Vibrations of Structures

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Lecture No. # 26

Applications of Modal Solution

In the previous lecture, we have discussed the modal analysis of beams. So, performing modal analysis is nothing but solving an Eigen value problem as we have seen; and what we obtained; we obtained the circular Eigen frequencies or the natural frequencies of the beam; and we also obtained the Eigen functions. These Eigen functions, they define, they describe the modes of vibration of the beam.

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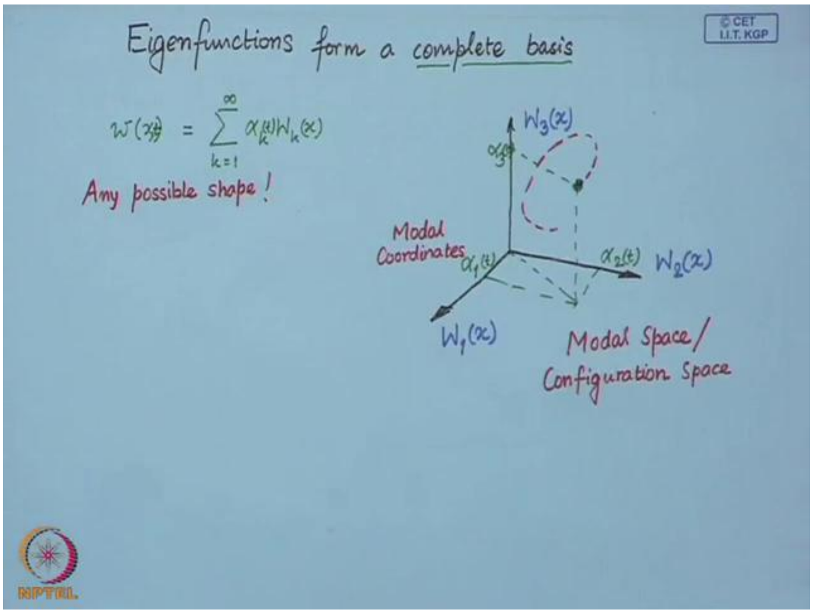

Now, these Eigen functions they form a complete basis for the shape of the beam. So, what we mean by complete basis is something like this. So what we mean by this is suppose, we have these Eigen functions. Let us call them… Now, there are infinitely many Eigen functions. Here I can draw only three and with a little stretch of imagination, you can think of this as an infinite dimensional space with each axis labelled with one Eigen function. So, any point in this space, in this infinite dimensional space with coordinate alpha 1 along W_1, alpha 2 along W_2, alpha 3 along W_3 etc.; so this describes the shape of the beam. So, any shape of the beam may be represented as a linear combination of these Eigen functions. Now, when we say this forms a complete basis, it means that any possible shape of the beam can be represented using these Eigen functions. So, any possible shape can be represented in this form; then we say this forms a complete basis. So, these Eigen functions, therefore, give us a good way of representing solutions which may also be functions of time. So, suppose if these are coordinates which may be general functions of time, in that case the motion of this point in this space may be represented through these expansion. Now, these alpha k's, they are functions of time; they form what are known as modal coordinates; and this space is known as the modal space or the configuration space. Now, this fact that any shape or any dynamical shape can be represented by this expansion allows us to solve a number of problems related to the vibration of beams; and this fact we have also seen in case of strings; and this is true in general for continuous systems.

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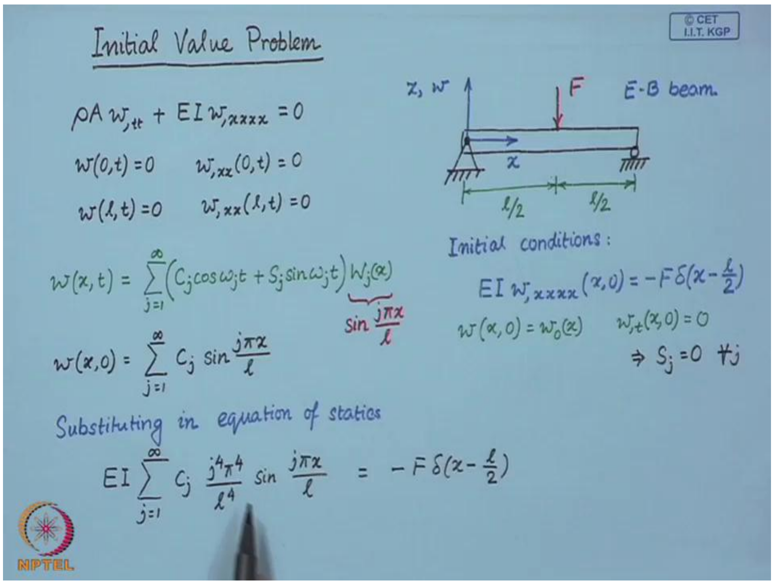

