Vibrations of Structures

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Lecture No. # 25

Modal Analysis of Beams

Today, we are going to discuss the modal analysis of beams. So, in the previous lectures we have been discussing about beam models, mathematical models of beams; and so today we are going to discuss the modal analysis which means that determining the Eigen frequencies, and the modes of vibrations of beams. So, we will begin with a simple case of uniform Euler-Bernoulli beam. So, as you know that the uniform Euler-Bernoulli beam is governed by a differential equation like this; and along with this we will have boundary conditions. Now, as we have discussed in our previous class that there will be four boundary conditions. So, suppose if we consider a simply supported Euler-Bernoulli beam then the boundary conditions, so this is the displacement equal to zero condition; this is the bending moment equal to zero condition; similarly at x equal to l, we have displacement zero and bending moment zero. So, let us consider this kind of a uniform simply supported Euler-Bernoulli beam. So, as we have discussed before, we search for a solution of this type which is separated in space and time; complex solution; and if you substitute this in the equation of motion, the differential equation, then upon some simplifications, this must be zero, where this prime denotes derivative with respect to x. So, this is the differential equation; along with this, corresponding boundary conditions we will have. So, we have a differential equation, an ordinary differential equation fourth order in w, which is the space function and we have these four boundary conditions, which define what is known as the Eigen value problem, abbreviated as EVP.

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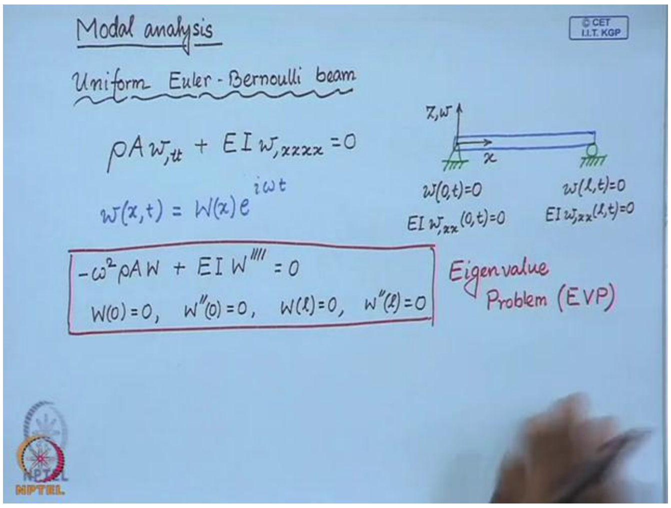

So, this is the Eigen value problem that we now have to solve. Now to solve this, we consider a solution of this form. So, let this be a solution of this differential equation. So, if you substitute this in the differential equation of the Eigen value problem… and that implies; so I have substituted this in this differential equation and made some simplifications to obtain this, which implies that… This beta tilde square, which is a unknown constant that is related to the circular Eigen frequency. Now, so we have, this can have two signs plus of minus; therefore this beta tilde can have four solutions, which we will represent as plus or minus beta and plus or minus i beta, where beta is… So, beta is fourth root of this quantity; and this beta tilde has these four solutions. Now, corresponding to these four solutions, therefore, the general solution of this function W(x) can be written as… So, in terms of this exponential function we can write the general solution, where these A_1, A_2, A_3, A_4 are arbitrary constants possibly complex. Now, we can also write this… So, by combining terms, you can also write this in terms of hyperbolic and trigonometric functions; so here again B_1, B_2, B_3, B_4 are now real constants. Now, here we have to determine or this general solution must satisfy these boundary conditions. So, if you put this, let us say, in the first boundary condition, so you will get… If you use the, second boundary condition… then this will get differentiated two times; then this will also get, all these terms will get differentiated two times. So, they will be beta square cos hyperbolic beta square sine hyperbolic minus beta square cos beta minus beta square sine beta. So, if you put x as zero… So, these are the two conditions in terms of B_1 and B_3; and you can easily infer from here that… so B_1 and B_3 must vanish for, we are considering the boundary conditions for the simply supported

