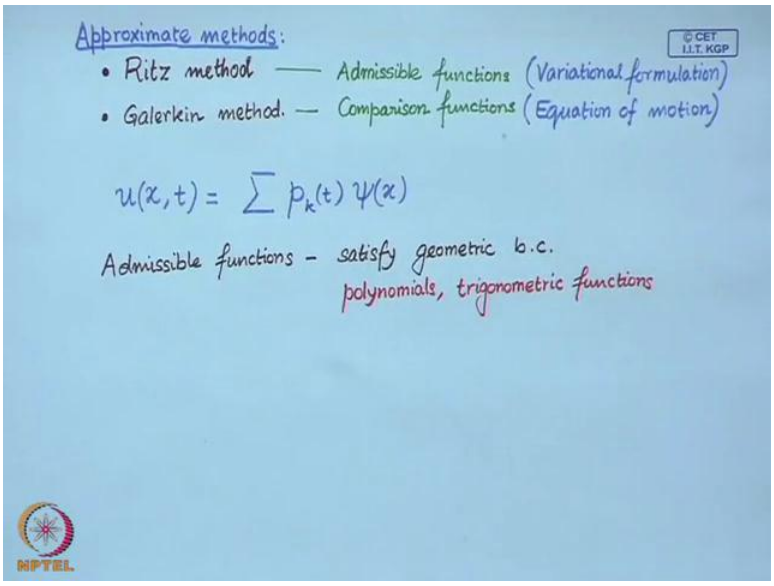
Vibrations of Structures
Prof. Anirvan DasGupta
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur
Lecture No. # 27
Approximate Methods

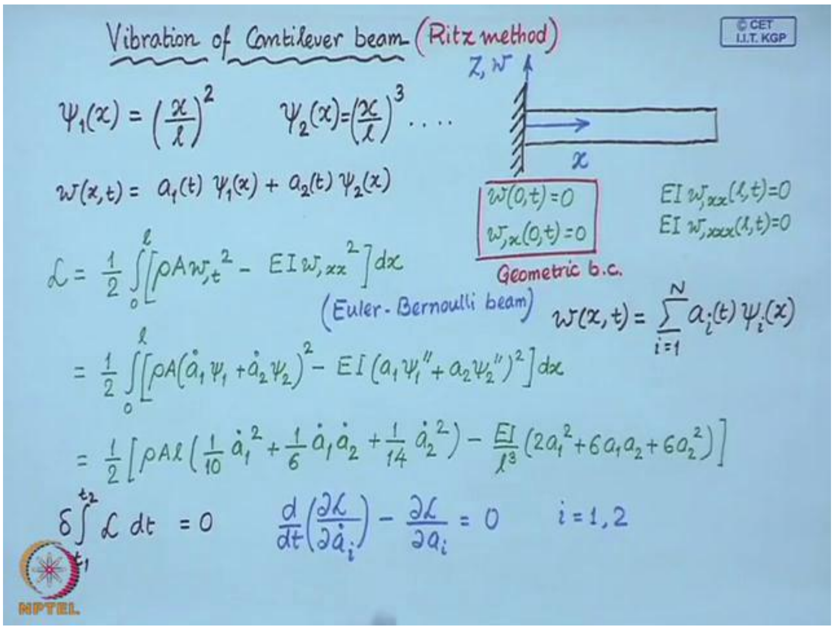
In the previous lectures, we have been discussing about the modal analysis of beams; and what we observe that even for simple configurations of beams or simple beam models, you can have fairly complicated Eigen value problem which you have to solve in order to accomplish modal analysis. So, one would be interested in knowing if they are approximate methods which can quickly tell us, give an estimate of the Eigen frequencies and the modes of vibration of continuous system and for example, for beams. Now, in our previous lectures, we have discussed some of these methods which are used for approximately performing the modal analysis; and as we have discussed that these methods can be improved to improve the accuracy of analysis. So, today we are going to look at some of this approximate methods applied to beams.