So, let us today look at two such examples. First we look at an initial value of problem for a beam. So, the problem that we consider… We have a simply supported Euler-Bernoulli beam. This beam is initially loaded with a force let us say F at the middle. So, there is a static force at the centre of the beam which is F. So, let me write down the equations. So, when we have; the problem we are going to address is like this. Initially the beam is displaced by the action of this force F applied at the centre of the beam and at time t equal to zero, this force is switched off. So, at the point this force is switched off, the beam is going to spring back. So, the equation of motion for the dynamics problem… So, this is the equation of motion and the boundary conditions… So, this is deflection is zero; this is the bending moment is zero at x equal to zero; and same way at x equal to l. Now, the initial conditions, we need to determine the initial condition which is the deflected shape of this beam. So, that will be obtained by solving the equation of statics. So, by solving this equation along with the boundary conditions that we have here; so if you solve this then you will obtain… and we assume that initially at time t equal to zero, the velocity of the beam is zero; and this is the shape. Now, this we can substitute in here and integrate out to determine the initial shape of the beam. But we are not going to do it immediately. We are going to first write the solution of this problem as an expansion in terms of the Eigen functions. We know that we can represent the solution in this form where these are the Eigen functions; and for a simply supported beam, Euler-Bernoulli beam, the Eigen functions are given by sine j pi x by l. Now, let us first solve this statics problem at time t equal to zero. Therefore, this is going to be the equation. So, at time t equal to zero, this expansion, so this is… so C_j times sine of j pi x over l; so at time t equal to zero, this is the expansion. So, if you substitute it here, so when we substitute in the equation of statics, this is what we obtain. From here, from this equation, we are going to solve these C_js; and this velocity condition will immediately tell you; so if you consider that this is the velocity condition which is zero, so from here, substituting in the velocity condition, this is going to tell us that all the S_js are zero. So, we only have these C_js. So, to solve this, we follow the standard procedure. We are going to multiply both sides by sine a pi x over l and integrate over the domain of the problem; so, we take inner product. So, because of the orthogonality, this is going to filter out, the k^{th} term.

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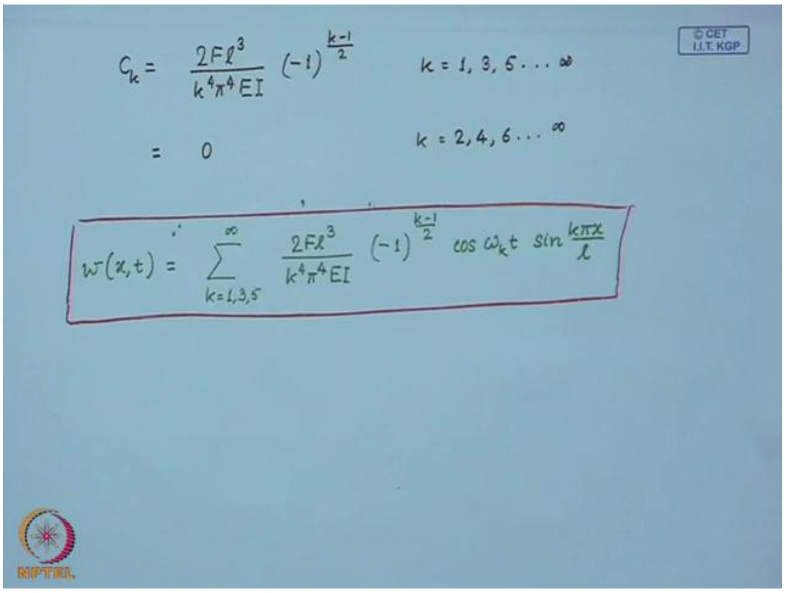

So, if we do that and simplify, then we obtain for odd values of k, we have C_k as non-zero given by this; and for other, so for even values of k, we have C_k equal to zero. So, finally when we have all these, we can write the final solution. So, this is the final solution of the initial value problem. So, this is function of space and time. So, one can animate this solution to determine the response of the beam.

