# Vibrations of Structures

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## Applications of wave Solutions - II

So, we have been discussing about the wave propagation solution in one-dimensional continuous media governed by the wave equation. So, our main motivation for studying this kind of problems is that we would like to understand transient motion or when one-dimensional structure interacts with boundary or an inclusion etc. or it goes and hits something. So, we have looked at examples, such examples in our previous lecture.

Now, today what we are going to study is motivated by, for example, a bouncing ball. So, when we drop a ball, we observe that it hits the floor, which is considered to be much more rigid than the ball itself, and it bounces back. So, I mean, in the in the real world of course there is the process is quite complex, because of internal dissipation, acoustic emission etc. But let us bring out the cracks of this problem. So, we would like to understand what happens, when one-dimensional continuous system, let us consider a bar, for example. So when a bar goes and hits another obstacle, which has a certain impedance, it could be an infinite impedance, if it is the wall or it could be a finite impedance.

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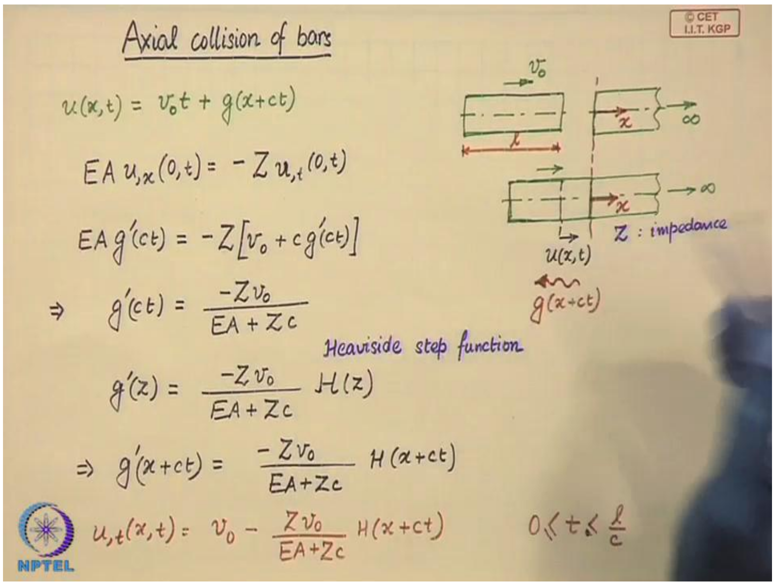


So, what we are going to study today is the axial… So, let me draw this figure. So this is a bar, this is a finite bar, which is going to go and hit a semi-infinite bar. So, in order to simplify the whole problem, we come to this very basic problem, in which two bars are colliding. So this is moving with an initial speed; let us say V towards this stationary semi-infinite bar and what happens after this that we are going to look at. So, we are going to fix our coordinates, coordinates system at this initial location. So, we consider this uniform finite bar moving with the velocity V and hitting this stationary semi-infinite bar, and we assume that all the motion is in the actual direction.

So, at any location x, the field variable for this finite bar is represented; so, the axial motion is represented by this field variable u(x,t). Now the motion of the bar of this finite bar just before collision… So, if I call this initial speed as $V_0$. Now as soon as they collide, what happens is this phase of the finite bar feels this semi- infinite bar or this obstacle, and this information starts propagating in this bar. So, there is propagating wave in the negative x direction, which carries the information, which tells this finite bar that this face has hit an obstacle.

So, this wave which carries the information will represent this with by g(x+ct). So this is the gross motion of the bar and above over and above this, that is this wave, which is propagating in the negative x direction as a result of the collision. Now, this wave goes, and so let us consider this, how this wave is being generated. So, we will assume that, we will consider that this semi-infinite bar has an impedance Z, this Z. We will consider special cases special values of Z later on.

So, the boundary condition that sets this wave into motion is given by… So, this is the force condition at this end. So, if the impedance is Z, then Z times the velocity of this phase of the bar must be the force and that force is being felt by this finite bar as negative of Z times the velocity. So, if you substitute this expression of U in here… so this is what we are going to get and from here we can solve.