Euler-Bernoulli beam; so B_1 and B_3, they vanish. Now, using the other two conditions, these two conditions… So, these are the two equations that remain; and if you want to have the non-trivial solution of B_2 and B_4, so if you want to have non-trivial solution of B_2 and B_4, so this I can write… So, this is what we have… So, if you want to have non-trivial solution for B_2 and B_4, then the determinant of this matrix must vanish; and that will give us the condition, so the determinant of this is… so it is minus two times; so that must be zero; this is the… So, vanishing of this determinant for non-trivial solution of B_2 B_4, this vector, leads us to this determinant equals to zero; and this is the characteristic equation for the simply supported Euler-Bernoulli beam. Now, this is, this sine hyperbolic zero is only zero at beta equals to zero, that gives us the trivial solution. So, for non-zero values of beta, because you see if beta is zero, the frequencies have to be zero, so for this to be zero, we must have sine beta l must be zero and that imples… Now, this beta l, sine of beta l equal to zero, that can happen at infinitely many values of beta, that can be indexed. So, these are the values of beta n for which all these boundary conditions will be satisfied.

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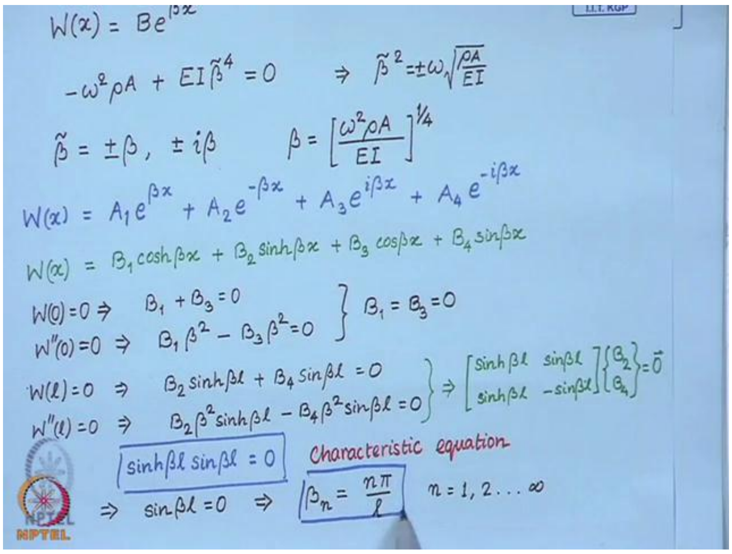

Now, you see beta is given by this expression, which has this omega. Therefore using these two… So, beta square is… and that must be… as we have obtained. So, this is indexed; so omega n now must also get indexed. So, we have now the circular natural frequencies of a simply supported Euler-Bernoulli beam. Now, this is for the Euler-Bernoulli beam.

Now, if you look at the differential equation of the Rayleigh beam model, then the differential equation of the Rayleigh beam model is… So, this is the differential equation of a uniform Rayleigh beam. So this is the rotary inertia term. So, if you follow the steps that we have just now done, then you will get the circular natural frequencies of the Rayleigh beam as… So, this is for the Rayleigh beam. So, we have this additional factor in the Rayleigh beam model, in the circular natural frequency of a Rayleigh beam model. Now, let us analyze this; now if you find, so we have this factor in the denominator, square of this factor. So, what is this factor? So, this is the length of the beam, the second moment of the area about the neutral axis I over the area of cross section of the beam. Now, this I over A is the radius of gyration's square; so under root of that is the radius of gyration. Now, length over the radius of gyration can be defined as the slenderness ratio. So, this length over the radius of gyration is the definition of the slenderness ratio. It indicates how slender the beam is. So, if the slenderness ration is very high, then the beam is very slender; the length of the beam is much larger than the dimensions of its cross sectional area. Now, this appears in, the square of this appears in the denominator of this term. So, you see if the beam is very slender, which means slenderness ratio is very high, then this term I can write as… Now, if the slenderness ration is very high then for the lower modes this term is negligible; say for n equal to 1,2 and if the slenderness ratio is for example 10, suppose the slenderness ratio is 10; in that case this factor is much, much smaller than 1; in that case this ratio is almost one. So, which means for very slender beams, the circular natural frequency is Rayleigh beam model should match with the Euler-Bernoulli beam; but then this term brings in the difference for higher modes. So for higher modes the frequencies are going to differ. So, as we go to higher and higher modes the frequencies, the circular natural frequencies are going to be different.

