Vibrations of Structures

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Lecture No. # 24

Beam Model - II

Today, we are going to continue our discussions on beam models that we had started in the last class. So, in the last lecture, we had looked at two beam models, under the assumption that the shear strain in the beam is negligible, or in other words that the beam is infinitely stiff in shear. Now, this assumption is very good, if the beam is slender which means that its transverse dimensions are much, much smaller than the length of the beam. So, under such assumptions, such an assumption, this shear may be neglected. So the Euler-Bernoulli hypothesis, which says that the plane sections perpendicular to the neutral fibers before deformation also remain plane and perpendicular to the neutral fibers after deformation. However, when the beam becomes thick or it its slenderness ratio, it goes on reducing; so, it becomes more and more thick in the transverse direction than in the length direction, in that case shear becomes important.

So, today we are going to look at a model which incorporates this shear in a very interesting manner. So, in the previous lecture what we saw that the shear force was introduced in the equilibrium equations. It was not quantity derived from the deformation of the beam, in terms of the material properties. But today we are going to look at this more advanced beam model which is, which goes by the name of Timoshenko beam model. So, we are going to look in to this modeling of the Timoshenko beam. So, the assumptions that we made in our previous lectures that the slope is small, the material is linearly elastic, homogeneous and isotropic; all those assumptions still hold, except that now we are going to introduce shear in our model. So, let us look at this Timoshenko beam model. So here, I have introduced what is the basic difference between the model we have previously discussed and what we are going to discuss today. So here, you can see an undeformed element of a beam which undergoes flexure or pure bending. So, this is what we have discussed in our last lecture. So, this is the neutral fiber or the neutral axis; and the cross section maintains this orthogonality with the neutral fiber. This is pure bending. Now, in this case, the slope is nothing but this angle psi that I have indicated.

So, when a beam undergoes pure bending, the slope of the neutral fiber; remember that we are always tracking the slope or the deflection of the neutral fiber. So, this slope is this angle.

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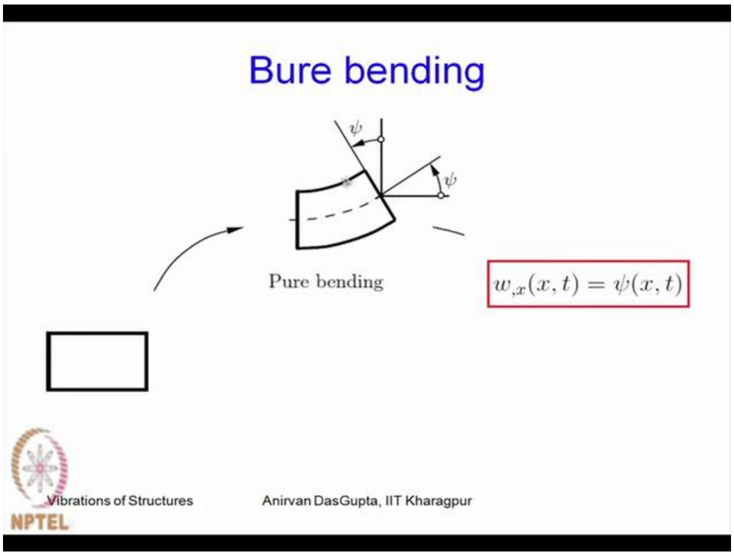

Now, let us look at the case of this simple shear, when this element undergoes simple shear. So here, this simple shear is a volume preserving deformation. So, this angle theta is the shear strain; and as you can see here that the slope of the neutral fibre or the neutral axis is nothing but the shear strain.

