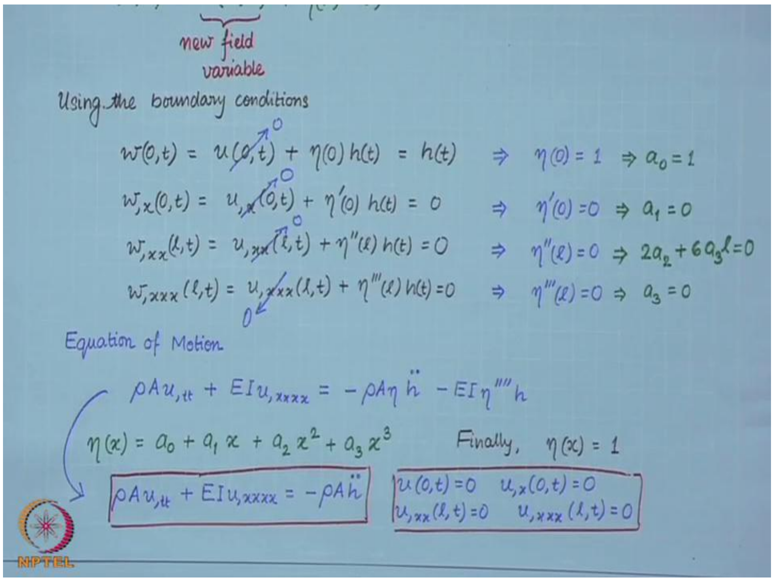

Now, let us try to make transformation of variables. So, let us see that our field variable w, we want to convert or transform to another field variable, let us say u x comma t, plus an unknown function of x times h of t, so, which is this function that we have as the forcing. Now, this structure is of course, motivated by the structure of the boundary conditions. Now, if you consider this new, so, this is the new field variable and this function is at as yet unknown. So, this is an unknown function. Now, let us use this transformation in the equation, in the boundary conditions. So, the displacement boundary condition… so, this must be h of t; the slope condition… So, prime here denotes derivative with respect to x that must be zero.

Now, the bending moment condition… zero and similarly the shear force condition at x is equal to l also zero. Now, we would like to have, so, this is our equation of motion. So, if you substitute this in the equation of motion that is equal to… So, minus of rho A eta h double dot, that comes from the inertia term and similarly this comes from the flexural stiffness term. Now, we would like to have homogenous boundary conditions for this equation of motion. Now, the field variable is u and the right hand side is now we have in terms of the time varying function h and as yet unknown function eta. So, if you want to have homogenous boundary conditions for u, which means that, this must be zero.

So, these are all zero. So, this implies eta of 0 is 1. Now, we have to construct the function eta which satisfies these 4 conditions. Now, to keep things simple, we can consider a polynomial in this form and if you now use this conditions. So, this will imply a_0 is 1. Eta prime 0 is zero; so, this will imply a_1 is zero. Eta double prime l, if that is to vanish then $(2a_2+6a_3l)$ must vanish and if the triple derivative of eta is to vanish, then a_3 must be zero. So, in that case a_2 is also zero. So, therefore, eta is nothing, but 1. So, this is the very simple example. So, in that case finally, and therefore, the equation of motion, so, what we obtain from here. So, this is the equation of motion and this is accompanied with, accompanied by boundary conditions that are all homogenous. So, these are the boundary conditions.

So, we have now a system which is first, but as homogenous boundary conditions. If you perform therefore, the modal analysis of the unforced system which is nothing, but normal cantilever beam, then you can use the Eigen functions of the unforced system to solve the forced vibration problem. So, we can easily convert any in non homogenous problem with non homogenous boundary conditions, two problems with homogenous boundary conditions. Next, let us look at this important phenomenon of damping.

So, as we have discussed before as well we can have two kinds of damping in continuous systems. One is the external damping which is more common and very easy to see and then there is an internal damping which is very actually hard to modal and also very less understood. This is because it depends on the material constitution, the internal structure of the material. Now, this internal damping, we are all considering throughout one dimensional elastic continuum. So, beam is also one dimensional continuum.

So, the source of this internal damping is usually friction between the layers of the, molecular layers inside the structure or inside the material. So, when we are saying we are considering one dimensional continuum, then there is no question of having layers. So, we must introduce this internal damping in phenomenal damping, in phenomenal logical way. So, to do that, we follow the Kelvin-Voigt modal.

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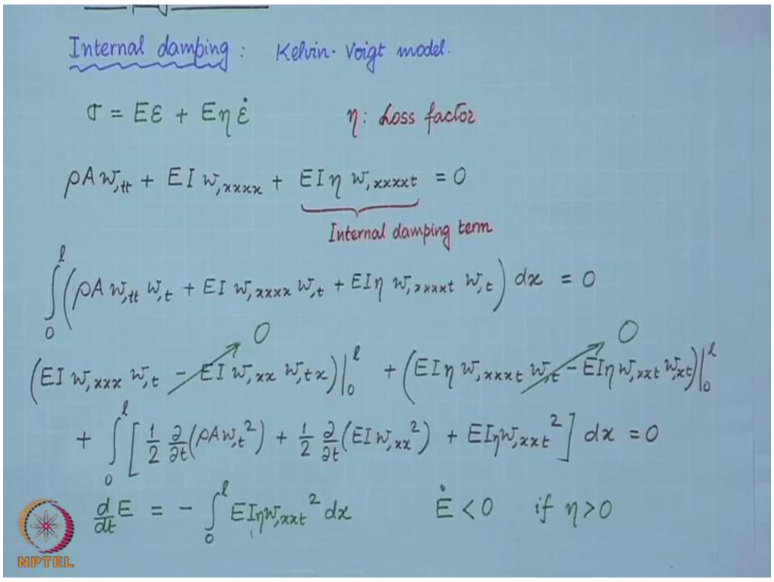

