Vibrations of Structures
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Lecture No. # 23
Beam models - I

On vibrations of beams; now before we discuss about vibrations of beams, we will discuss the beam models, the way to model beams as an as a one-dimensional as an elastic continuum in one dimension. So, what are beams? So, you already have lot of idea about beams, because this kind of structural element is so ubiquitous, and you find it everywhere, and you have studied about beams in mechanics. So, you all have some idea about the mathematical, some mathematical aspects of beams. So, how do we define a beam? Because beam is also one-dimensional elastic continuum, and we have also discussed about strings, which is also one- dimensional elastic continuum, so how do we distinguish between a string and a beam? Now, a beam is one is a one-dimensional elastic continuum, which can resist or transmit bending moment and shear.

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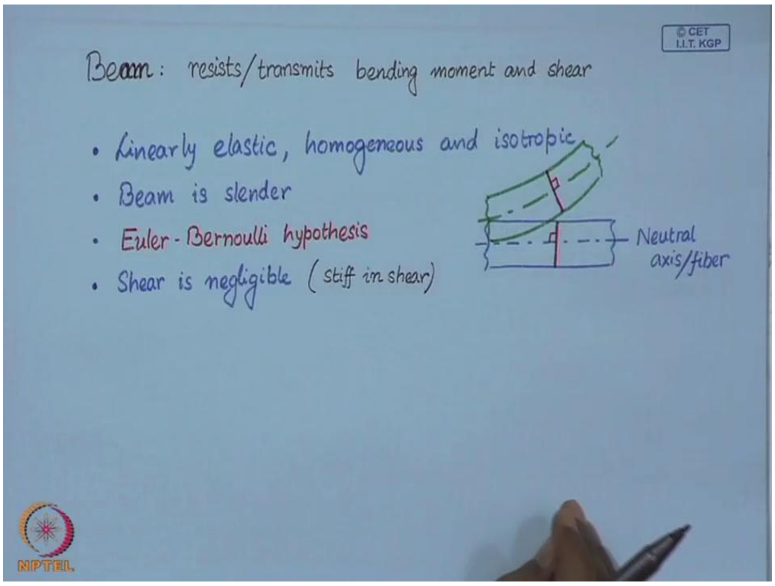

So, beam can resist or transmit bending moment and shear, and it is a one-dimensional elastic continuum. So, this is a distinctly different from the definition of a string, which cannot resist bending; and such examples we have seen in our previous lectures. Now when we mathematically model beams, we must make certain assumptions; because first of all, we are considering it to be a one-dimensional elastic continuum, and we are also saying that it can resist bending moment. So, we must, in order to take care of these various things, we have some simplifying assumptions in our model, for our mathematical model.

So, let us see what are these assumptions; the first assumption that we make is about the material, which is we will restrict ourselves to linear elastic material, which is of course and also homogenous and isotropic. The second assumption that we make is that the beam is slender. So, in today’s lecture, this will be an important assumption, under which there is this well known Euler-Bernoulli hypothesis. This Euler Bernoulli hypothesis holds. So, what this tells us is that suppose I have a beam, so this is a section of beam, and there is something called neutral axis, a neutral fiber; when this beam deflects, so it takes up shape something like this.

So, if you take a section of this beam before deflection, a plane section of this beam before deflection, which is perpendicular to this neutral fiber and neutral axis, then in the deformed configuration, this section remains plane and remains perpendicular to the neutral deformed neutral axis. So, this is Euler-Bernoulli hypothesis. Then we make the final assumption in the model that we are going to discuss today, that shear is negligible or another way of saying this is that the beam is infinitely stiff in shear. So, we will assume that the shear strain in the beam is negligible, and another way of saying this is that it is infinitely stiff in shear. So, now we with these assumptions, let us get down to modeling of the beam.