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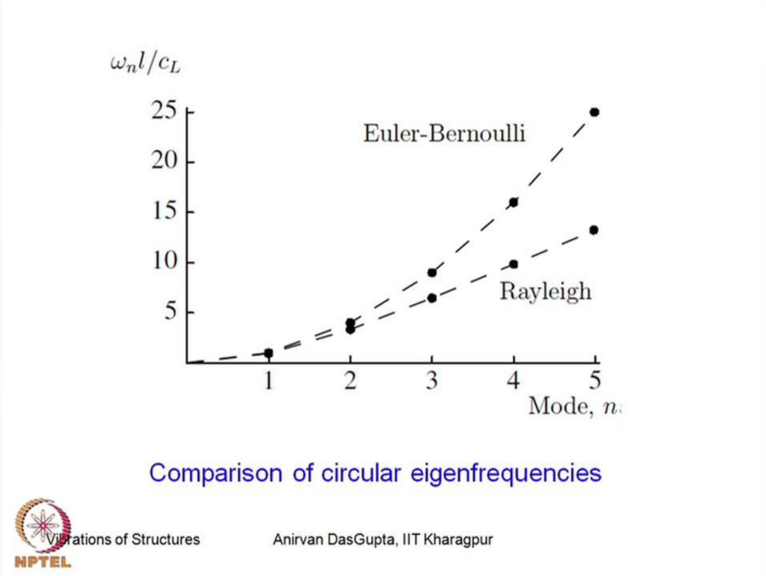

So, let us look at comparison of these circular natural frequencies of the Euler-Bernoulli and Rayleigh beam models, so, as you can see at lower modes they are very close; but as you go to higher and higher modes, the difference is appreciable. Now, let us understand why this additional term that comes in our circular natural frequency, I mean, see the source of this term is obviously the rotary inertia. So, that we can very easily see if you look at the differential equation of the Rayleigh beam, then we can very easily see that the source of this term is the rotary inertia term; now so which means the rotary inertia term is effective for higher modes. Now, to understand this, let us see what happens in the higher modes after we look at the Eigen functions. Now, if you look at the Eigen functions if you do the analysis for Euler-Bernoulli or Rayleigh beams, then... So, the Eigen functions are obtained; so, we have these expressions of beta n; we will put in here; and solve for B_2, B_4; then you will find that B_2 must vanish. So, only B_4 can exist.

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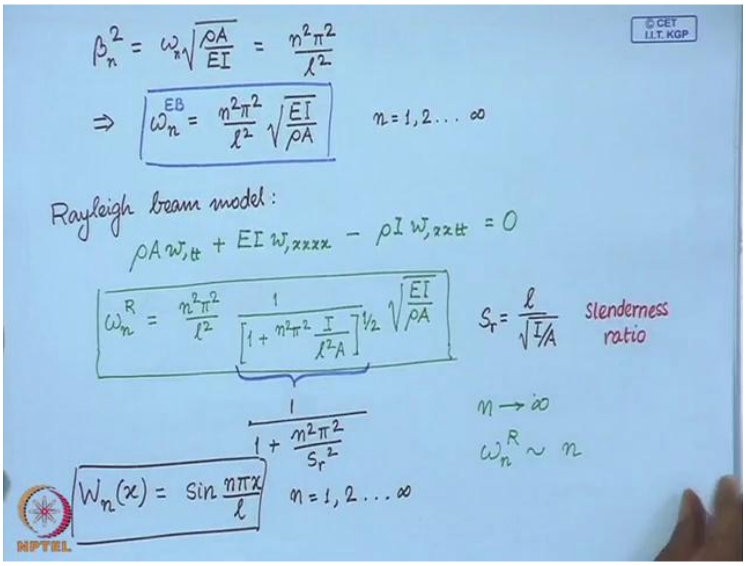