(Refer Slide Time: 06:49)



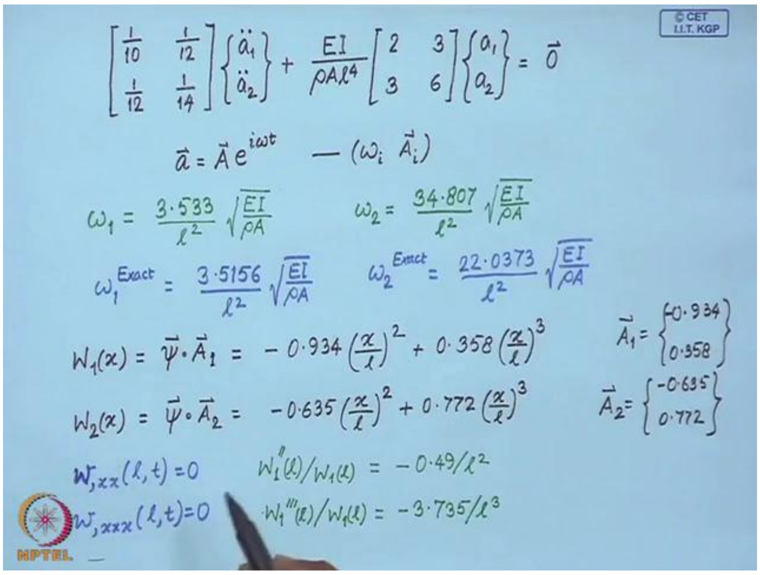
So, the first example that; so let me first enumerate the various methods that we have discussed, that we will use also in the case of beams. So, for example, we have used the Ritz method. So, these are all… So, the Ritz method; we have also looked at… Now, in the Ritz method what we need? We need admissible functions. So, we expand the solution in terms of the admissible functions. On the other hand, in the Galerkin method we use the comparison functions. So, suppose we have a field variable u that we expand in terms of these special functions, which in case of Ritz method are admissible functions. On the other hand, in the Galerkin method, these are comparison functions. Then in the Ritz method, we use the variational formulation. So, we substitute or replace our field variable directly in this variational formulation; while in the Galerkin method we work with the equation of motion. So, these have its own advantages and disadvantages. For example, for Ritz method, it is sometimes tricky to consider non-potential or non-conservative forces; while it is much easier for the Galerkin method. On the other hand, for the Galerkin method, these comparison functions have to satisfy all the boundary conditions of the problem, which are more difficult to construct; while these admissible functions must satisfy only the geometric boundary conditions. So, this is an advantage of admissible functions; and these can be very easily constructed using polynomials or trigonometric functions or such elementary functions. Now, today we are going to look at the applications of the Ritz method in beams.

(Refer Slide Time: 16:34)



