Vibration of Structures

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Lecture No. # 21

Applications of Wave Solution – I

Today we are going to look at some applications of the wave propagation solution that we have been discussing over the past few lectures. So, by applications I mean applications in our, what we observe in daily life. So, we are going to discuss today two examples; one is that of a bar which is started with an impulsive force.

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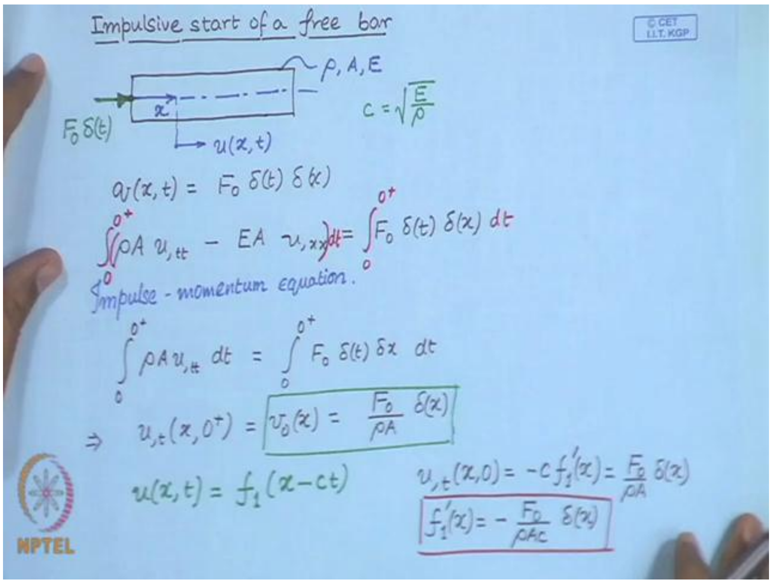

So, suppose you have a bar – a finite bar. So, this is our first example. We have finite free bar which can move in this axial direction, but it is restricted in the in the vertical direction. So, this bar can move only in the axial direction. So, this is a uniform free bar which is given a force of strength F_0, and which is an impulse - an impulsive force. We consider the actual displacement field of the bar to be represented by u(x,t), and the properties rho, A, and E.

So, the force distribution is actually represented by… So, what we have been writing, the force distribution, therefore is F_0 delta t, an impulsive force applied at x equal to 0, the way we have considered our coordinate systems. So, we have this force. So, our equation of motion for this bar reads… So, this is the forcing. Now, we write down the… So, to obtain the initial condition on the bar, we write down the impulse momentum equation. So, this is obtained by integrating this equation of motion over time - over a very small time interval. So, we integrate this equation over a very small time interval 0 to 0 plus; so, the time just after the impulse has been applied; and we know that this term over this very small time interval is not going to contribute to this integral. So, the terms that are going to contribute… and what we obtain immediately is… So, this gets integrated once and the velocity at zero is zero. So, the velocity just after the impulse that is what we obtained which is represented by v_0 of x; so this is the distribution of velocity just after the impulse. So what this means is that velocity distribution is like this; at x equals to zero there is this velocity, whereas everywhere else, almost at points of the bar the velocity is zero except this layer at x equals to zero, which has a velocity of F_0 by rho A. So this gives us our initial condition on the bar. Now, when we disturb the bar like this, we give an impulse, what we expect and by this time whatever animations I have shown you, you can guess that when this bar is disturbed there is wave front that starts propagating. This point of the bar that immediately does not know that there is an impulse; so this whole part of the bar is stationary, whereas this layer has acquired a velocity and that information is going to propagate, is going to propagate at a speed c which is square root of E over rho. That is the speed of propagation of axial wave. So, we have… The wave field in the bar therefore can be written as… Let me write it as f_1, there is a positive travelling wave front in the bar; so our axial displacement is guided by this propagating wave front. Now, we will apply the initial conditions. Let us differentiate this first.