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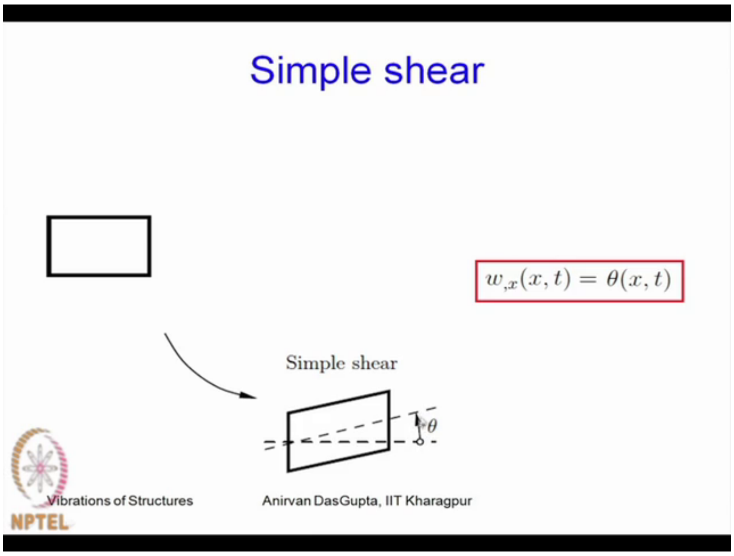

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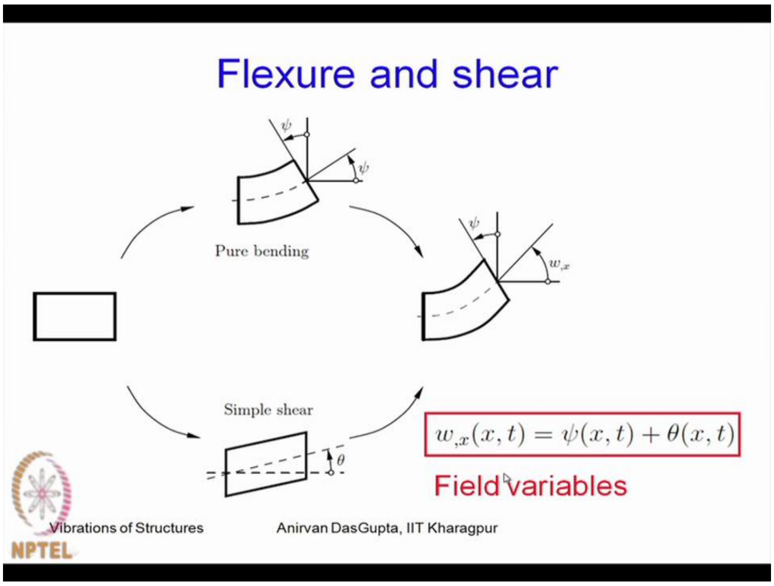

So therefore, when we put these two things together, which means the beam undergoes both flexure and shear; so it is under both bending and shear. So, the total picture looks like this. So, this is the pure bending picture; this is the simple shear picture and the net result is shown here. Now, you can see that this shear, because of the shear the angel does not change as you can see here. So, this was vertical; so, this remains vertical. So, which means, but this angle will not change, the angle measured from the vertical; this angle remains psi whereas, the slope of the neutral axis changes; and this is what we have. So, the net slope of the neutral axis, del w/del x is the angle, because of pure bending and the shear strength, so thus the total angle.

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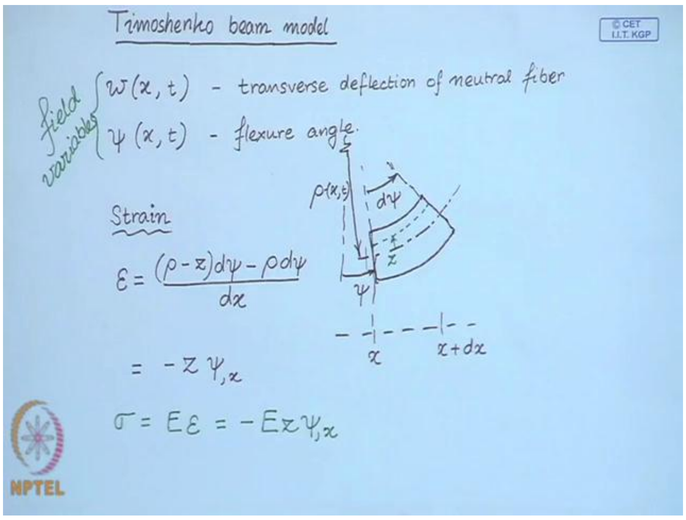

So, we are representing the slope of the neutral fiber in terms of this flexure angle and the shear strain. So, we can consider our field variables when we denote or represent the deflection of the beam, we can use any two of them. So, what we will do is, we will choose w and psi as our field variables.