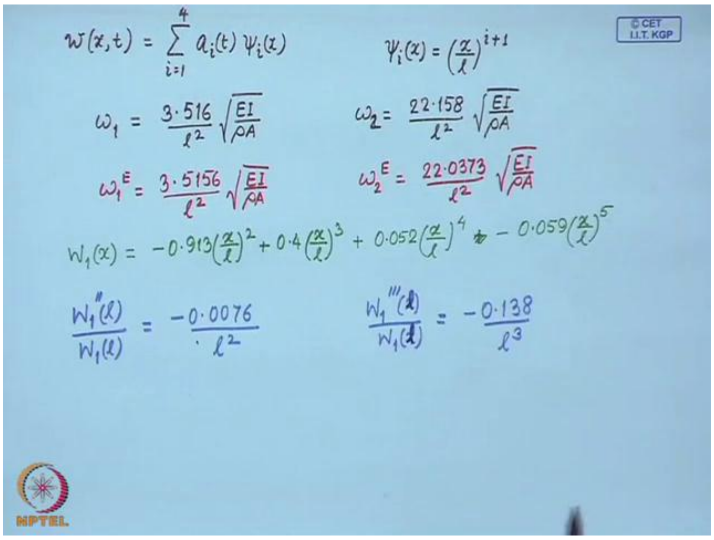
So, the first problem is that of a vibration of a cantilever beam. So, let us consider this cantilever beam. Now, the boundary conditions of this beam; w have discussed this before. We have the displacement of this point is zero; and also the slope has zero for all time. N the other hand, at the free end of the cantilever beam, we have the bending moment to be zero; and we also have the shear force at the free end to be zero. Since we are applying Ritz method, and in the Ritz method we use admissible functions which must satisfy the geometric boundary conditions; now, these are the geometric boundary conditions. So, whatever admissible functions we choose, they must satisfy these conditions. So, let us consider admissible functions. So if I consider a function like this; so remember, we are going to use this, we are going to use an expansion like this. So, we must choose our admissible function which must satisfy the geometric boundary condition of the problem. So, if we consider this to be let us say linear in x; so at x equal to zero, this is satisfied; psi 1 at zero must be zero. But when we look at this boundary condition, which is a slope condition, so del/del x of psi 1 at x equal to zero must also be zero. But if we choose a function like this then this boundary condition will not be satisfied. So, from these considerations, one can easily come to the conclusion that this must be one of the functions that can be used as an admissible function. Then we can use the higher powers of… etc. So, let us first begin with only two terms expansion, so which means… So, we will first use this two term expansion. So a_1 and a_2 are two temporal coordinates. Next, we will introduce in the Lagrangian which reads… So, this is the Lagrangian of an Euler-Bernoulli beam. So, let us consider an Euler-Bernoulli beam. The Lagrangian is given by this; and we substitute this expansion here and what we obtain… and if you simplify this further, if you substitute these expressions of psi 1 and psi 2 and peform the space integration that means the integration over x; these are the polynomials in x; so they can be integrated out very easily. The final result… SO, this is the Lagrangian that you have. Now, this is the Lagrangian of the discretized system with coordinate a_1 and a_2. Now, we can write down Hamilton’s principle for this.

(Refer Slide Time: 25:31)



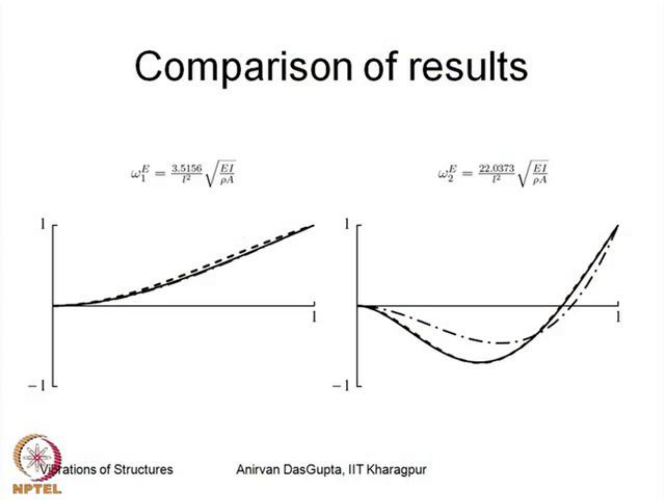
So, this will give us the equations of motion; and you know that this will lead to the Euler-Lagrange motion which… So, these are the equations for the two coordinates a_1 and a_2; and when you derive these two equations, they are of the form… So, this is the discretized equation for the cantilever beam. Now, we perform the standard modal analysis for this discretized system; and we can calculate the Eigen frequencies, the circular Eigen frequencies and the modes of this Eigen vectors which can be used to determine the modes of vibrations. So, let us first look at the Eigen frequencies. So, when you do this calculation… So, this is the first circular Eigen frequency for the cantilever beam calculated from this discretized equations of motion; and the second one is obtained like this. Now, if you do the exact calculation which we have discussed before, so this is the exact; this turns out to be… Now, you can make a comparison. While the fundamental circular Eigen frequency compare very well with the exact, the second circular Eigen frequency, this is on the higher side. As we have discussed before that the Ritz method gives us an upper bound on the Eigen frequencies. So, when we calculate by this method, we will get this omega 2. Now what this tells us is the actual Eigen frequency is less than this value. Similarly, here also you can see that this value is less than the exact Eigen frequency. So, this is the property of upper bound of the Eigen frequencies from the Ritz method. Now, let us look at the Eigen functions. Now when we substitute here, we are going to calculate omega and A, so the Eigen pairs. So, we are going to get these Eigen vectors; and using these Eigen vectors, we are going to construct our Eigen functions using the expansion that we have used. So, we do a dot product. So, the first Eigen vector which we get corresponding to omega 1, A_1; so if you dot product with the vector of the admissible function, you get the first Eigen function; and this turns out to be… So, this A_1 vector is actually… So this is the A_1 vector; we take the dot product with this psi vector. Similarly, the A_2 vector was actually this. So, this is our Eigen function, second Eigen function. Now, as we have discussed that these admissible functions do not satisfy, they are not required to satisfy the natural boundary conditions, which in our case of this cantilever beam, these are the moments and the shear force and the bending moment being zero at x equal to l. So, let us look at, so what we have is this must be zero; but since this was the bending movement condition, this was the shear force condition. Now, let us see how well these Eigen functions satisfy these conditions. So, if you calculate for example, W_1 double prime at l that turns out… and divide this by W_1 at l that turns out to be… and similarly… So, we are trying out with the first Eigen function; we take double derivative of that and see how close to zero this is. Now, as you can see, with the increasing length of the beam, this is going to go to zero quite fast and similar for the shear force. Similarly, you can do for W_2, the second Eigen function. But since we are more confident about our first Eigen function, here I have taken the example of our first Eigen function.