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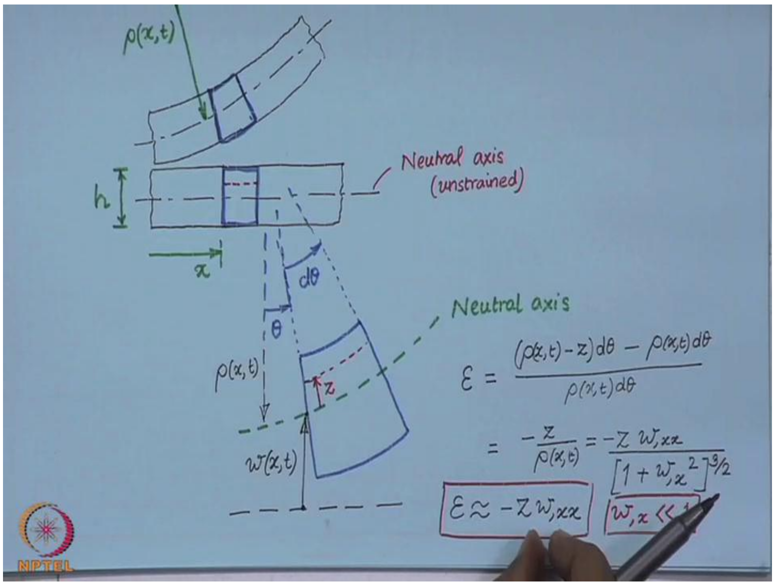

So, once again let me draw this section of the beam. So, this is our neutral axis or neutral fiber. Now this is neutral, which means it is unstrained, it remains unstrained. So, this axis in the undeformed configuration and in the deformed configuration, it remains unstrained. So, once again let us consider this element, which upon deformation… So, this is an element which undergoes deformation. Let the depth of this beam or height of the beam we denoted by h, and once this beam deforms, let us assume that the radius of curvature at this point is given by… at time t. So, the radius of curvature at x, at time t is rho of x and t. Now let us look at this element; let us draw the free body diagram of this element. So let this element be of angle d theta, and this plane, so this is an exaggerated figure; so this angle is theta and the angular length is d theta. Now, let this be the neutral axis.

Then at a distance z measured from the neutral axis or the neutral fiber, let us look at another fiber. So we can write, so first we are going to find out the strain in these fibers of the beam. So, as I mentioned that this neutral axis is unstrained. So, let us look at another fiber, which is at a height z from the neutral axis or the neutral fiber. Then I can write the strain as so the length of this fiber; so first I have to write the length of this fiber. Now this radius of curvature is rho, so length of this fiber can be written as, so deformed length of this fiber is rho minus z d theta. So, that is the length of this fiber. Its undeformed length before deformation as you can see, this is same as rho times d theta.

So, since this length remains unchanged. So, this in the undeformed configuration was of the same length as this.

So, therefore, its initial length was rho times d theta and the initial length, so therefore, this turns out to be… Now 1 over rho, 1 over the radius of curvature is known as the curvature, and the expression of the curvature in terms of the equation of this neutral fiber, so if you represent this equation of this neutral fiber in terms of the deflection of this neutral fiber from the undeformed neutral fiber, so w(x,t) is the displacement of this point from the from the undeformed configuration. So, the equation of this curve is given by w(x,t) at any time t. So, in terms of the equation of this curve, the curvature as you know can be written as the double derivative with respect to x of w divided by 1 plus del w/ del x whole square rise to power 3 over 2.

Now we have consider that, we will consider a in these derivations that the slope of the beam is small. So, under that assumption, we can neglect, we can drop higher powers of the displacement variable w or the derivatives of the displacement variable or the field variable; we can drop the higher order terms and we can write the strain in this fiber as minus of z times the second derivative with respect to x of w. So, w is our field variable. So this is under the assumption that… So, we make this assumption that the slope of the beam at any location is much smaller than 1. So, under this assumption, our strain simplifies to this expression.