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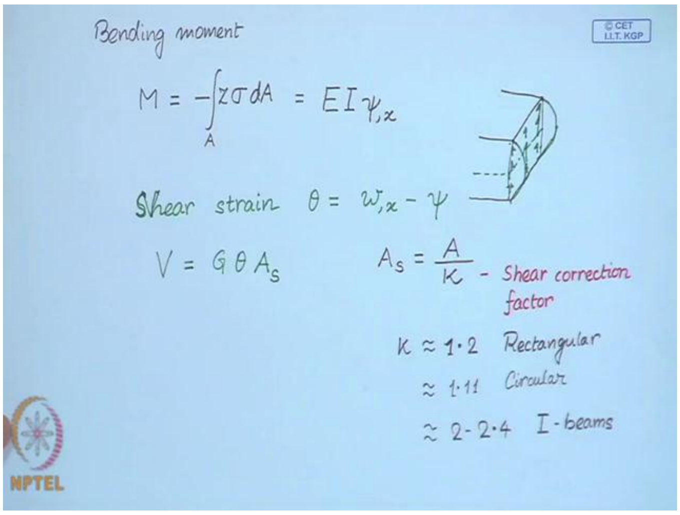

So, this is the transverse deflection; this is the flexure angle. So, these are the two field variables that we will use. Now, let us then start representing our deformation in terms of these field variables. So, the first thing is strain; so the fiber strain in the longitudinal direction of the beam… So, if you once again consider a fiber at a height z from the neutral plane or the neutral axis, then following our discussion from the previous lecture, we can write the length of this fiber; so if rho is the radius of curvature, then rho minus z, which is this radius, times this angle is d psi, so this initial angle; so we are considering an element between x and x plus dx. So, the strain can be written as the current length of this fiber minus its initial length divided by, so that is the length of the element. So, this can be written as minus z del psi/ del x. So this is in terms of the angle by which the section has been rotated. So that is the strain in this fiber. So, if z is positive and del psi/ del x is positive, then you can see that this strain is positive. Then we can write, stress using Hooke’s law; so that becomes, so here with this expression of stress, we follow the steps that we have carried out in our last lecture; we determined the bending moment negative of sigma times dA, this is the force; so arm cross, so z cross the force; that integrated over the area; and then if you substitute this expression of sigma then we can easily see as we have done before, the bending moment turns out to be EI del psi./ del x.

Now, the next thing is to determine the shear force. Now, in the models that we have looked at the previous lectures, the shear force was introduced at the equations of equilibrium considering that the beam element is infinitely stiff in shear. But today as we have introduced the shear strain, we know that the shear strain theta is nothing but del w/ del x minus psi; so we can represent the shear strain in terms of our field variables w and psi. Then, I can write the shear stress acting on, so the shear stress is G times the shear strain, where this is the rigidity modulus, times the area of the cross section; that is going to give us the total shear stress. As you know that in a beam, what we learnt in mechanics that the shear stress is not uniformly distributed over the cross section. So just multiplying it with the total area of cross section is actually going to over-estimate the shear stress. So, the shear stress will be actually lower than just the product of, the shear force will be less than just the product of the shear stress and the area of cross section. So, what we do to remedy this problem and to keep the formulation very simple, we introduce a shear correction factor. We calculate a corrected area where A_s is the actual area of cross section of the beam divided by a factor kappa, where kappa is known as the shear correction factor. So as you know that say for example, for the case of a rectangular cross section, the shear stress distribution is parabolic; so it is parabolic; it is maximum at the neutral fiber and reduces and goes to zero at the top and bottom fibers, because there is no shear stress at the top and bottom surfaces. So the distribution of shear stress on a rectangular cross section is parabolic. Now, to take care of this pack, we introduce the shear correction factor; and calculate the reduced area; and multiply this area with the shear stress to determine the shear force. So, we want to keep the expression of this shear force simple; so for that reason we have introduced this shear correction factor. Now this shear correction factor typically take up values of 1.2 for rectangular cross section; it is approximately 1.11 for circular cross section; and it is about 2 to 2.4 for beams with I cross section, for I beams. These are some typical values that are used in various situations. Now here, we now have a very simple expression of the shear force acting on the cross section; therefore we can write this as… So, that is the expression of our shear force. So, we have these forces and moments.