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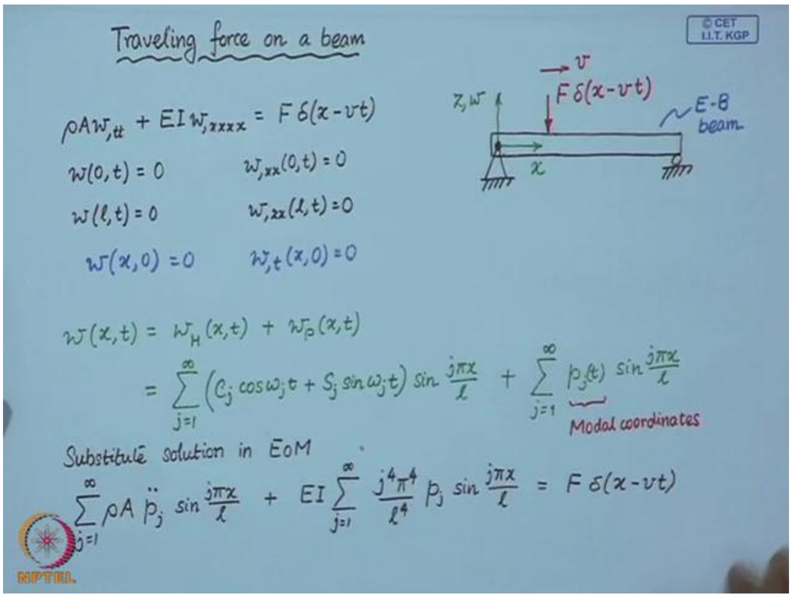

Now, let us look at another problem, which is the problem of a travelling force. Now, we consider, once again, the simply supported Euler-Bernoulli beam carrying… So, this beam is carrying a force which is travelling with a speed v. So, this problem is important in case of let us say bridges where you have travelling loads. This is a simplified version here. Here, we are considering a constant force travelling on Euler-Bernoulli beam. So, the equation of motion of this system… this along with the boundary conditions… and we also consider the initial conditions to be zero. So, which means the beam is undisturbed before the force enters the span of the beam. So, this is a forced vibration problem with general forcing. So, we can write down the solution of this problem as the homogeneous solution plus the particular solution. Now, we already know that this homogeneous solution can be expanded in terms of the Eigen functions and can be represented in this form. As this is a simply supported beam, therefore the Eigen functions are sine j pi x over l plus… So, this particular solution also we can expand in terms of these Eigen functions along with these modal co-ordinates. So, these coordinates capture the dynamics because of the forcing. So, these coordinates capture the dynamics of the forced motion. Now, when we substitute this solution form in the equation of motion; so this is the homogeneous solution; so this is going to vanish; sp this is going to now contribute. So, this gives us…

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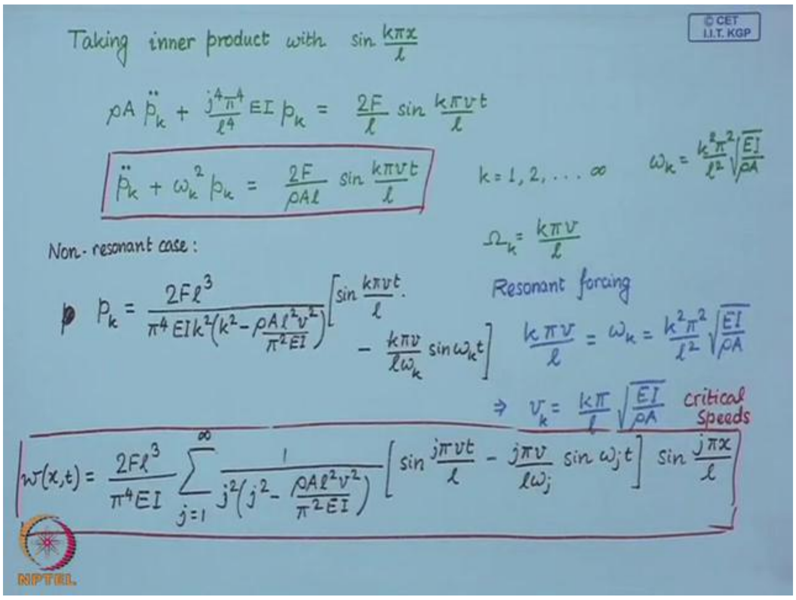