So, that is what we have… So, f_1 prime… So, we have this f_1 prime. Now, we must integrate this to obtain f_1. Now, the integral of this Dirac's delta function is Hemiside step function and there is going to be a constant of integration. Now, if we use the condition that the displacement of the bar throughout is zero even after this impulse; just after the impulse is applied, the bar has taken a velocity distribution like this but its displacement is still zero. So, which means that if you use this condition then you can solve for this integration constant and then this will imply that C_I is… which you can easily see from here, because this f_1, u(x,0) is nothing but f_1 and that must be equal to zero. Now this at zero is one. So this becomes, the integration constant becomes F_0 by rho A c. So therefore f(x-ct) which is u(x,t) becomes… So, we have replaced this x by (x-ct) and that is u(x,t) and by substituting this here, and finally replacing this x by (x-ct), I obtain this wave field. But remember that this solution is valid as long as this wave has not reached the right boundary of the bar. So, which means this the validity of this solution is from 0 plus to l over c. This is the time taken, l over c is the time taken for this wave front to reach the right end of the bar. So, this is the solution that you will observe when t is less than l over c and just after the impulse has been applied and t is less than l over c.

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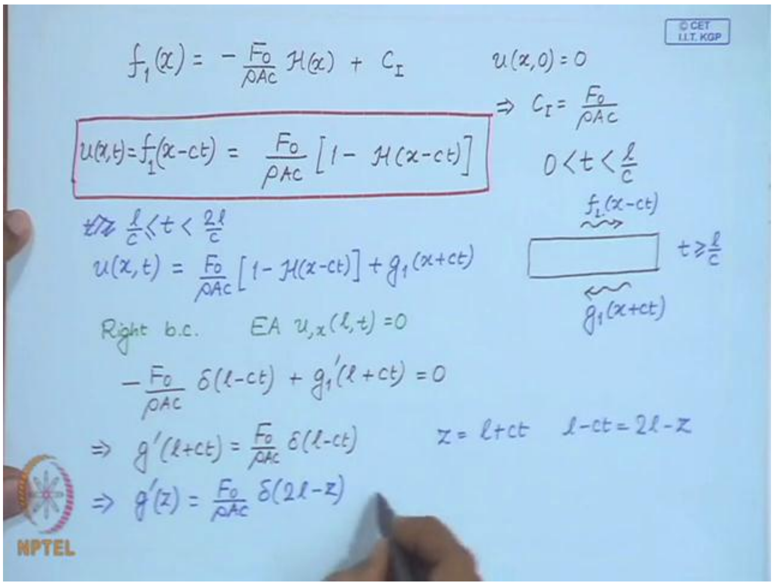

Now, this front is going to go and hit the right boundary of the bar. So, here there is a front which is travelling and that is going to hit the right boundary and going to reflect back. So, this was f_1. So, this is f_1 that is incident on the right boundary and what gets reflected, I will represent as $g_1(x+ct)$. So, this takes place at t equal to or greater than equal to l over c. So, as long as this wave has not reached this, this is not generated; this is generated after this wave hits the right boundary. Now, then our wave field… So, if t is… what you obtain is... So, this is the solution when the wave goes and hits the right boundary. This is valid for t greater than l over c, but less than 2l over c, when this solution goes and hits this boundary again. Let us now restrict to what happens when it is just reflected. Now, this boundary is a force free boundary. So, the boundary condition… So this is a force free boundary; this is the boundary condition. When you substitute this in here, what you obtain is… So, that implies… Now, let me make a substitution, let me substitute this l plus c t, because we want to have g prime of z; so, z as l plus c t and that means this l minus c t can be written as 2 l minus z.

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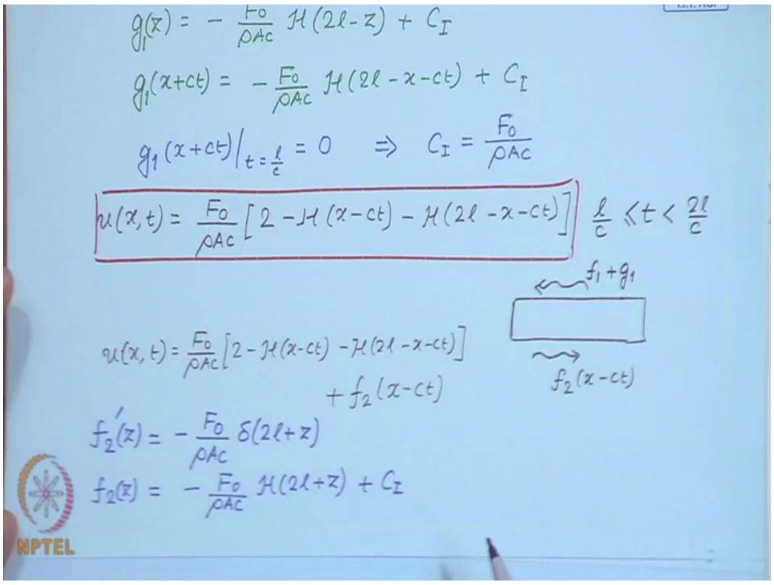