So, g prime c t is given by this expression. Now to take care of the causality, we can write g prime z, where this is the Heaviside step function. So, this is to take care of the causality of this process. So, therefore… So, this is the derivative of the wave that propagates in the negative x direction. So, one can write down, actually the velocity of the bar at any point… so that is the velocity wave that propagates in the bar, in the negative x direction. So, now this solution is valid till the time l over c, so l is the length of this bar; so l over c is the time taken by this negative traveling wave to reach this free end of the finite bar. So, let us see what happens at the free end. So, the free end of course, there is a force free condition. So this is our finite bar. There was a wave propagating in the negative x direction, and once this wave reaches the free end of the bar is going to reflect back in the bar. So, let us represent the reflected wave from the free end as f(x-ct). So, therefore we can now represent the wave field in the finite bar by this expression. Now at the left end of the bar, the boundary condition, so at the left end the boundary condition… Now see the co ordinate system has been put here. So this is the force free condition at x equal to minus of l. So, this is our condition and once we substitute this expression in here, and simplify, we are going to obtain this expression in terms of g prime and f prime. Now g prime is already known to us, and if you substitute that expression of g prime that we just now derived and we can write this f prime z. So, let me make the substitution. So, let me define this small z as minus l minus c t, in that case the argument of g which is minus l plus c t becomes minus z minus 2l. Now, since we want f prime x minus c t, and we already have this expression of g prime… So this is the expression of f prime which is the positive travelling wave after this wave reflects from the force free end of the bar. Therefore the velocity can now be written as… and hence, once we substitute these expressions… So that is the expression for velocity after the first reflection from the left boundary of the finite bar. Therefore now let us look at what happens at 2l over c; which means this wave g reflects here and f is produced; now this f has reached this end of the bar. So what we are interested in looking at the velocity at time 2l over c. So, in this expression, we have the velocity here. So, in this expression, we put this condition of 2l over c and we can calculate and that turns out to be… Now, EA or E can be written as… we know that E over rho is c square; so if you substitute this here and make some simplifications… So, this is the velocity field in the bar. Now let us analyze this expression. You see when Z is infinite, which means the semi-infinite bar is actually a rigid wall; so, Z is infinite, so the impedance of this obstacle is infinite, which means it is like wall, rigid wall; in that case, the velocity field… so, if you take this limit, so which means that at all points of the bar the velocity is $V_0$. In other words the bar rebounds back, so it was coming with an initial velocity $V_0$ and it hits the wall, and after time 2l over c, it rebounds back with the same velocity with which it came. So, this is the case, when this is the rigid wall.

Now, we can consider the case where Z is finite now this there can again be three cases; the first case could be rho A c is greater than the impedance of the bar. So, which means the impedance of impedance of the bar is greater than the impedance of this wall or the obstacle with which it is colliding. In that case, you see this this ratio is positive, which means that now, the bouncing is such that the velocity is still in the in the direction in which it was initially moving, because in that case $u_{,t}$ is going to be positive; which means after collision, the bar is still going to move to the right. In the case where the impedance of the wall and the bar they match, in that case this bar, the finite bar comes to a complete halt, complete rest. And the other case when a the impedance of this bar is less than the impedance of the wall, then this ratio is negative, which tells us that there will be a bounce back of this of this finite bar.

So, now we have, most of the times if the conditions are suitable, if the damping etc. are minimal, then we will observe this phenomena where the wall is much more rigid than the bar itself of the object that is incident. So, in that case, it is going to rebounds back. When in other cases where the impedance is less than the impedance of the wall, which is more likely to happen actually, then it is going to bounce back, but not at the incident velocity, it is going to going to lose that velocity, so where is it going to lose the energy? So, it is going to lose the energy in this wall or the semi-infinite bar as we have considered. So because this impedance is now finite, some part of this energy of this finite bar, which is incident here, is going to go into this wall or the obstacle.

So, some of part of the energy is going to be radiated in to this obstacle. In this case, where there is complete impedance matching, in that case the bar is going to come to a complete rest, and in this case it is going to go into the obstacle. So, even after the collision, it is going to enter the obstacle, because the impedance is much less comparing to the impedance of the bar. So, these are some of the interesting observations that we can have. So, here we also observe that collision, even though its seems to instantaneous, it is not; as we can understand that this takes place after the time interval of 2l over c, where c is the speed of axial waves in this finite bar. So, the wave, the disturbance wave must go travel this length of the bar twice before the rebound can take place, if at all there is a rebound. Now, so this example can be used to understand the phenomenon of collision of objects. So, here again using the wave solution, we have looked at this transient phenomenon.