So, this is what we obtain and if you simplify this by taking inner product on both sides with sine k pi x over l, we can write this as… So, using orthogonality, we multiply this by sine k pi x over l and integrate over the domain of the beam which is from zero to l; now on account of orthogonality, taking this inner product is going to filter out the kth

term in this expansion. So on the right hand side, as we are multiplying this Dirac delta function with sin k phi x over l and integrating, this x gets replaced by this v t. So, then we can rewrite this. Now by I am dividing throughout by rho A; so, this j power four phi power 4 by l power 4 EI by rho A that is nothing but square of the k^{th} circular natural frequency of the beam. So, this defines the dynamics of the k^{th} modal coordinate and this can be written out for all the modes. So, we have the dynamics of all these modes. So, here of course… Now, this is a forced vibration problem for discrete system. Now, you see all these modal coordinates are decoupled. So, they can be solved independently; and we know the general solution for this system. So, we can easily the write the general solution for p_k and construct our solution of the beam. Now, here you can note that it is a harmonic forcing. So, the frequency of the harmonic forcing, let us name it omega indexed with k, so, that is the circulate frequency of the harmonic forcing. Now, there can be velocities for which this harmonic forcing equals the natural circular frequency of the k^{th} mode. So, in that case, we will call this forcing as the resonant forcing. So, for resonant forcing for any k suppose, if you have this condition and from here we know that this is… So, we can find out the velocity for which the k^{th} mode is resonant. So, let us index this also with k. So, this is the velocity which will send the k^{th} mode into resonant. Now, for simplicity let us first assume that this is not resonant. So, for non-resonant case, we can write the solution; so let me write the k^{th} coordinate. Now, this solution also satisfies the initial condition of the beam which is it is undisturbed at t equal to zero. So, the initial shape as well as the initial velocity of the beam is zero. So, using this solution, therefore… So, this is our solution for the response of the beam under a force travelling at a non-resonant speed; so which means the speeds do not take any of these values. Now, if the speed of travel is one of these resonance speeds, which are also called critical speeds; so if you have a force travelling at a critical speed, in that case the solution of this is modified which can be easily written out and we have studied this case for the string.

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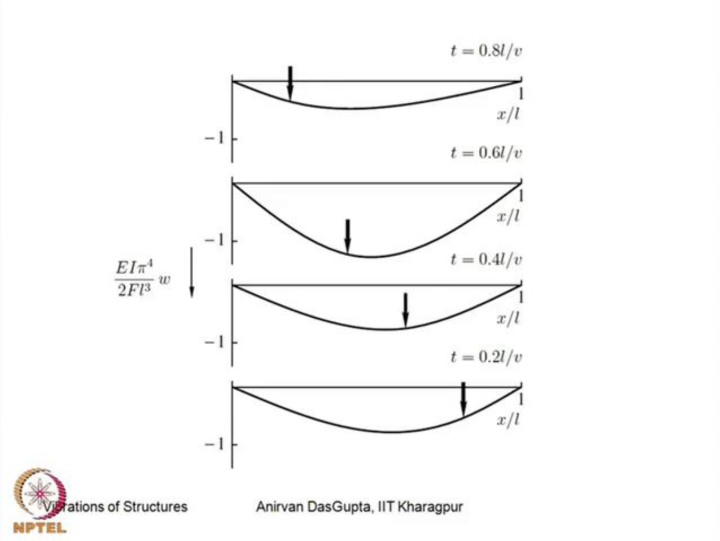

Now, let us look at this solution at certain time instant for two speeds, one is a low speed and other is a high speed transport over a beam. So, this figure shows snapshots at certain time instants for the force travelling on beam. So, you can see how the beam deflects at various time instants. Now, this is for low speed.