So, therefore, the Eigen functions will all be the sine functions. So, these are the Eigen functions and these Eigen functions are same for the string. So, let us look at these Eigen functions. So, we have simply supported Euler-Bernoulli or Rayleigh beam. So, this… The second… So, this is the Eigen function of the third mode. The mode shapes are defined by these Eigen functions. So, you have two nodes here, here one node and no node here in the first mode. Now, this is same as that of a string. Now, let us see what happens as we go for the higher modes. If you calculate the curvature, then the curvature is increasing; as you are going to higher and higher mode, the curvature is increasing; and this rotary inertia term; so more is the curvature more is the rotation of this element. So, as you go to higher and higher modes, this term brings in the effect, the increased effect of the higher curvature and of course of the higher frequency. So, as you go to the higher modes, this term becomes substantial and that then starts effecting the Eigen frequencies which is not so for the lower modes. Now, you see if the n is very high, suppose n tends to infinity then you can easily see that the frequency becomes proportional to n; so if n becomes much, much larger than 1, then this becomes proportional to n; so this term becomes much, much greater than 1; so we can drop this one; so square root of this will be n; here there is an n square; so the Rayleigh beam’s circular natural frequency will become proportional to n as you go from higher and higher modes, whereas for the Euler-Bernoulli beam, it increases as n square as you can see here.

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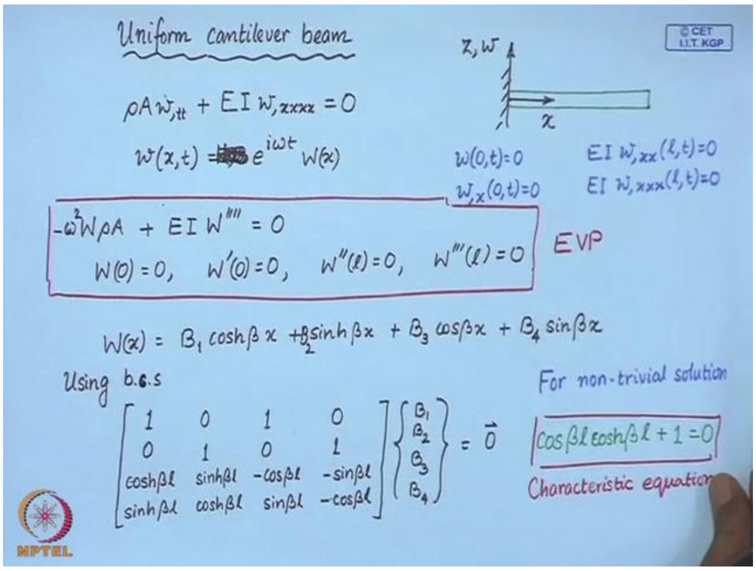

Now, let us discuss a uniform cantilever beam. This is another kind of boundary condition which is quite common. Now, in this case the boundary conditions are given by… So, on the built in end, we have displacement zero, and slope also zero; so both of them are geometric boundary conditions on this end. On this end, we have the bending moment and the shear force as zero; so, that gives us… Now, the equation of motion remains the same. So, we are considering an Euler-Bernoulli beam with these boundary conditions. Once again we search for separable solutions, and we consider… So, we have a space part and a time part. So, when we substitute; the differential equation for the Eigen value problem remains the same. Now, with this case of the… So that gives us the differential equation of the Eigen value problem along with the boundary conditions, which we have now. So, that is the Eigen value problem for the cantilever beam. So, the general solution is like this; and when we use these boundary conditions, we obtain the following four conditions. So, these conditions can be simplified and written in this form. So, for non-trivial solution of B_1, B_2, B_3 and B_4, the determinant of this matrix must vanish; and this leads us to, so if you calculate the determinant of this matrix… and this is the characteristic equation for the cantilever beam. Now, these are all transcendental equations. So, you have to solve them numerically. Now if you plot these functions after some rearrangement, if you plot these functions, then we can have a graphical approximation for the solutions as shown in this picture.