So, then this I can write… So, g prime is given by this expression, and if you integrate this… so, if you integrate this expression, again C_I is integration constant. So, g(x+ct)… Now, we will use the condition that at t equal to l over c, this solution is zero. So, because just at the time it is l over c, this is zero; this is just been generated or just before that it is not generated. So, it is only the left, the wave has just reached the initial, has about reach the right boundary and then you can substitute in this and you can very easily see that can solve for the integration constant, and then finally the solution. So, this is the solution after the wave has reflected once from the right boundary, and this solution is valid in the time interval l over c to 2 l over c.

Now, once again this wave is going to come back to the left boundary. So, this is left - negative traveling wave and that is going to be come back to this boundary and this is going to be a reflection. So, this is f_1 plus g_1, I am not writing the arguments and what I generate is f_2(x-ct). So, this f_1 was a positive traveling wave, g_1 was negative traveling wave, the superposition is travelling now this; and what it generates at this boundary is f_2. So, once again you consider this solution… plus f_2(x-ct); and when you do this calculations, then what you obtain following the steps that we have done above. So, you will come to this expression; f_2 prime of z is given by this expression. So, f_2 of z is… and so, therefore…

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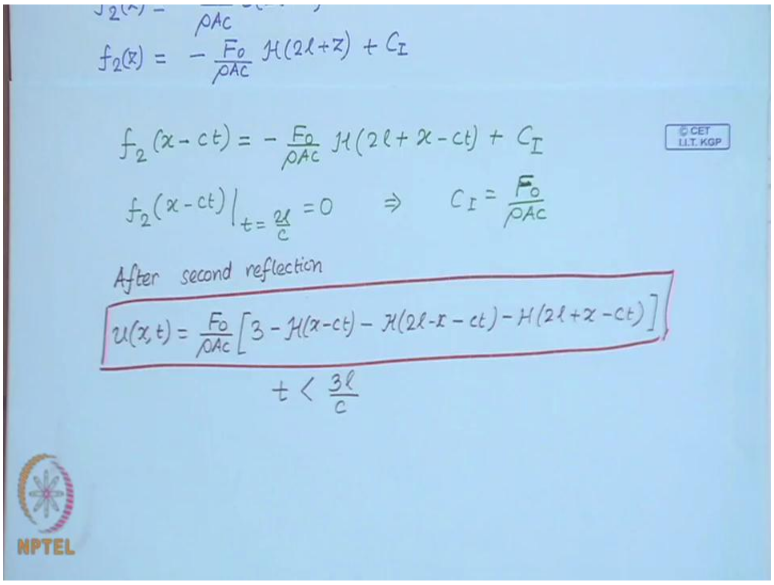

Now, again we use the condition that this is zero, f_2 is zero when t is just 2l over c; and that gives us once again the integration constant which is F_0 by rho A c. So, therefore after the second reflection… Now, this is the solution which is valid till again you have, this wave reaches again the right boundary. So, this is how you can keep solving these reflection processes. Now, if you consider the solution and plot the propagation of this wave, then you will find that the motion of the body is something like this.

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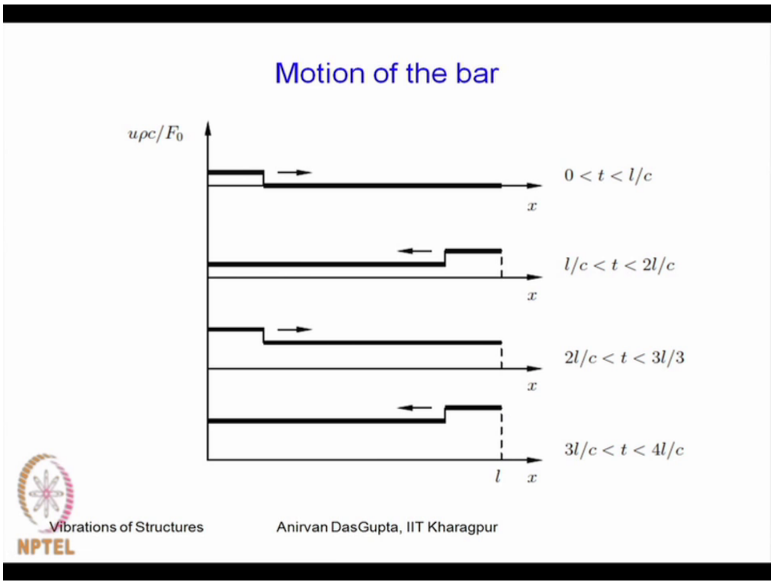