Now, so this is the strain in this fiber at a height z measured from the neutral axis, and thus we have the strain in terms of the deflection of the neutral axis, which is what we are going to track, when we represent the deflection of the beam. So, the deflection of the beam will be represented in terms of the deflection of its neutral axis.

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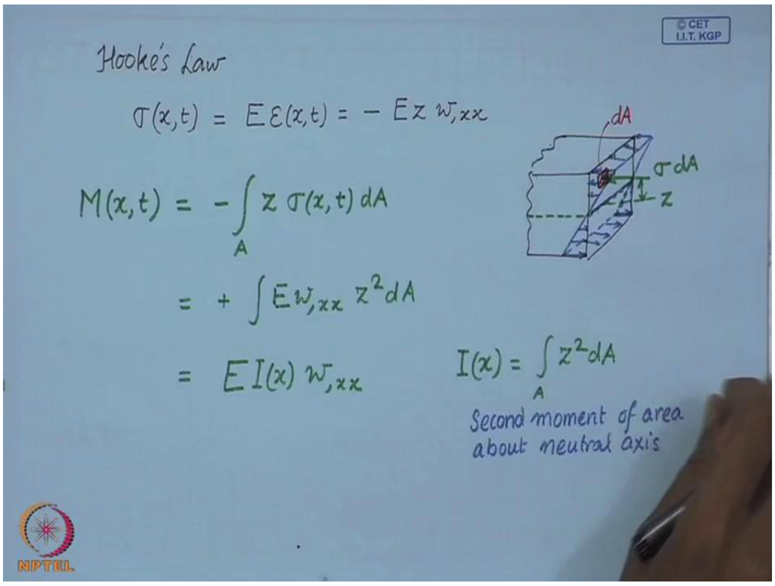

So, once we have this expression of strain, we can write, we can bring in the constitutive relation using Hooke's law; and we can write the stress in the fiber at a location x at time t… So, this is the axial stress or stress in the in the axial fibers of the beam at a location z measured from the neutral axis. Now we have… So, this stress as you can see here is linear in z. So, let us consider a cross section of this beam. So, this is the neutral fiber plane, and if you represent the stress, then… so if z is positive, we are measuring z from the neutral axis positive upwards.

So, now what we want to find out is the moment that comes, because of this stress distribution on the cross section of the beam. So, moment at any location x, at any time t will be given by… So, if dA is a little area, elemental area on this cross section, then the force acting on this area is given by sigma times dA and the moment, moment about the neutral axis. So, we are going to find out the moment about the neutral axis, because of this stress distribution is given by… So, this arm, so z cross this force and if you do that calculation that turns out to be negative of… and this has to be integrated over the total area A.

So, if you now substitute these expressions, so this negative and this negative turn out to be positive. Now Young's modulus and the curvature, the approximate curvature does not depend on this area integral. So, that can come outside and what we have here is…

where I, this is known as the second moment of the area about the neutral axis. So, this is the moment that is acting on this cross section.

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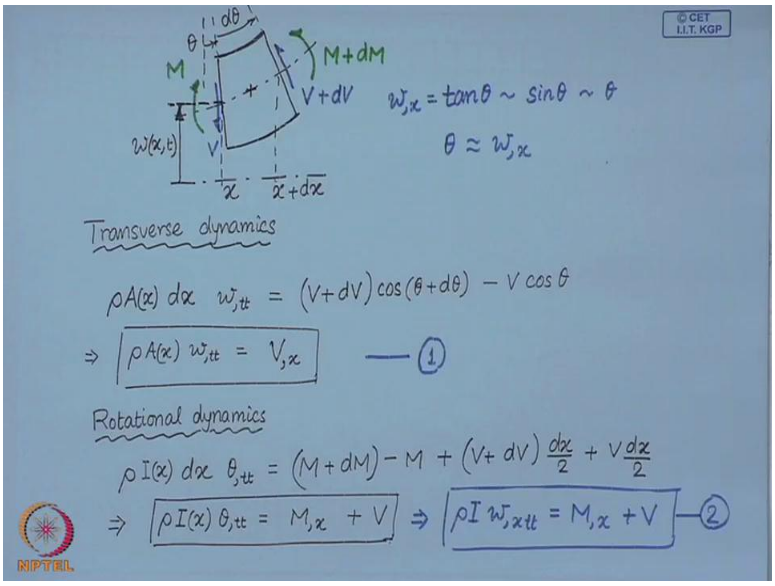