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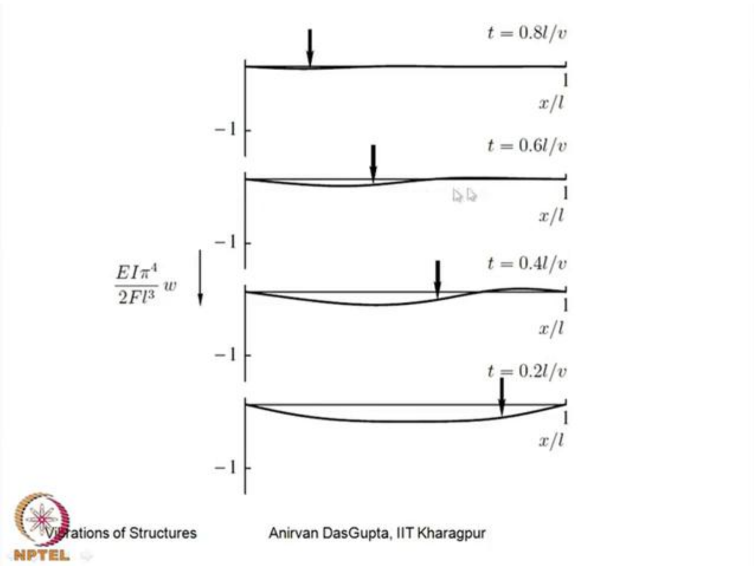

This figure shows the same thing when the speed is high. Now, here you can see a difference from the previous figure. In the previous figure, the deflection of the beam is completely on the negative side, below the equilibrium position. Now, here at certain time instant, this beam goes above the equilibrium line. Now, let us see this solution through animation. Now here, the velocity is, this is a low velocity that I have considered in this animation. So, you can see that the beam is always on the negative side. This black spot indicates the location of the force. The force is of course on the beam and this is of course an exaggeration of the deflection. Now, there are two things that has to be remembered when you see this animation. The first one is that this is the slow motion of what is happening. The second thing is that once this force leaves the beam, the response the beam is not shown in this animation. So, this is, this animation is looped and you see only the response of the beam when the force is on the beam and travelling on it at constant speed which is indicated below. Now, let us see what happens when we increase the travel speed. So, here v is l times omega 1 into pi over 4; so, this is the higher speed, higher than the previous situation. Now, you can see that here for example, the beam goes above. So, here beam is going above this the equilibrium line. Again this is the slow motion of what is going to happen. One more thing to notice in this solution which is different from the previous solution is that the deflection of the beam is much smaller in this case. Since, this force is travelling at a much higher speed, the beam gets less time for deflection; but this is what we observe when the force is within the span of the beam. So, you can again see, the disturbance propagating forward and it is reflecting back.

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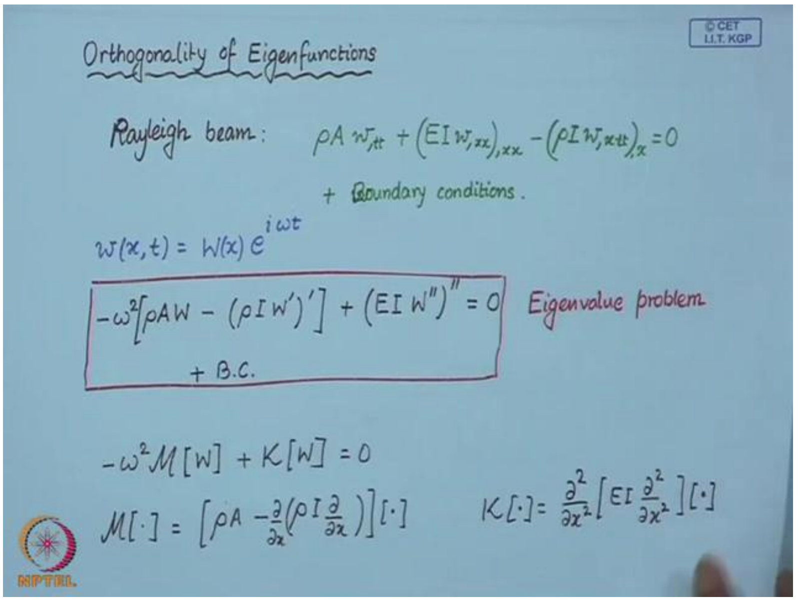