Now let us next look at what happens in a travelling string or vibrations in a translating string. So, we have discussed this travelling string problem before. This travelling string occurs in a loom or a rolling mill etc. So, we can approximately model a travelling thread line as a travelling string. Now, we are going to look at the wave propagation through a travelling string, how the solutions can be represented using these travelling wave solutions and finally the scattering of the waves in a travelling string. So, first let us consider this string which is travelling at a speed v. So, we have derived the equation of motion for a travelling string. Now, here I have shown boundaries, but since we are going to consider an infinite traveling string, we are going to look at waves propagation in traveling strings and understand the solution in terms of these propagating waves, so, which means that, even if I have a disturbance in a in a finite problem string, we are going to look a transient solution; that means, when this disturbance has not reach the boundary, so initially we are going to look at solutions, which do not interact with any boundary; later on we will see what happens when there is a boundary and there is an interaction.

So, to begin with we do not have boundary condition. So, if you consider infinite traveling string, then we do not have boundary condition. Now let us look at this little transformation. Let me define zi, a new coordinate zi as equal to t, another coordinate eta as x minus beta time t, where beta is a constant. Now let us recall this equation in these co ordinates zi eta. So, here the field variable was w(x,t); our new field variable is w tilde of zi, eta; and to determine the equations governing this new field variable, we must replace our derivative operators del del t as del del zi, del zi del t is 1. So, that is del del t; similarly, del del x is nothing but del del eta. Now when we replace these derivatives here, then and we make simplifications, this is what we obtain; now if you make a choice, if beta is taken as v, then the equation reduces to the wave equation, but now, in these coordinates zi eta. Now we already know that the solution of the wave equation can be written as… and if you go back to the original coordinates, so this implies that in the original coordinates… So this is the wave propagation solution or the d'Alembert's solution for the traveling string. So, now, you can see that there is a positive traveling wave, whose speed is c plus v. So there is a positive traveling wave with speed c plus v and there is a negative traveling wave with speed c minus v in the string.

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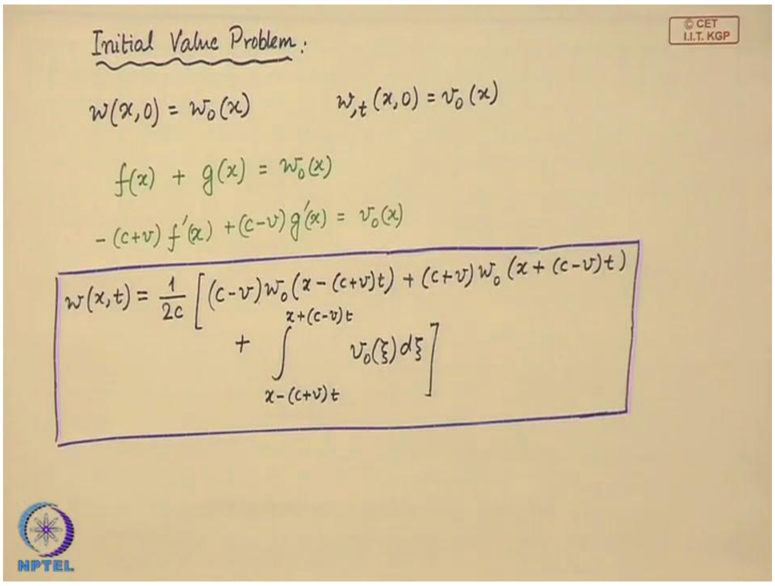


Now, then let us look at the initial value problem. We have looked at this, before we are look at this problem for the static string. Now we can once again look at the solution of the initial value problem for the traveling string. So, we normally have displacement, initial displacement distribution on the string and velocity distribution. Now from using the solution, which we have written out here; so when I use this solution here… and similarly from the velocity condition… So, we have to solve these equations for f and g in terms of $w_0$ and $v_0$. So, this we have done before; we have to integrate this second equation with respect to space, and then solve for f and g. If you do that and make the simplifications as we have discussed in one of our previous lectures, the final solution can be easily obtained. So, this is our solution for the initial value problem. So when we have initial displacements, so these are the contributions in the solution and initial velocity contributes in this way.