So, let us look at this model for internal damping. So, first we are going to look at internal damping. So, to model internal damping, we use the Kelvin-Voigt model. So, in order to implement this model, we modify the constitutive relation of the material. So, we know, we have used that for our beam model the Hooke's law given by the stress equals the Young's modulus times the axial strain. So, this was the axial stress in the fibres of the beam. Now, in addition to this the stress because of the mechanical straining, we add the second term which is dependent on the strain rate. So, the stress not only depends on the amount of straining in the fibers, but also the rate at which the fibers are being strained. Here, this eta is known as the loss factor. It is written in a special way in order to match this term. So, some e times eta times epsilon dot. So, if you use this model to derive the equation of motion, then you can easily see that the equation of motion… So, these are the two terms that we already have in the Euler-Bernoulli beam. Now, because of this additional term, you can expect that, there is going to be… So, this is the new term, because of internal damping.

Now, to see that this is really a damping term, so, we multiply this whole equation with the velocity del w/del t and integrate over the domain of the beam, that is the length of the beam. So, we obtain… and integrate this and that must also be zero. Now, we integrate by parts this term two times. So… and similarly this term also integrated by parts two times and the rest of the terms in the integral, I can write this term as… similarly after integrating by parts this term 2 times, so, this is what we are going to get. Now, we use the boundary conditions say for example, in a simply supported beam, then at l the velocity must be zero, at and the bending moment must be zero. So, this term will vanish; similarly, this term will also vanish. So, using the boundary conditions, you can show that these boundary terms will actually vanish. So, therefore, we are left with this integral equal to zero. Now, you can easily identify that these terms, these two terms are nothing but the total energy, the rate of change of total energy, half rho A del w/del t whole square is the kinetic energy, this is the potential energy. So, this is the total energy and this term, I am going to take on the other side. Now, there is an eta as well.

So, now E is positive, I is positive, if eta is also positive and the squared quantity, so, this is the positive quantity, so, integrated over the length of the beam, so, this must be positive quantity, if eta is positive and therefore, the energy, the rate of change of energy is negative, which means that energy always reduces. So, if this loss factor is positive, then this term is going to drain the mechanical energy.

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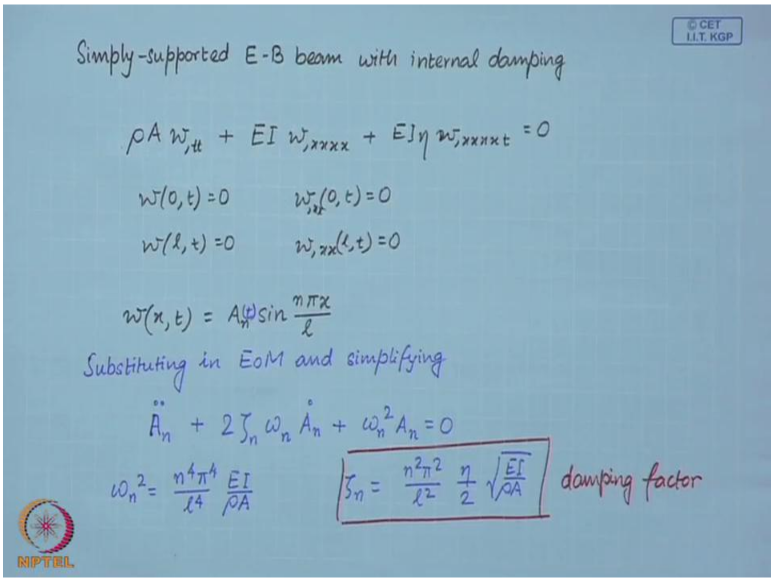

Now, let us look at now, simply supported Euler Bernoulli beam with internal damping. So, the moments are zero. Now, we have seen that for an undamped beam, the modal vibration is given by… Let us, for a moment, assume that this beam is vibrating in this n^{th} mode. So, if I substitute this in the damped equation, then I can rewrite; if you substitute this in the equation of motion and simplify; if you simplify then… So, this can be in general function of time, we can recast the equation of motion in this form where… So, this is what is interesting to observe.

So, if you consider that the beam is vibrating in the n^{th} mode, then the equation of motion for that mode, the modal coordinate, modal dynamics is governed by this equation, where, the natural frequencies as expected, but there is a damping factor which we find is proportional to n square. So, higher the mode, higher will be the damping factor; so, which means that the higher modes get effectively damped, because of internal damping. You can repeat this analysis with external damping and you can easily show and this we have see before as well in the case of strains and bars, that with external damping the lower modes are more effectively damped than the higher modes, same confusion can be drawn for this beam vibration, where we find that internal damping is affectively damping the higher modes and the external damping will effectively damp the lower modes.

So, to conclude, today, what we have discussed we are considered two topics from beam vibration, three topics. So, first we have considered the non homogenous, we have considered non homogenous boundary conditions. We have considered internal damping in beams and in the beginning we also looked at the beam under tension; so, how the string and the beam behaviour get delineated. So, that was another topic we looked at. So, with that we conclude this lecture.

Keywords: tensioned beam, boundary condition homogenization, internal damping.