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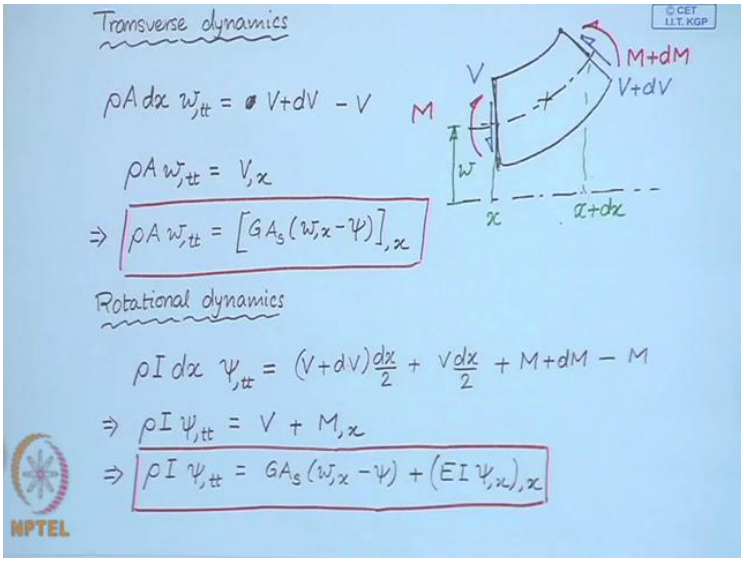

Let us look at the free body diagram of the beam element; so this is an exaggerated view. So, this is the transverse deflection at x at time t. Here, we also have this shear force and the bending moment. So, if you write down the transverse dynamics for this element, so if rho is the density of the material, A is the area of cross section; then rho times A is the mass per unit length; times dx which is the length of this element is the mass of this little element multiplied by the acceleration that must be equal to; here again considering the slopes are small, we can write this as this, so the forces in the transverse direction, V plus dV cosine of this angle, which is approximately equal to one minus V times cosine of this angle, which is also very small; so this is what we have; so dividing throughout by dx… Now, we know that this expression of V; so we have this expression of V; so introducing this expression here… So, this is our equation of transverse dynamics for this element. Now, the next thing is the rotational dynamics. So, rho times the second moment of area about the neutral axis; so this we have discussed previously as well. So, this is the moment of inertia about the neutral axis per unit length; so this times the length of the neutral axis would give us the moment of inertia about the neutral axis, which is perpendicular to the plane of the paper; so this times the angular acceleration; now this angle is the rotational angle is psi; so this is the angular acceleration. So, this must be equal to all the moments about the, so this on the right hand side we must write all the moment about the center of mass of this element; so we have because of this shear force, and because of the bending moments. So, we obtain, dividing over dx and considering the dV/dx of smaller order, then we obtain the equation of motion. Now, once again substituting these expressions of shear force and bending moment in here, the bending moment is E times I times del psi/ del x; and this is what we obtain as the equation of motion of the rotational dynamics. So, these two equations now; you see we have two variables w and psi; so we need two equations of motion. So, these are the two equations of motion in w and psi. Now they are coupled partial differential equations. Now, here in general this area of cross section can be a function of x; similarly this second moment of area can be a function of x for a beam with varying cross section. For a beam with uniform cross section, these equations will of course simplify. So, this is the equation of motion of the transverse dynamics. So, this is for a uniform Timoshenko beam.