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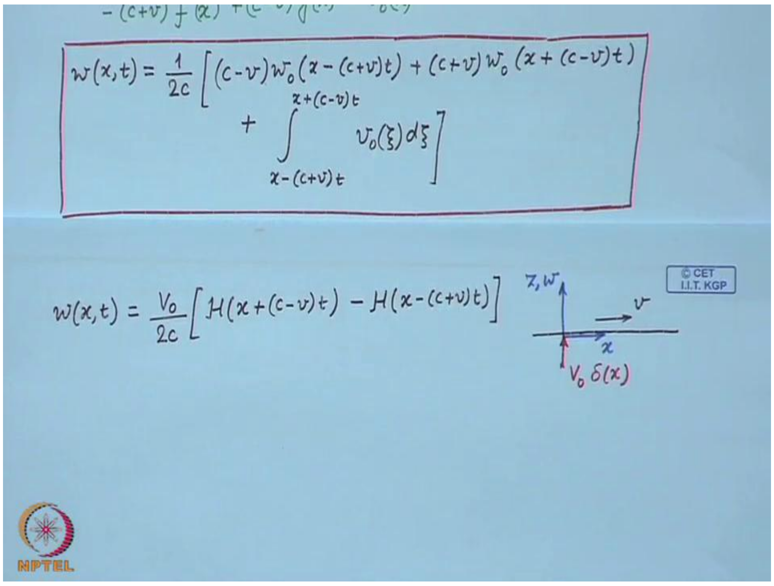


Suppose we… let us taken an example. Consider a traveling string, which is given an impulse; so given an impulse at x equal to zero. Now we have discussed this before. If you give an impulse, it sets a velocity conditions on the string. So, it sets a velocity conditions on the string at the point, at which this impulse is given. So, let us consider the velocity distribution as $V_0$ delta x, where delta is the Dirac delta function. So, if you use this in the solution; so here is our solution; so, what we obtain… so initial displacement is zero, so $W_0$ is zero; we have this initial velocity distribution, which is the Dirac delta function.

So, this is obtained as… So, integral of the direct delta function is the Heaviside step function. Now this has to be calculated at these two limits. So, this is our solution. Now let us look at this solution, what happens for… so for different… So for different conditions, so velocity conditions v; let us look at the evolution of the solution as time progresses, but you can see that so this is the difference of two unit step functions and as time progresses, there is a gap at which, the step stepping takes place.

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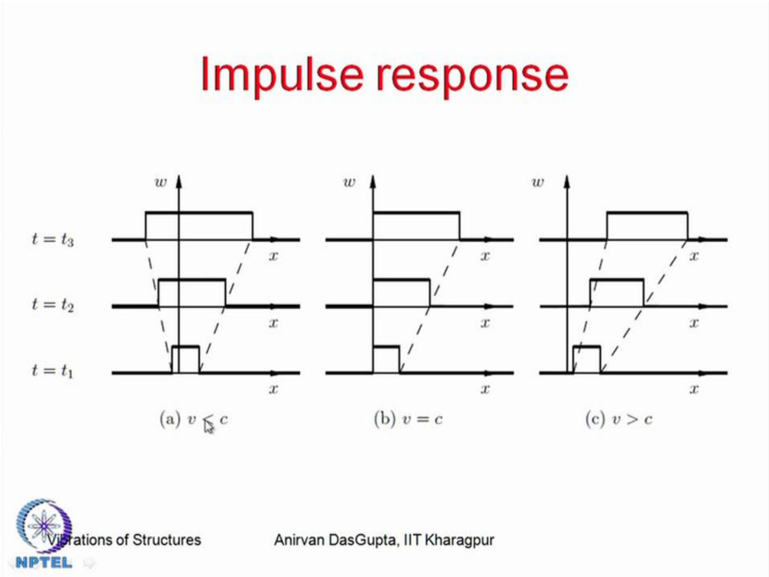


So, let us look at this figure, where I am considered three regimes. Here the velocity of translation is less than the speed of propagation of transverse waves in the string. So, in that case, as the time progresses, the wave fronts, they are propagating in the two directions, though in the positive direction, which means the direction of string travel, the speed of propagation of the front is higher than in the backward case as we expect. Similarly when v is equal to c the backward front does not propagate at all, because its velocity is now zero; while this forward wave now propagates at the two times the speed of transverse waves in the string. Now v is greater than c, then whole thing is convected, the whole disturbance is convicted to the positive side. So, this front is also convected to the in the positive x axis direction.