So, now let us once again look at the free body diagram of this little element of the beam and introduce the interaction forces. So, we have the shear forces at these faces and the bending moments. So the sign convention what we follow, so this is the positive shear, force and bending moment; and this, so this is the undeformed neutral axis. So, this deflection is w(x,t),. Now let me write down the transverse dynamics, the equations of transverse dynamics of this little element whose free body diagram, I have drawn.

Now, so from Newton’s second law, if rho is the density of the material, and if A is the cross sectional area at this location x, then rho times A is the mass per unit length. So, into dx, which is the length of this element, then this is the mass of this little element, times its transverse acceleration must be equal to… So, I can write… where what I have indicated before, so this angle is d theta and this angle is theta. So, I have taken projection of these forces in the transverse direction, and if you divide by dx and consider that cos theta is approximately equal to 1, then… So, this is the equation of transverse dynamics for the beam element. Now along with the transverse dynamics, now I will also write the rotational dynamics.

So if rho is the density of the material, and if I is the second moment of area about the neutral axis, then rho times I gives us the moment of inertia of this element about this axis which is perpendicular to the plane of the paper. This is per unit length, moment of inertia per unit length times dx will give us the moment of inertia of this element times the angular acceleration; so, the angular acceleration can be written as the double time derivative of this angle theta of the element and this must be equal to the sum of all moments about the center of mass of the element. So, we are writing the rotational dynamics about the center of mass. So, that turns out to be…

So, this implies upon dividing throughout by dx… Now, this theta is somehow represented in terms of w. Now, we know that this tan of theta for small slopes is almost equal to sine of theta which is almost equal to theta; and tan of theta is nothing but the slope of the beam. So, therefore theta is approximately the slope of the beam. Therefore from here, I can rewrite this equation as… Now you see in these two equations, the bending moment M, we have represented in terms of the stress and which was calculated in terms of strain, and which was calculated in terms of the deflection of the neutral axis our field variable w. Now this V, which is the shear force is determined from these equations of this equilibrium, because this element is not shearing. So, this has to be determined from the equations of equilibrium and that is what we have in these two equations. So we will eliminate this shear force between these two equations, and if we do that…

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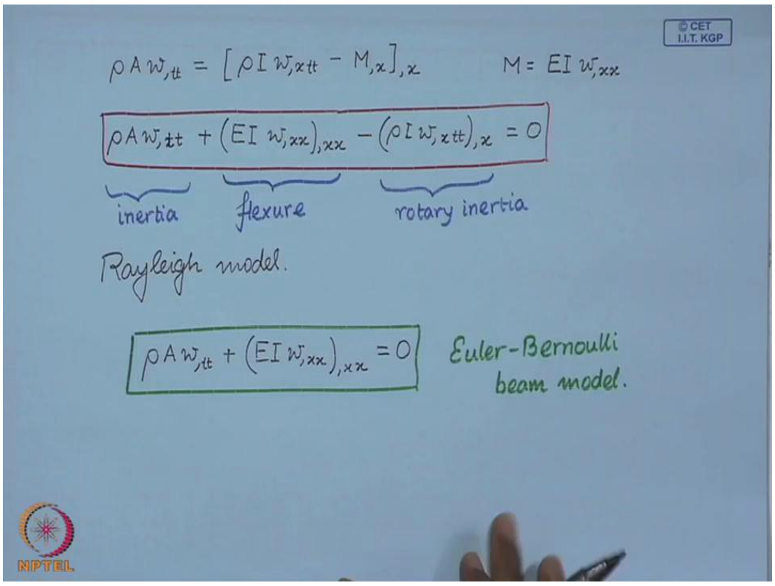