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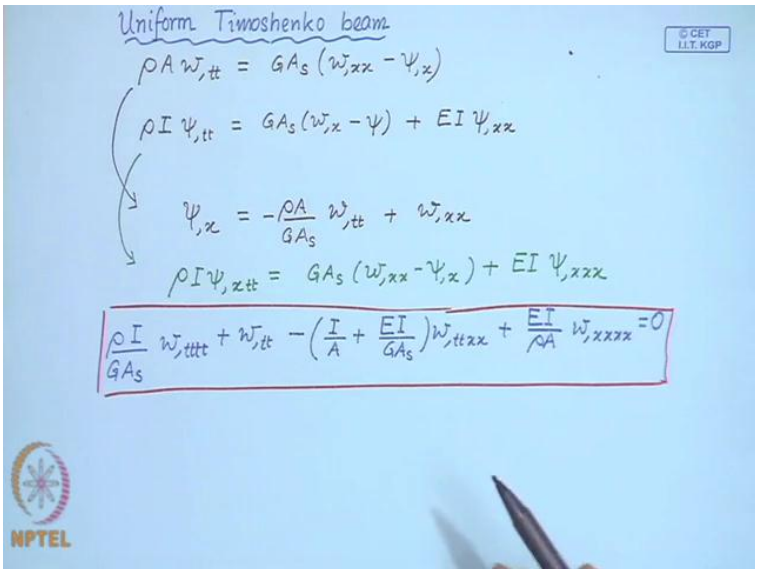

Now, here it is possible to eliminate psi between these two equations to obtain a single equation. Now to do that, so here, if you differentiate this equation with respect to x, then you have del psi/del x in this equation; and del psi/ del x can be solved from here. So, from here, I can write… and differentiating this equation with respect to x… Now, we are going to substitute, eliminate del psi/ del x in this equation. So, if you do that and simplify… so this is what you are going to get. Now this equation is in terms of the field variable w. Now, here notice that here you have fourth derivative with respect to time. This has fourth derivative with respect to space. So to solve this equation, you need to have three initial conditions and boundary conditions; so you will need four initial conditions and four boundary conditions; but that usually you do not have, I mean four initial conditions you definitely do not have. The reason is that, the reason that this equation is correct, but when you solve this, this requires a little care, because it is going to generate four constants of integration, for the time integration and four constants of integration for the special integration. Now to solve for these eight unknowns, which you actually do not have; so what you will have to do is the general solution of this has to be once again substituted in these equations to find out additional constraints. So, when you want to solve this you have to be careful; you have to do some additional steps. Usually this is used to determine what we discussed in one of our previous lectures, and what we will going to also discuss, this is use to determine the dispersion relation for Timoshenko beam. So, that is the equation of motion. Now, as we have discussed previously, we also would like to have variational approach for deriving the equation of motion of a Timoshenko beam. The reason being this keeps us, our boundary condition directly which we have not discussed as yet. So, we see look at the boundary conditions from the variational principle, and also I mean, the variational principle gives as very powerful way of approximately solving the system as we have seen before.

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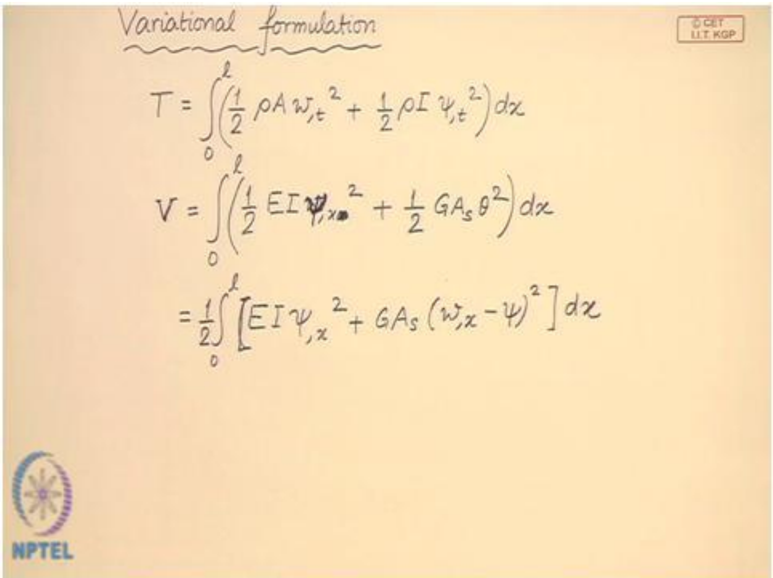