So, as soon as you give the impulse, the left end of the bar undergoes, so it acquires a velocity and because of this velocity, a displacement wave follows. So, this is the displacement, axial displacement of the bar; it is shown in the transverse direction, but this is the axial displacement of the bar; and this front propagates and it propagates at a speed c as we have seen. Now, this portion of bar yet does not know that there is an impulse given to the bar, the left end. So, this is in the equilibrium position, and this wave propagates, hits the right boundary and the right boundary undergoes twice that deflection, because now it is a force free boundary. So, there is a compression wave. So, this is a compression wave actually. So, when it hits the boundary, it relaxes and it reflects back. Now this bar here is under tension. So, this displacement front now propagates to the left again at the speed c. Again it reflects and this reflection takes place. So, what is happening in effect? So, that is very important. So, here this is the displacement after at in this time region you see, the displacement stays like this and here this is the twice the displacement. So, what we find the displacement is continuously increasing, which means the bar is actually propagating.

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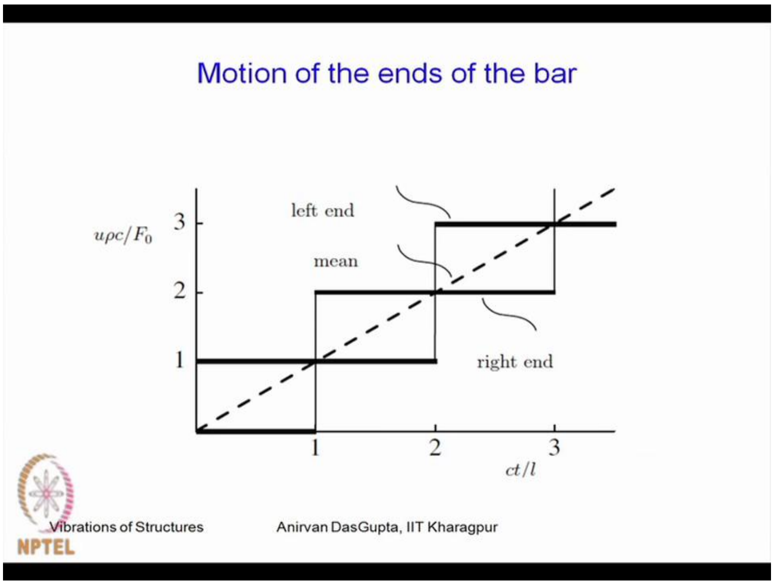

So, to see this, here I have plotted the motion of the ends of the bar. So, this is the left end of the bar. So, as soon as the impulse is given; so, this is time; as soon as the impulse is given the left end of the bar gets the velocity and the displacement follows. So, left end of the bar remains displaced at a constant value as we have seen just before; right end of the bar does not know at this time l over c, the right end of the bar knows that there is a displacement front. So, it gets displaced twice this distance and its stays here. So, this is the right end of the bar. Here, the wave again reaches the left boundary and the left boundary takes a displacement jump.

So, like this the two ends of the bar, they are moving, but now they are moving in discrete steps as we can see here. Now, this is the mean motion of the bar. So, let us see, what is happening? You have displaced, you have provided this impulse to the left end, the wave propagates and this end gets, this end will undergo twice the displacement; then again the wave propagates back and pulls this. So, the bar is constantly moving like this. Now, this is transient motion, but when we do it in I mean real world, suppose you hit a bar, you hit a bar, you find that the bar is almost moving like a rigid body, if it is a frictionless surface it will keep moving at a constant speed.