So, we have looked at these two examples related to the application of the modal solution. So, before we close this discussion, let us quickly look at one of the important properties of these Eigen functions which is the orthogonality. So, we have been using this property for all our calculations; but let us now formally look at this property form the equation of motion. So, let us consider a Rayleigh beam. So, for the Rayleigh beam, the equation of motion… So, this is the equation of motion for a Rayleigh beam; and along with this we have the boundary conditions. Let us say the boundary conditions… So, suppose we have simply supported boundary conditions. So when we do modal analysis, we search for the solutions of the form; so, we look for solutions of this form which is complex and separated in space and time. So, if you substitute this kind of solutions in this equation of motion, we obtain… So, this differential equation, where the prime denotes the derivative with respect to x; so this plus the boundary conditions, they complete the description of the Eigen value problem. Now, for a moment let us write this in a compact form, which this operator M… and the operator K can be represented in this form.

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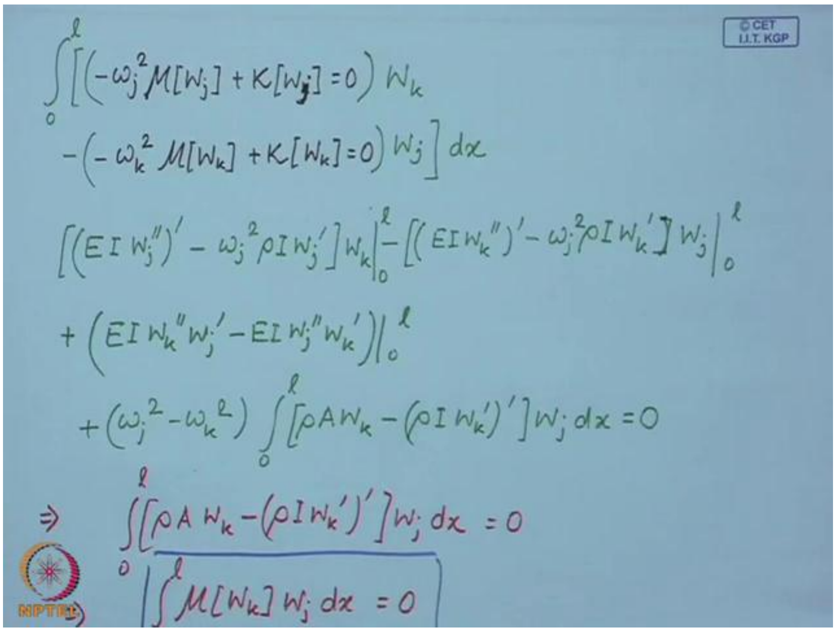

Now, let us write this for the j^{th} mode. Now, for the jth mode, this is going to satisfy this equation. For the k^{th} mode, this is going to satisfy this equation. Now, what we do, we multiply the first equation by W_k, the second equation by W_j; we subtract one from other and integrate over the domain of the problem. So, if you do that and simplify… These simplification steps we have discussed also previously. So, this is what we obtained. Now, if you use the boundary conditions of the problem, suppose simply supported, then this is going to be zero at both zero and l; and also the bending moment is going to be zero at both zero and l. So, these boundary terms, they all vanish if you use the boundary conditions. What you are left with… and this you can see… So, this is the orthogonality condition for the beam. So, you see that the Eigen functions are orthogonal with respect to the inertia operator.

So to summarize, we have today looked at some applications of the modal solution in solving the initial value problem, and solving the forced vibration problem; and we also looked at the orthogonality conditions of the Eigen functions. So, with that we conclude this lecture.

Keywords: modal analysis, initial value problem, travelling force, Eigen functions, orthogonality.