So, let us look at the variational formulation of the Timoshenko beam. So, in the variational formulation, we write down the kinetic and potential energy expressions. So, the kinetic energy is one half mass per unit length, times the velocity square. So, this is the translational kinetic energy plus one half rho times, I is the moment of inertia per unit length, times the angular velocity square; now this is per unit length; this is kinetic energy per unit length; so, multiplied by a small length of the beam, and integrating over the length of the beam will give us the total kinetic energy of the beam. Now, the potential energy.... So, here I am using the same symbol V; so, potential energy; so that is given by one half E I, the second derivative of w with respect to x square; so, this is the potential energy due to flexure. So, this we have written in terms of psi, a single derivative; so, psi single derivative and that is the expression that we have for the stress, in the stress and strain. So, the product of those two will give us the potential energy. So, this is per unit length; over the area we have already integrated; and in addition to this term we have the potential energy, because of the shear, the shearing of the element, which needs this; and these are all per unit length over the area, we have already integrated, so, this times the small length and integrated over the length of the beam. So, that is the potential energy expression. Now, this theta, so we have expressed theta in terms of our field variables w and psi. So, this is our potential energy expression in terms of w and psi; and this is kinetic energy expression again in terms of w and psi.

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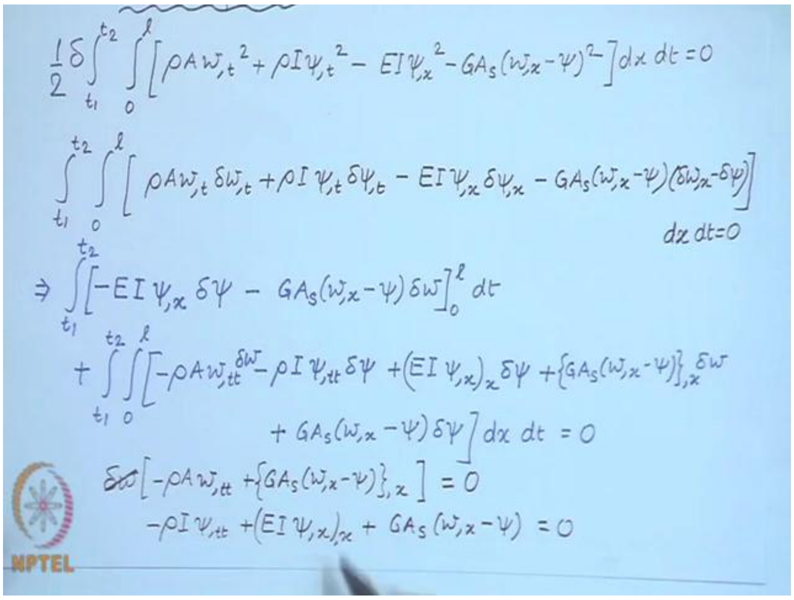

Now, we apply the Hamilton's principle which says… So, the variation of the kinetic minis the potential, this must vanish. So if take this variation… So this is the variation of the kinetic energy, translational kinetic energy. So, this must vanish. Now, we want to take out this variation delta w out, common from here. In order to do that, we have to integrate this parts with respect to time and also integrate this by parts with respect to time; and these steps we have discussed many times before. So, those terms at t_1 and t_2

must vanish. Now these terms, for example this term and this term, we have to integrate by parts with respect to space; so when we do that… So because of this term, we have this and … because of the boundary terms… So this term get differentiated with respect to x… So, this must vanish; again using the conditions of vanishing this terms on the boundary and over the domain, again we invoke the same argument to obtain the equations of motion. So, we have to collect terms of delta w. So from here, the coefficients of delta w… So, we can say that for arbitrary variation delta w if this integral has to vanish then this must be zero; similarly, the coefficient of delta psi… So, these are the two equations of motion that we have obtained before as well. Now, from here, we now look at the boundary conditions.