(Refer Slide Time: 30:29)



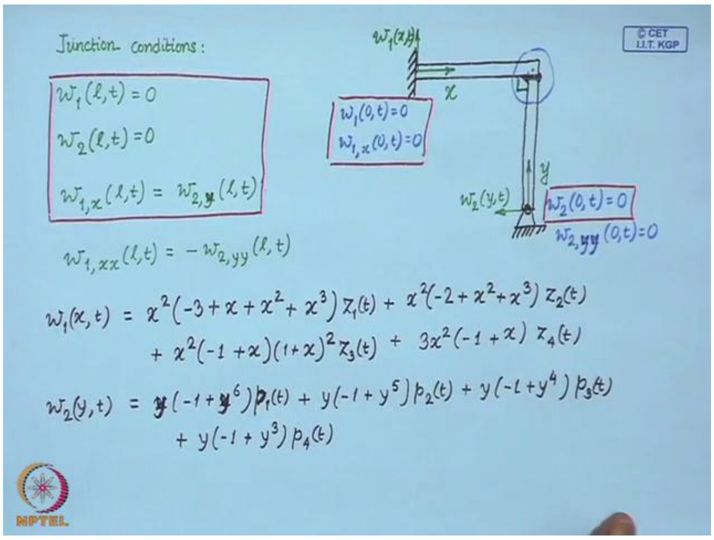
Now, this may or may not be satisfactory for our purpose. Now what we can try out is that we can increase the number of terms in our expansion. So in the second example, I have considered four terms in the expansion and I have taken the admissible function in this form; so psi 1 is square of x over l and I have gone up to four terms in this expansion; and when I calculate following the procedure that we have just discussed, if you calculate this Eigen frequencies, this turns out to be… and remember the exact was… Now, you can see with four term expansion, we are pretty close to the exact solution; and now once again if you calculate the first Eigen function, this turns out to be… Similarly, you can calculate the second Eigen function, third and the fourth. Here, we are focusing on the first Eigen function. Again let me calculate this ratio which will tell us how far the natural boundary conditions are satisfied at the free end. So, these are at l. So, now you can see with increase in the number of terms in the expansion, even the natural boundary conditions at the free end which are the bending moment and the shear force, they are also going to zero quite rapidly. So, as you increase the number of terms in the expansion, you are going to get the accurate solution for the Eigen frequencies as well as the Eigen functions will also get more and more accurate; and they will automatically satisfy, they will tend to satisfy the natural boundary conditions which you have neglected while doing this expansion.