So, this was del V/del x. So, this is… and moment, the bending moment we have determined this… so therefore finally… So, if I rearrange this equation… so this is finally our equation of motion of the beam. Now here this term is known as the flexure term; this term is the rotary inertia term; and this is the normal transverse term or translation inertia term. So, this model of the beam is known as the Rayleigh model of the beam or Rayleigh beam model.

So, in the Rayleigh beam model, we have inertia term, the flexure term and the rotary inertia of the element. For very slender beams, this term can be neglected and in that case, this is known as the Euler-Bernoulli beam model. So, the Euler Bernoulli beam does not have this rotary inertia, which is present in the Rayleigh beam model. Now when you have forcing, then instead of this zero on the right hand side, you have the force distribution. So, these are two very simple models for beams. Next, we are going to discuss the variational formulation for beam dynamics.

So, as we have seen before that this variational formulation gives us a very powerful alternative way of deriving the equations, and not only that we also get, the we have which we are not talked as yet, which are the boundary condition. So, we will also get the boundary conditions, the possible boundary conditions for the problem, and this variational formulation also leads us or gives us some very powerful techniques for approximately solving, discretizing the equations of motion as we have seen in our previous lectures.

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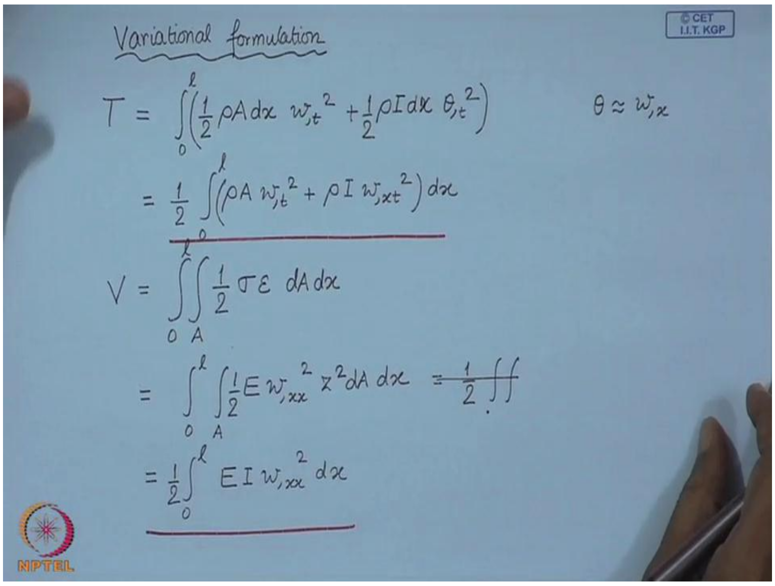

So, let us discuss the variational formulations for the beam model. Now so, the first thing that we do in this variational formulation, we write down the kinetic and potential energy expressions. So, the kinetic energy of beam can be written as the translation kinetic energy, so rho A is the mass per unit length, so into dx will give us the mass of a little element, times the velocity square. So, that is the translational kinetic energy plus, so this gives us the second moment of the moment of inertia of the little element, times the angular velocity square and half, and this when integrated over the length of the beam will give us the total kinetic energy. Now using this approximation, we can now rewrite.

So, that is the kinetic energy expression of the beam element. Now the potential energy of the beam can be, we know that the potential energy per unit volume is for a linearly elastic material is given as half the stress times the strain; so over the volume, so dA is the small area of the cross section, and dx is the small length of the element. So, if we integrate this over the area and over the length, then we should have the potential energy. Now if you insert the expressions of stress and strain in this expression, then this is what you are going to get; and therefore, there is of course this one half. So, this Young's modulus and curvature square this will have nothing to do with this area integral, so they can come out and what we have is z square dA integrated over the area, and that we already know is the second moment of the area about the neutral axis of the beam. So, these are the expressions of the kinetic and potential energies.