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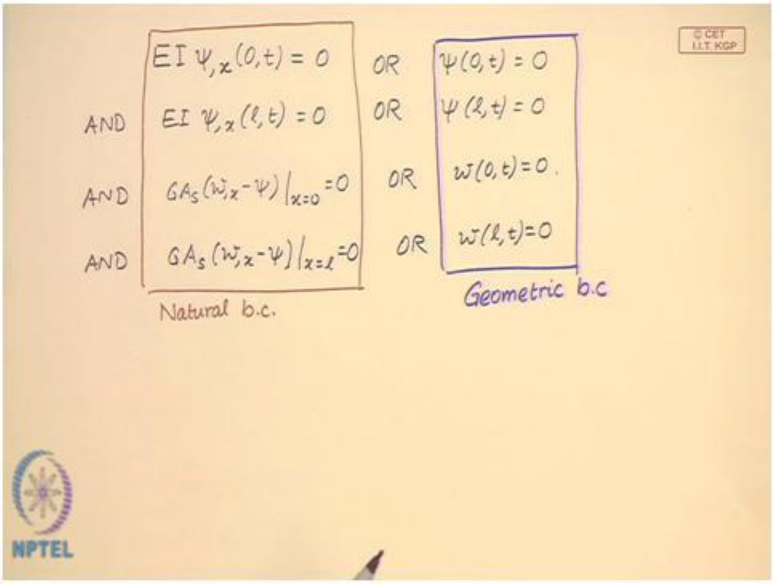

So, let us look at the boundary conditions which we have obtained from these boundary terms. Now, we have two field variables and corresponding to these two, we have the... So for example for this… so, this equal to zero or psi; so these are connected by or, these are conjugate. So, this is the angle and this is the bending moment, and at l... so, these are the possible boundary conditions obtained from here, and from here; and similarly, at x equal to l. So, these are the possible boundary conditions, that we have, that we can usually have, there are other possibilities. Now, as we have seen before as well, this set of boundary conditions, these are the geometric; while this set, these are the natural boundary conditions. So, these are conditions on the bending moment or the shear force etc.; and these are on the boundary, the angle, the rotation or the transverse deflection, and these are conjugate; these are, one is the conjugate of the other. So, this deflection is the conjugate of the shear force; similarly this angle, rotation angle is conjugate of the bending moments. So, they come in or; and... So, these are the possible boundary conditions we have. Now, this Timoshenko beam is still has, it makes an assumption that the cross section which was plane before deflection, that remains plane, though it need not remain perpendicular to the neutral fibers. So, this is a kind of limitation of Timoshenko beam, but then you can very easily relax this by introducing more complex functions for your deflection field. So, let us review and see what can be done.

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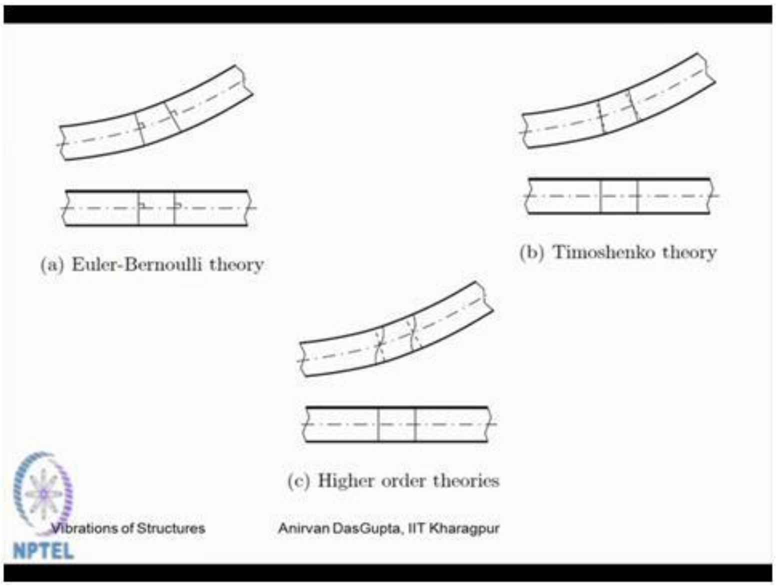

So, in this figure, I have shown this Euler Bernoulli theory. So, what it says is that plane section will remain plane, as well as perpendicular to the neutral plane, neutral axis. So, this must perpendicular here, and still remains perpendicular. In the Timoshenko theory, this was plane section still remains plane, but now it need not be perpendicular to the neutral axis. Now, you can then think of higher order theories in which plane section need not at all remain plane. So, there is warping of the cross section. So, if you want to introduce or want to have higher order theories for beam, for the deflection of the beams or vibration of beams, then you have to relax this flatness condition of the cross section. So, we introduce the warping of cross section. So, to do that here you can see the kinematics of deformation can be written out in this way.