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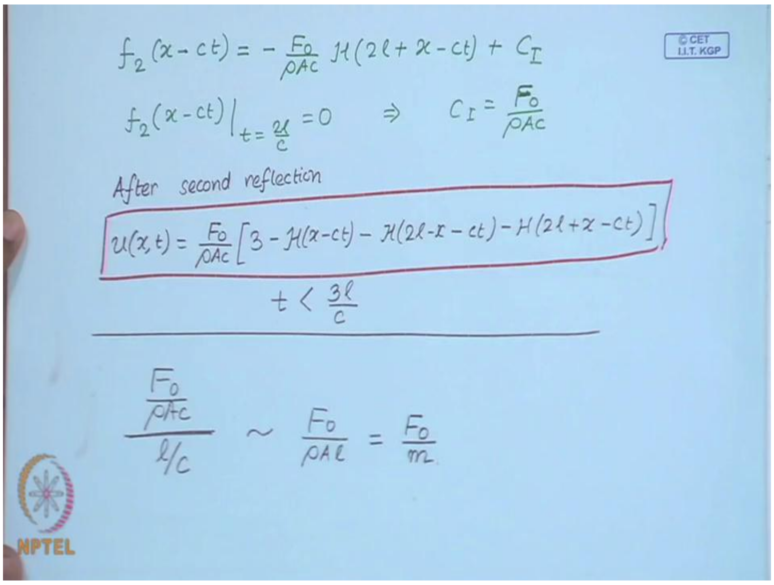

Now, if you do this calculation, just little bit of calculation further over this, then you see this, the whole bar shifts a distance. So, this is the distance traveled in time l over c, if you look at these plots, then you will find the distance traveled is F_0 over rho A C. So, at every step, you have this displacement, and the displacement takes place in this time interval. So, the average speed of the bar is F_0 by rho A l. Now, A into l, area of cross section into length is the volume into density of mass. So, this becomes F_0 over m.

Now, that is the result that you will obtain from; so, this m is the mass of the bar; so, this result you will obtain from particle dynamics, if you have a particle of mass m and if you give it an impulse, this is the velocity it will acquire. So, what we observe is the mean motion of the bar is what we obtain as impulsive start of a particle or a rigid body. But inside what is happening as we now know that is not a steady motion with speed F_0 over m, but its transient motion. Always there is a transient and the ends are moving in discrete steps; and that is how this this bar actually moves rather than like a rigid body. So, what we see as the rigid body motion almost rigid body motion is actually can be understood in terms of the wave propagation in the system in the bar.

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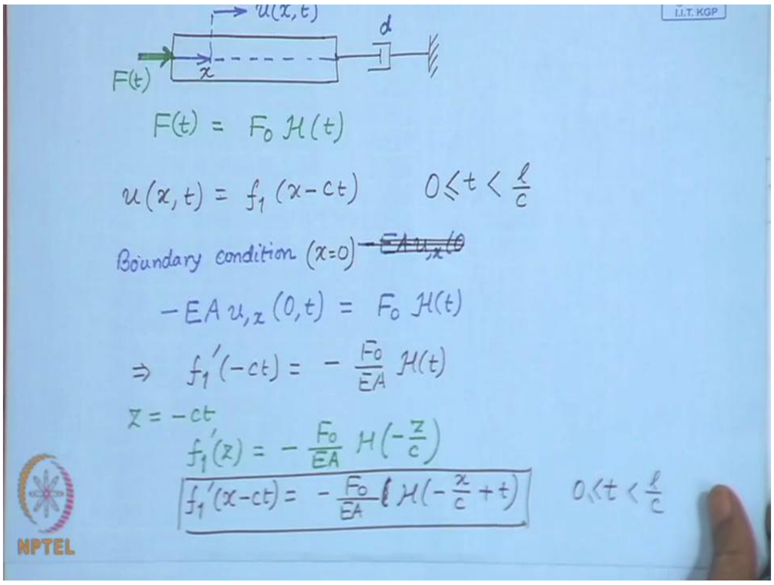

Now, let us go to the second example. The second example is that of a once again a bar with a boundary damper. Now, here I have a bar which is connected to viscous damper; damping coefficient is d and this is given a force. So, what you are going to study is step forcing. So, this force F is actually of magnitude F_0 and it is a unit step. So, this is a heavy side step. So, this is a unit step function. So, you give suddenly switch on a constant force at time t equal to zero. So, time t equal to zero, you switch on a constant force like this. So, again what happens is as soon as you switch on wave starts propagating in the bar. So, we have a positive traveling wave propagating in the bar and this is going to the situation as long as this wave does not hit the right boundary.