(Refer Slide Time: 31:07)



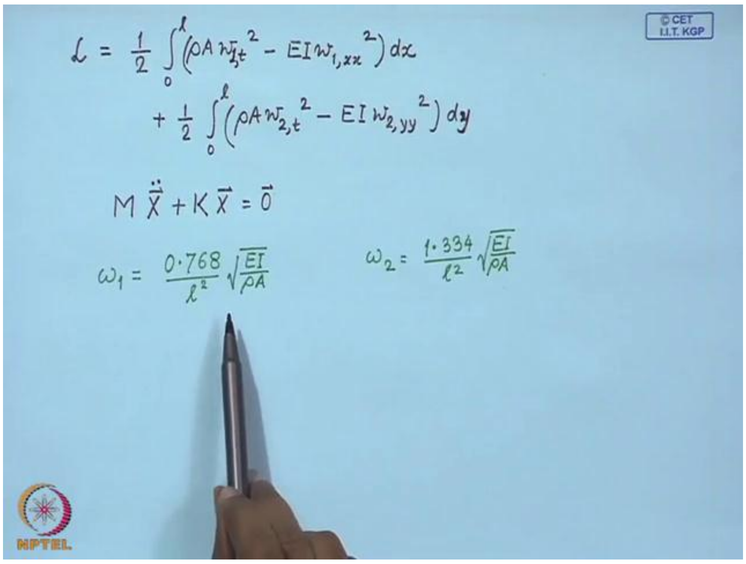
Now, let us look at these Eigen functions which I have plotted here. So, this is the first Eigen function. The solid line is the exact and this chain dotted line is with two term expansion; and with four term expansion, you have this dotted line. So, you can see the Eigen functions, they also tend to go close to the exact Eigen functions which we have discussed in one of our previous lectures.

(Refer Slide Time: 41:46)



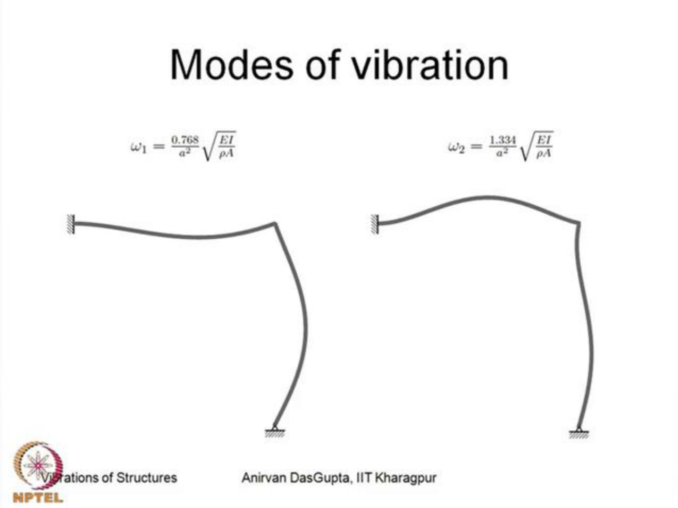
Now, let us go over to our second example. This example is of plane frame. So, let us look at this plane frame. So, we have this plane frame constructed out of two beams which are welded at this point. For simplicity, we consider that the lengths of both these beams are the same. So, we have l and l. Now, here we have a built-in end of this frame and here it is a pinned end. Now, so, these are essentially two beams which have a junction. So, we must, we contrived them like that. So, let us consider the coordinate here is x and the displacement in this direction for this beam, horizontal beam is represented by W_1, and this coordinate is y and the coordinate and the field variable for this vertical beam is W_2. Now, we intend to determine the Eigen frequencies and modes of vibration of this frame. Let us first write down the boundary conditions. So, at this built-in end… So, these are the boundary conditions at the built-in end. At the pinned support, we have the displacement as zero and the bending moment… The coordinate is y; so, this is zero. Now along with these boundary conditions, we also have this junction. So, what are the conditions at this junction? So, the first condition, if we consider this beam, the horizontal beam, there cannot be any vertical displacement of this beam at this point assuming that this beam is axially rigid; so there is no axial displacement of this point. In that case, the horizontal beam cannot have any displacement. Similarly, by similar reasoning, for this vertical beam cannot have any displacement in the horizontal direction. Now, since this point is welded, so these two beams are welded at 90 degree, so under deflection as well this angle has to be maintained, which means… So, this is the slope condition. These two slopes, they must maintain a certain relation. The second condition is on the bending moment. So, there must be equilibrium. So, from these conditions, we can obtain this bending moment condition at this junction. Now, we have all the conditions required for this plane frame. Now, let us identify the geometric boundary conditions. Now here… So, these are the geometric boundary conditions. Now, we have to satisfy, since we are following the Ritz method, we have to satisfy these boundary conditions; and the others, the natural boundary conditions are not so much essential. So, now let us consider this expansion. I will write out this expansion which has been constructed using polynomials. So, there can be various ways of constructing this expansion, these individual polynomials which satisfy these geometric boundary conditions. So, these are… I have considered these expansion; they are the admissible functions. But they can be constructed in various other ways.