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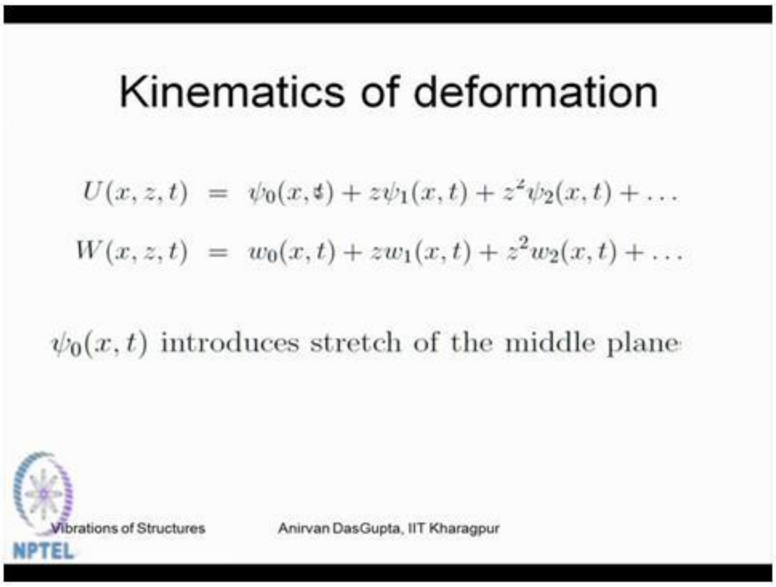

So, where capital U is the deflection in the axial direction; so it is being represented in terms of these functions psi 0, psi 1, psi 2, etc.; and expanded in terms of this variable z which is the location of this fiber from this point from the neutral axis. Similarly, this transverse deflection is also expanded as the series as powers of z, where we introduce these field variables w_0, w_1, w_2. Now, here this psi zero introduces the stretch of the middle plane.

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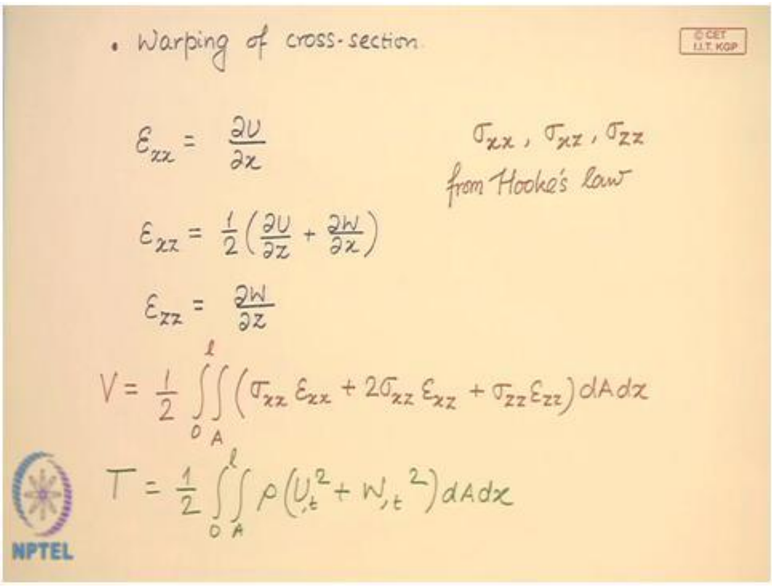

So, using this kind of this deformation field, then you can write down the strains in terms of these fields. For example, the axial strain can be written as say del u/del x; the shear strain can be written as one half del u/del z plus del w/del x; and the strain in the z can be written as del w/del z. Now, with these expressions of strain, you can write down the potential energy, for example…where this sigma xx, sigma xz and sigma zz are determined from the Hooke's law. So, once we have this, we can write down the potential energy, similarly kinetic energy can be written as one half… Now, once you have this expression then you follow the variational principle to derive the equation of motion. So, you can use this; so, by taking different kinds of expansion for the field for our field variable capital U and capital W, you can device higher order theories for beams. So, what we have looked at in today's lecture, we have looked at the Timoshenko beam model which uses or introduces shear in the beam which is, and this theory is valid for even thick beams; and we have briefly looked at how we can device higher order beam theories. So, we conclude our lecture here.

Keywords: Timoshenko beam, shear deformation, variational formulation, higher order beam models.