So, this is going to be the solution as long as the wave does not hit the right boundary. Now, as soon as hits the right boundary, we have this boundary condition. Well, let us first consider the condition at the left boundary. So, what we have is the boundary condition at this left boundary is... So, the force is compressive. So, for that reason we have this negative sign. So, this is going to be the boundary condition at the left boundary at x equal to zero. So, this is… Now, if you substitute this expression - this solution in here, and simplify… So, this is going to be f prime. Now, let us define z as minus c t, then we obtain f_1 prime… So, this is what we have, and therefore, f_1 prime x minus c t is… and this solution… this is the solution as long as time is less than l over c.

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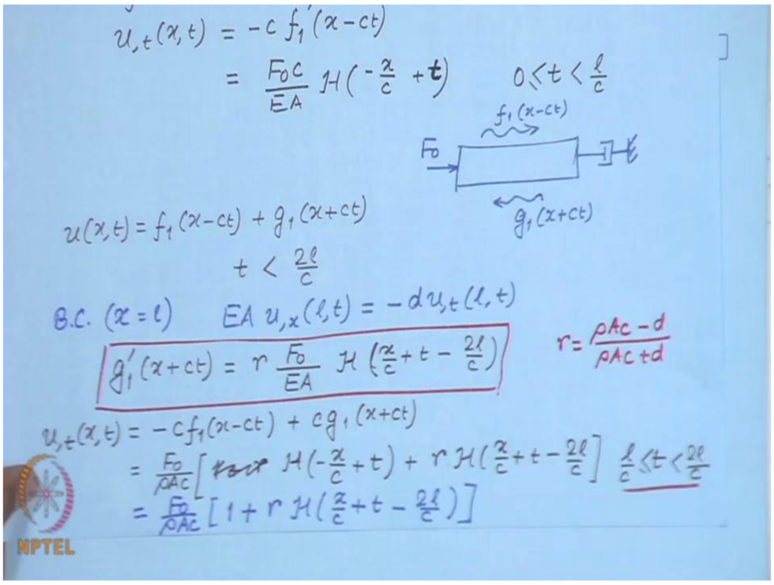

So, using this now we can write the velocity of the bar. So, if you consider the solution form that we have, then the velocity of the bar is given by… and therefore this becomes by substituting this expression here. So, this is positive traveling wave and… So, this is of course, between zero to l over c. Now this is going to go and so, this is of course between 0 to l over c. Now, this is going to go and… So, the force is now F_0, this wave is going to go and hit the right boundary where there is the damper and there is going to be a reflection. So, if I call this if I call this as f_1, I will call this as g_1. So, the total wave field therefore, in this situation… and this is... So, this solution form is valid till this wave does not hit the left boundary once again.

So, till that time this is going to be the form a solution. Now, the boundary condition at the right end at x equal to l is given by this expression. Now, once you substitute this form here and use… So, once you substitute this here and you do the simplifications using this expression of f prime, f_1 prime, because what you are going to get here is f_1 prime, there are derivatives on both sides. So, you will get f_1 prime and you will replace that with that expression, then finally, you obtain g_1. So, on simplification you will obtain this expression of g_1 where r is… So, this factor r is given by this ratio. So, as you can see r is always less than 1. So, this is the damping coefficient and this is the impedance of a semi infinite bar. So, this you can obtain easily. So, then once again you can write down the velocity of the bar. So, velocity of the bar is given by minus c f_1… and that turns out to be… So, let me first write down… Now, this time is from l over c to

2 l over c. Now, for this time interval you will find that this is always 1. So, this gets simplified. So, this is the velocity of the bar in this time interval.

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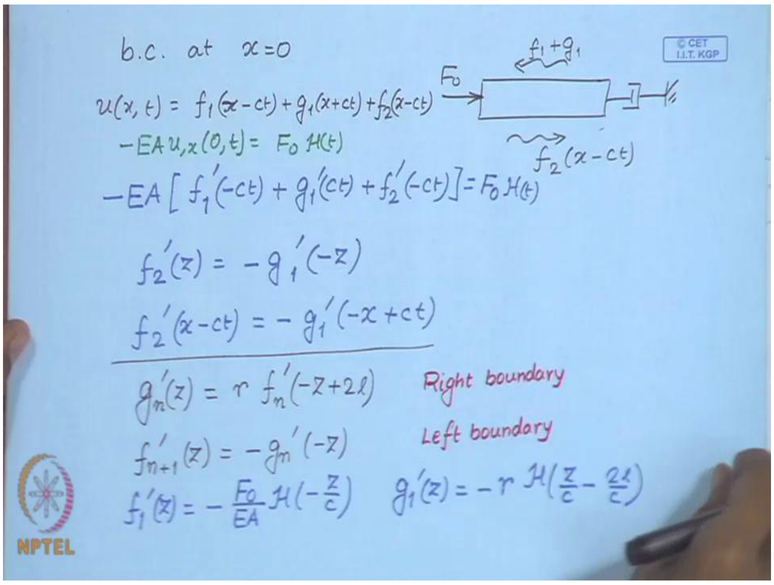