(Refer Slide Time: 45:06)



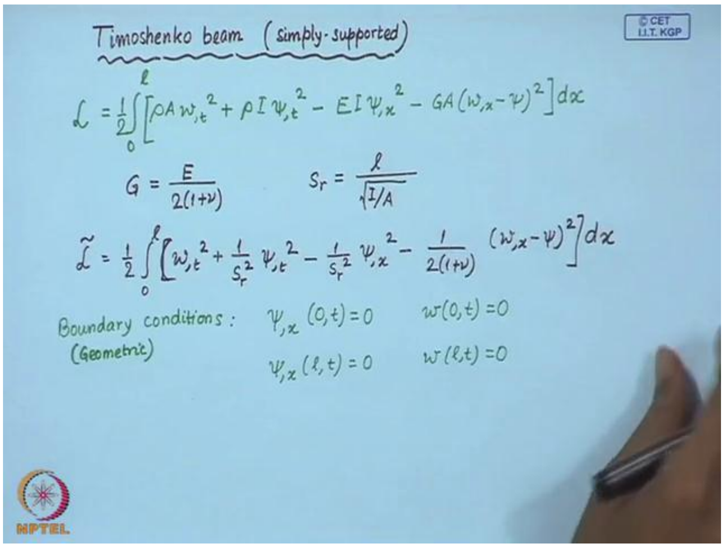
Now, using these two expansions for the field variables, we write out the Lagrangian. So, we have written out the Lagrangian for these individual beams; and when we substitute this expansion in the Lagrangian and integrate out the space part; so here we integrate over x and here we integrate over y and we obtain the discretized Lagrangian, from where finally, as we saw in the previous example, we are going to get the discretized equations in this form. Now these are the matrices, the mass matrix and the stiffness matrix; and we again perform the modal analysis for this discretized system; and if you do that then the result for the first two modes… So, these are the first two circular Eigen frequencies of the system which are calculated using this Lagrangian and the expansion that I have discussed just now.

(Refer Slide Time: 45:17)



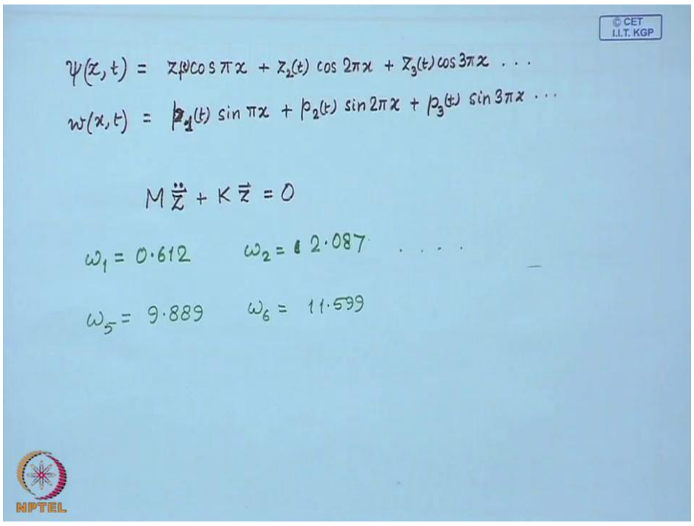
So, this figure shows the first two modes of vibration of this plane frame. So, you see that this is the fundamental frequency and the corresponding mode of vibration. So, we can see that this angle of 90 degree is being maintained in both these cases since we have chosen our admissible functions which satisfy both these boundary conditions already. So, this was an example of a plane frame.