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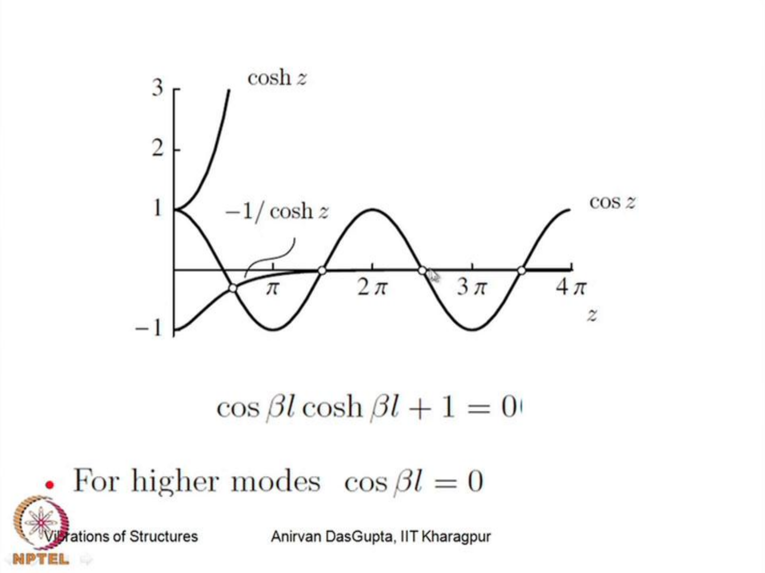

So, what we have done is we have written this cos beta l; so this is the characteristic equation; we have written this as cos beta l is equal to minus 1 over cos hyperbolic beta l; and we are plotted this cos beta l, so cos z, this is cos z; and one over minus 1 over cos hyperbolic z. So, these two functions are plotted; and the intersections of these two functions are marked by this unfilled surface. So, these are going to give us the circular natural frequencies forever cantilever beams. So, let us see what we have done. So, essentially, we have plotted these two functions and wherever they match, that gives us the solution of z, which is the solution of beta l. Now, let us look at this figure once again. Now, you can see for higher, now each of them as I have indicated that they will tell us the circular natural frequency; so for the higher modes, so higher frequencies, you see this one over cos hyperbolic z, this falls off very fast to almost zero. So,, these points can be written as zeros of this function, cos z equals to zero. So, for higher modes, you can use this characteristic equation, this simplified characteristic equation cosine beta l equals to zero, which will, so this is… Therefore, I can write this beta, so beta l is approximately this. Now, what I will do is in order to get this, I will add… Here, beta get indexed with n; so here this is some error; so this beta into l will deviate from the zeros of cos beta l equals to zero by this numbers e n. Now, when I have added this e n , this small error, then I can claim that this is the solution of this; and for e n, let us say e1; if you calculate the first solution of this characteristic equation, then it will deviate from this value by this amount. If you calculate for the second mode beta 2 l, then it deviates from this value; if you calculate for the third mode then it deviates from this value by this amount. So, as you can see this is rapidly falling; this e n, the correction is rapidly diminishing; so for higher modes, it is sufficient to use 2 n minus 1 over two times phi as beta n into l. Now, if you, so then beta n square is therefore… and as you know this beta, we have looked at this; so beta n square is nothing but rho A over E I times omega, square root of that times omega n; so, this implies… So, this is the expression of the circular natural frequency of a cantilever beam.

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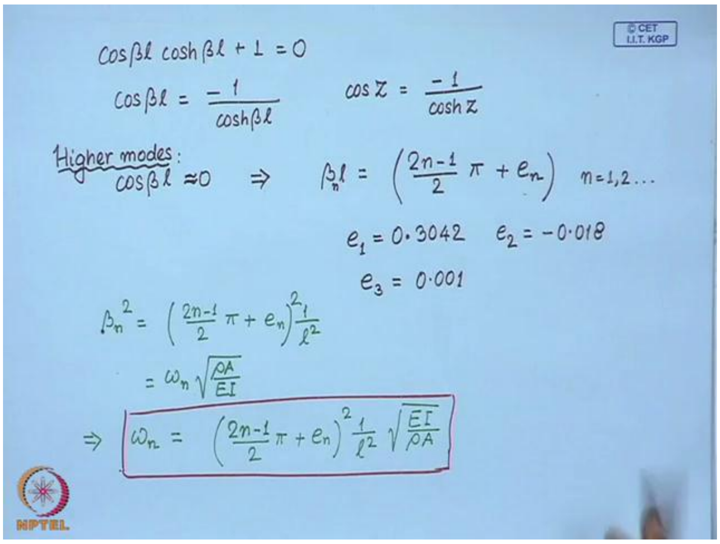

Now, corresponding to these circular natural frequencies, one can calculate the Eigen functions from the general solution that we discussed. So, these are obtained as…