So, in the case of this infinite string, if you take the string at a certain point, if you hit it, then you expect to see this front spreading out in the string depending on the speed of propagation compared to the speed of propagation of transverse waves in the string. So, these are the conditions, his is what you will observe in the case of an infinite string. But suppose now, this wave or a disturbance in the string goes and interacts with a boundary. So, let us study this case of scattering from a boundary in the case of a traveling string.

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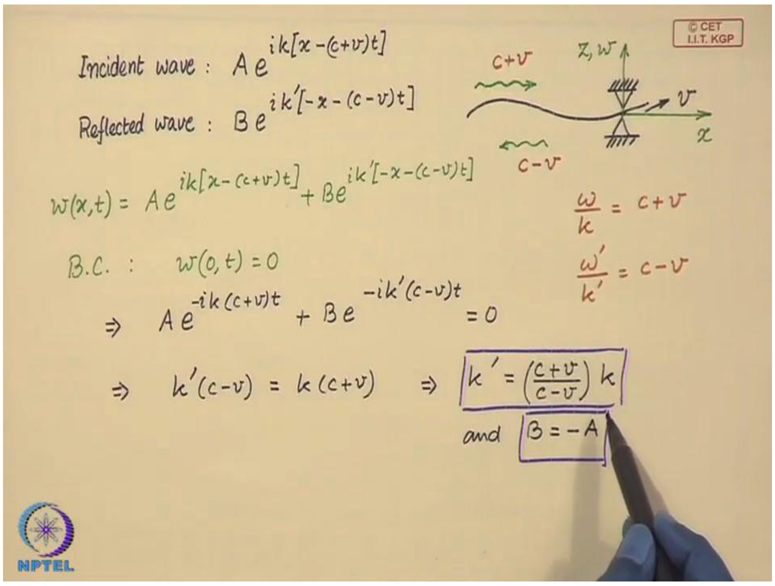


So, this is the string traveling at speed v. Now we consider that there is the incident wave. So, this is the semi-infinite string and there is a wave incident from minus infinity, so harmonic waves, a positive traveling wave incident at this boundary; and what we expect is that there will be reflected wave once this hits the boundary. So, there is going to be a reflection. We are going to study how this reflection process takes place; so, what is a reflected wave. Now as we have discussed just now that the speed of a positive traveling wave is c plus v; on the other hand the speed of negative propagating wave is c minus v, so these are the phase speeds in the positive and negative directions. So, we expect that this, since this phase speed is the ratio of the frequency is to the wave number; so, if for the incident wave it is c plus v, then we expect, in general, let us assume that it is omega prime over k prime is c minus v, unlike in the case of the static string, where they must be equal, but now since the speeds in the two directions are different, we have to be careful that these may be in general different. So, therefore if the incident wave is written out like this, then… so this is k x minus omega t. So, omega I have replaced as c plus v into k. Now the reflected wave, you can see that it can be represented like this, where I have replaced this omega prime t as c minus v k prime, and k prime has been taken out common.

So, therefore the total wave field in the string can be represented like this. Now the boundary condition is given by the zero displacement condition at x equal to zero. So, this implies… Now this has to vanish, this sum has to vanish for all times. So, this can happen, only when… and… So, we obtain this interesting result that the wave number of the reflected wave gets modified upon reflection from this phase boundary in the case of a traveling string. This is unlike the case of a stationary string, while there is the phase inversion of the reflected wave. So, there is a phase change of pi upon reflection, which is as usual. We have also seen this in the case of a static string, but this is what is very interesting, the wave number gets modified upon reflection.

So, today we have looked at two examples, and we have studied these two examples using the wave propagation solution. First one was the reflection process collision of bars, by which we understood the reflection process, we can understand the reflection process of let us say ball on rigid floor or similar collisions. So, we have seen how there is a finer time of contact, in which actually the waves travel in this object, while it is in contact with the wall or the floor. The second example was that of a traveling string, we have looked at the at the wave propagation solution in the case of traveling string; and we have also looked at the solution of the initial value problem; and finally, we have looked at the scattering process in a traveling media, in a traveling string, and we had seen how the wave number can get transformed upon reflection, which is an unusual dynamical behavior. So, with that we conclude this lecture.

Keywords: wave propagation, axial collision of bars, translating string.