(Refer Slide Time: 50:19)



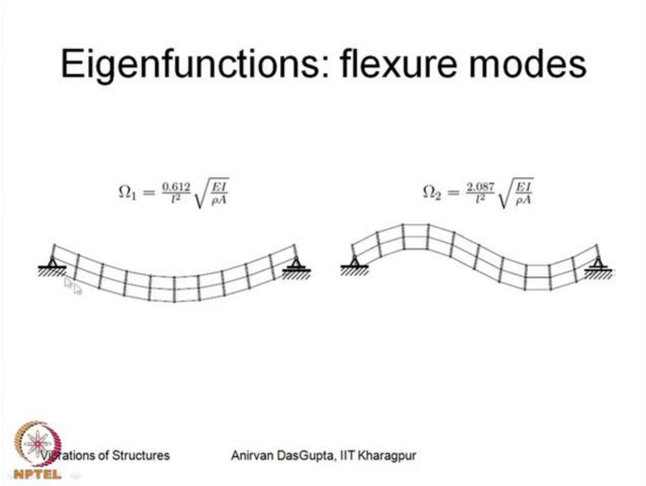
Next, we look at this Timoshenko beam which is little more sophisticated model for beam, which consider also the shear deformation of the beam. So, we will consider a simply supported Timoshenko beam. Now, if you recall the Lagrangian; the Lagrangian for this Timoshenko beam is given by this expression. Now in order to simplify this, we use the definition of the shear modulus; and we also define the slenderness ratio. In that case, the Lagrangian get simplified. So, with this definition and we can take the material constant out and simplify; so this is actually L tilde; and now let us look at the boundary conditions of a Timoshenko beam. So, the boundary conditions are obtained through variations. We can write them as… So, these are the geometric… which we need to satisfy when we are performing the Ritz analysis. So, in order to satisfy these boundary conditions, we can choose…

(Refer Slide Time: 50:27)



We can expand these field variables; for example, psi can be expanded as… and W can be similarly expanded… So, you can similarly see that these boundary conditions can be satisfied; so psi can be satisfied using an expansion like this; whereas… So, if the length is l, one can use an expansion like this. So, using the admissible functions, we can expand these field variables; and after applying the variations etc. we will obtain the discretized equations of motion. Now, if you perform the modal analysis of these discretized equations, then the first non-dimensional circular Eigen frequency is obtained as .612; second one is obtained as 2.087 and so on; the fifth is obtained as 9,889; the sixth is obtained as… Now, there is a reason why I am writing 1, 2 and then 5,6; of course there are other circular Eigen frequencies in this range. But let us look at the Eigen functions which are shown here.

(Refer Slide Time: 54:10)



So, this is the first circular Eigen frequency. You can see the mode of vibration in the first mode for Timoshenko beam; and similarly this is the second circular Eigen frequency and the second mode of vibration; so, these two look very similar to the normal beam. Now, let us look at these fifth and the sixth. Now, here there is hardly any transverse displacement; it is very small, not visible in this figure. These are actually the shear modes of the Timoshenko beam; and these frequencies are substantially higher.

So, what we have looked at in this lecture today; we have discussed about the approximate methods for modal analysis for discretization, which says we can discretize equation of motion of the beam; and we have used the Ritz method for discretization. One can also use the Galerkin method in a similar manner. The only thing is in the Galerkin method, since we use comparison functions, so they are little more cumbersome for calculation to construct. On the other hand in the Ritz method, we have seen the admissible functions are very easy to compute; and if you increase the number of terms in your expansion, then you can also satisfy, the you can also make this natural boundary conditions which are neglected while constructing the admissible functions; so, you can make these natural boundary conditions also to be zero; so, the satisfaction of the natural boundary conditions; so with increase in number of terms, you can satisfy these natural boundary conditions better. So, with that we conclude this lecture.

Keywords: Ritz method, beam vibrations, plane frame vibration, Timoshenko beam.