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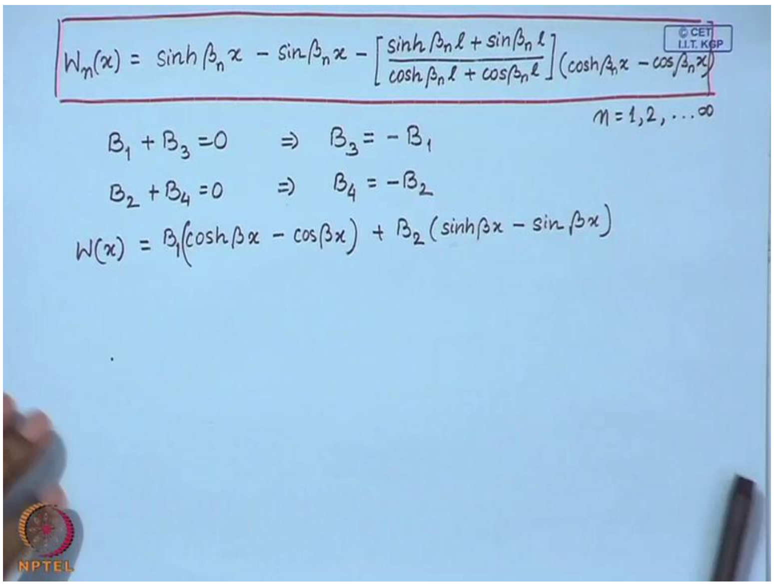

Now, let me just indicate how this has been obtained. So, this was what we had. Now, the first condition is, the first condition that we have from here is that… So, that implies B_3 is minus B_1… Similarly, from the second condition… Now, we have used this, the third condition… So, therefore, so with these two conditions, we can write… So, that is proportional to this; and we have used the boundary conditions and from there as you can see we have solved this; we have obtained this Eigen function. So, this term appears here. So, we have consider beta 2 as 1, and at x is equal to l we have used the boundary condition to obtain this factor B_1. So, this is the Eigen, these are the Eigen functions; again these are indexed.

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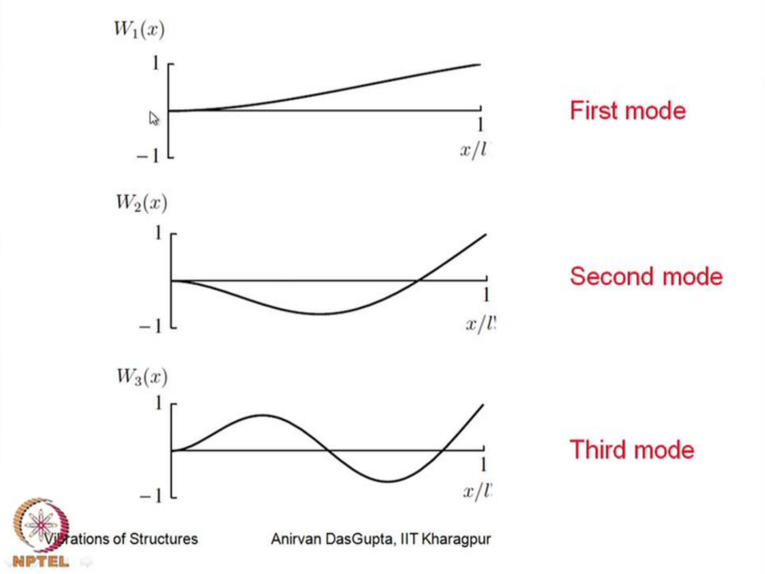

So, let us once look at these Eigen functions. So, this is the first mode, the first mode characterized by this Eigen function. This is the second mode; this is the third mode. So, you can see in the third mode there are two nodes; in the second mode there is only one node and no nodes in the first mode; and the slope here is zero, the displacement and slope as we expect.

So, we have discussed the Eigen, the modal analysis of beams which is essentially again solving an Eigen value problem. So, we have looked at these two examples of simply supported Euler-Bernoulli beam and we have also look at the simply supported Rayleigh beam and we have understood the effect, role of rotary inertia on the, effect of the rotary inertia on the circular natural frequencies of the Rayleigh beam. Finally, we have looked at this Euler-Bernoulli cantilever beam. So, with that I conclude this lecture.

Keywords: modal analysis, simply supported beam, cantilever beam, Eigen functions.