Now, when this wave reaches at the left end once again… So, what we have is… So, we have this wave propagating, I am not again writing the arguments. Now, when it reaches this left end, so this constant force is already acting, then what we have is another reflection of a right propagating wave. Now, the boundary condition at this end… so, the wave field is therefore… Now, if you use the boundary condition at the left end of the bar. So, here… So, what you will obtain finally, when you do all this substitutions. So, at x equal to zero, so, this is the boundary condition at x equal to zero. So, this is going to be the… So, so the boundary condition at the left end was… So, this was the boundary condition at the left end. So, from there we can obtain this expression, and finally when you use the expression of f_1 in the here and that will cancel of with F_0 H(t), because that was how we solve for f_1. So finally, when you simplify, this is what we are going to obtain and therefore… So, we have solved for f_2 prime x minus c t. So, once you… So, since we already have this g_1 prime, so we can, we know what is f_2 prime.

So, you see we are… So, we have solved one reflection at the right boundary and the second reflection at the left boundary, and if you continue this process, you can keep solving one after the other, and it so happens that these reflected they follow a recursion; it is like this that at the right boundary. So, this is at the right boundary. So, the reflected wave that is generated at the right boundary is in terms of the incident wave and the reflected wave that is generated at the left boundary… So, this is for the left boundary; and just the at the start f_1 prime, we have already solved this, f_1 prime is this and g_1 prime… So, once we know f_1 prime and g_1 prime, we can use this recursion to find out the wave at any n^{th} reflection.

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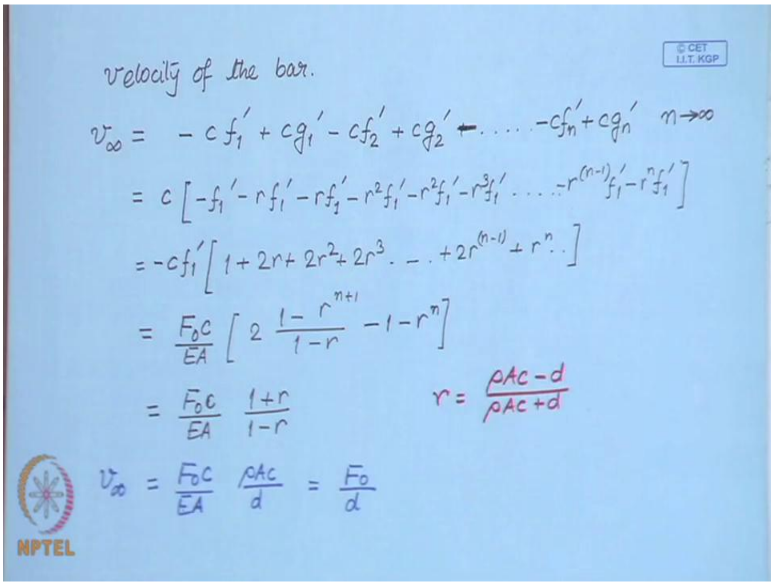

So, using this recursion relation, we can therefore write down the velocity. The velocity of the bar as time progresses and it reaches steady state, because ultimately this bar under constant force F_0 is going to reach a steady state. So, let us denote this steady state velocity as v infinity, then that should be given by… So, as we find velocity it is negative c of f_1 prime plus c g_1 prime, this is what we have discussed. So, for every right propagating wave, this is going to have a coefficient minus c. So, here there will be minus. So, like this… So, when n tends to infinity this series, if you sum up then you will get v infinity.