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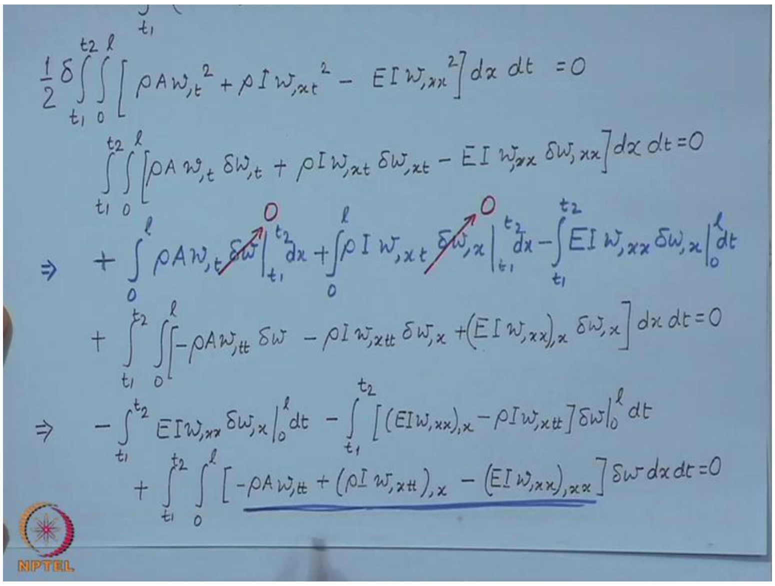

Now, when we derive the equation of motion using the Hamilton's Principle we write it like this mathematically; so this turns out to be… So, this is the statement of the Hamilton's Principle. So, when you take the variations… Now, we will integrate by parts, this term with respect to time, this term once with respect to space, and once with respect to time, and this term twice with respect to space. So, if you do that and rearrange, then what you are going to obtain… So, when we integrate by parts with respect to time, this term… so and this term once with space and once with time; so let me first integrate with respect to time and this I have to integrate with respect to space. So, this will get a derivative with respect to time with a negative sign, this will get similarly then you can check, so this, these are the boundary terms and… So, this is what we are going to get. Now, here as we know that the variation must vanish at these two time points, so these terms will vanish. So, we will be left with this as a boundary term. Now, here still we have this space derivative which we can once again integrate by parts with respect to the space and if you simplify… So, these terms are going to vanish. Now, if you integrate by parts once again and do these simplifications, then you can check… So, these are the boundary terms and… so this is what we are going to get.

Now we invoke our statement of the variational formulations, which says that the boundary variations and the variation over the domain, if this whole thing has to vanish, then these two must vanish individually. So the integrant must vanish for arbitrary variations over the domain, and that as you can see will give us the equation of motion. So, this integrant is going to give us the equation of motion, but we had been derived before; now look at the boundary terms. So, equation of motion, we have already written; let us concentrate on the boundary terms.