And if you now use the recursion relations that we have just now discussed, then this turns out to be... So, this goes on. So, we have this infinite series; and if you simplify this further… and this can be summed up. We have this expression of f_1 prime. So, using that we can write this as… This can be cast as a geometric series and summed up. So, this when n goes to infinity, when n goes to infinity, so, this sum simplifies and as you know that r is less than 1. So, the definition of r… So, since r is less than 1, so, we can make this simplification and this therefore…

So, if you substitute this expression of r in here. So, this turns out to be... So, now E by rho is c square and this c square this cancels of, so, and A cancel of, so, this becomes F_0 over d. Now, this is of course, the expression of velocity we know, that if to a dashpot if we gave a force F_0 and if the damping coefficient is d, then the velocity that we will have is F_0 over d. So, this velocity is as expected. But then this does not happen all of a sudden; there is a transient and this is the transient that we observe in this plot where I have plotted the solution as it reaches the steady velocity.

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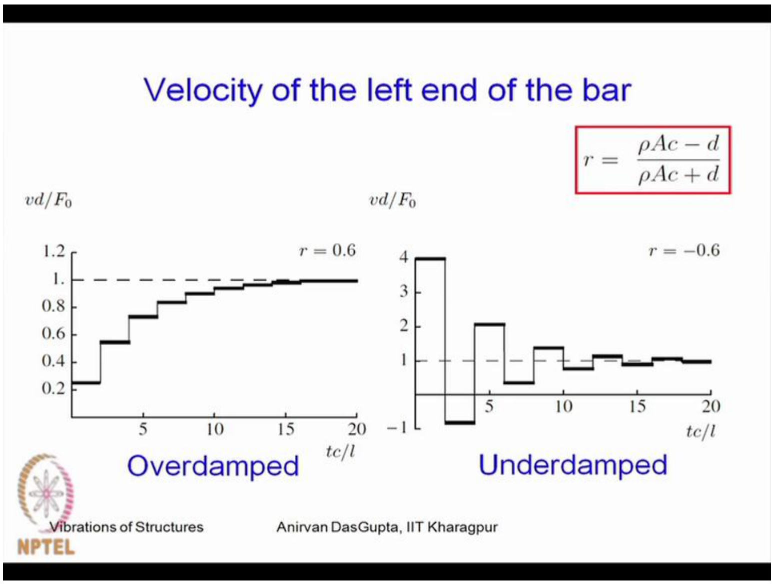

So, whatever we have discussed now, I have plotted here in these two dash parts two values of r. Now, r… So, I have written the definition of r here. So, r can be positive or negative depending on the value of d let us say. So, if d is less than rho over c, then r is positive; and in that case what we observe is something like an over damped behavior. So, the velocity… So, what we are talking is the velocity of the bar with time. So, the velocity that is achieved in this steady state is something like an over damped system. On the other hand, if r is negative which means d is greater than rho A C, which is the impedance of the semi infinite bar, what we find is, there is an oscillation, oscillatory behavior before it is settles down to this steady value - steady velocity F_0 over d. So, what we conclude from here, so, why we are calling… See, when d is less than the impedance of the bar, then we obtain an over damped kind of a response.

So, what is happening is, since d is less. So, the disturbance that goes to the right boundary and hits this damper is observed. Whereas, if the damping is more than the impedance of semi infinite bar, there is a reflection and this reflection is this reflection is more than what we the initial disturbance was. So, there is some oscillation before the bar settles down. So, if the damping is more, it is more towards the hard damper, then we have we have reflections which cause oscillations. Whereas, if the damping is the damping coefficient of the dash part is lower than the impedance of the bar, then it absorbs the energy and there is no reflection, and there is a reflection, but then the kind the total response of the bar shows something like a over damp behavior.

So, in this lecture, we have looked at these two examples where we have applied, the wave propagation solution to understand, the behavior - the transient behavior of two bars with different kind of conditions. In the first example, we discussed a bar with which is started with an impulse, and in the second case, we have step forcing of a bar with boundary damping. Now, what we conclude therefore, with these two examples is that in the steady state what we observe as the almost like a rigid body motion through our naked eye that is what we observed. That actually can be understood from the propagation of waves in the bars or in the systems.

So, for any system, any continuous systems like this be it a string or a bar, the steady states motion, which at least appears to be steady state motion, actually is something like a transient motion, because of this wave propagating back and forth in the system; and this is what we have also observed when we looked at the animation of a string carrying a moving load. So, the motion of the string there also was something what we have observed today is intermittent. It is not continuous as it appears to us.

Keywords: wave propagation, impulsive start, step forcing, boundary damping.