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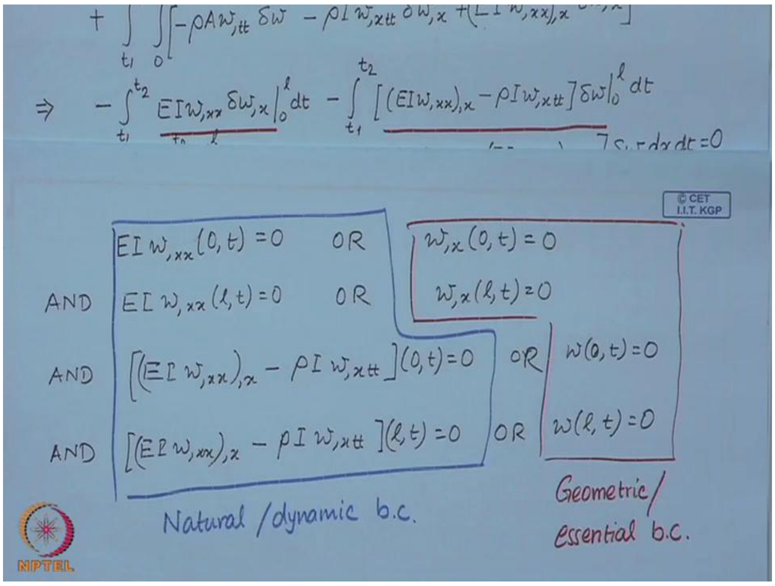

So, so here we can have… So, these are the boundary terms. So, we can have this at zero to be zero or we can have the slope at zero to be zero are fixed, then at l again or and these two are connected by an And; so, you can have this and this or this and this or this and this so combinations. So, this is from the first boundary term; from the second boundary term, which must again be connected with an And or so here we have this displacement And… so these are the possible boundary conditions.

So, this as we can recognizes the bending moment just zero or the conjugate of the bending moment is the angle with the deflection that must be zero or this is the shear force, now this is an additional term, because of rotary inertia. So, this must be zero or the displacement, which is the conjugate of this, by conjugate I mean, the product, which give us energy or work. So, either force or its conjugate this is displacement is zero, either moment or its conjugate the angular displacement must be zero; now this set of boundary conditions are the geometric or also known as the essential boundary conditions, and this set is known as the dynamic boundary conditions.

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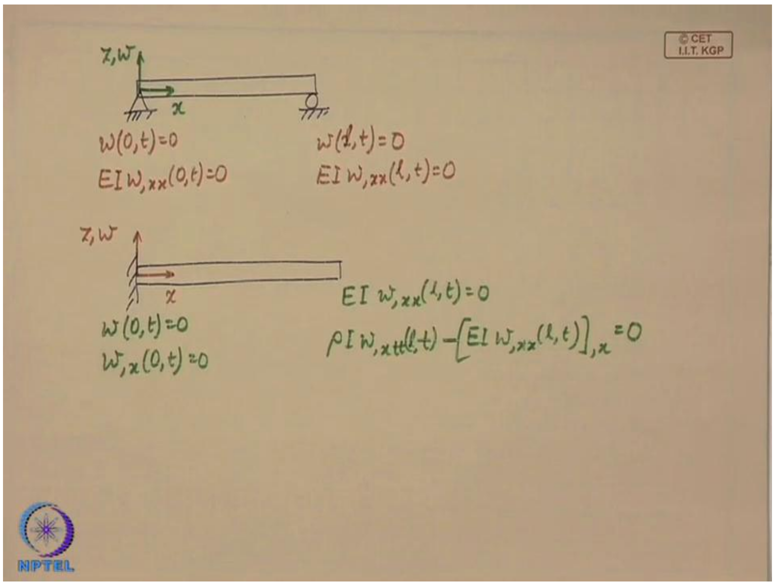

Now, let us quickly look at an example. So, we know that we have simply supported beam like this. So, the boundary conditions here, and the bending moment… Similarly here, you again have the deflection to be zero. For cantilever beam, the boundary conditions here, deflection zero and you know that slope is also zero. Here we have the shear force, and the bending moment to be zero. So, these are two examples that we have considered, where we have written out the boundary conditions. There can be other examples and we will discuss these in the subsequent lectures.

So, what we have discussed today; we have looked at some models of beams transverse dynamic of slender beams; and we have derived the equation of motion for the Rayleigh beam and Euler-Bernoulli beam; and we are looked at the variational formulation from where we can also derive the equations of motion. So, we will continue this discussion in the next lecture.

Keywords: beam models, Euler-Bernoulli beam, Rayleigh beam, variational formulation.