Vibrations of Structures

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Lecture No. # 20

Scattering of Waves

So, we are discussing in the last few lectures, we have been discussing about wave propagation in one-dimensional continuous media; and we have looked at the d'Alembert's solution and the initial value problem; and when we look at the behavior of a system under some initial conditions; for example, we have looked at a string with initial velocity condition or a string on which there is a traveling force. So, we have observed that the behavior of the string looks like a transient behavior, there is a, the motion is not continuously changing as we would see with our naked eye, it seems that it is almost continuous. But then when we look at this motion in slow motion, then we find that the motion is little different; it seems like a transient motion, there is some propagation of information back and forth in the in the medium, in the string or in the bar. So, today we are going to look at this phenomenon of scattering of waves, and this will help as in understanding what happens when a wave front goes and hits a boundary and or an obstacle or an impedance for example.

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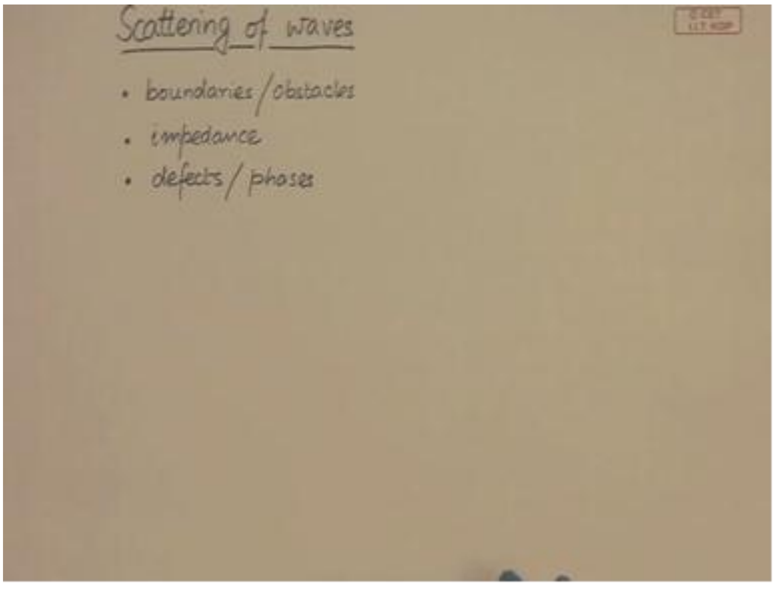

So, we are going to look at scattering of waves. So, scattering can take place at boundaries or obstacles, it can take place at impedance - some finite impedance, let us say boundary damping or I mean it can take place at defects, so, which are defects or phases. When there is a difference of phase inside a material, then scattering will take place; and this scattering is used is of great practical importance for characterization and evaluation in non destructive evaluation of materials.

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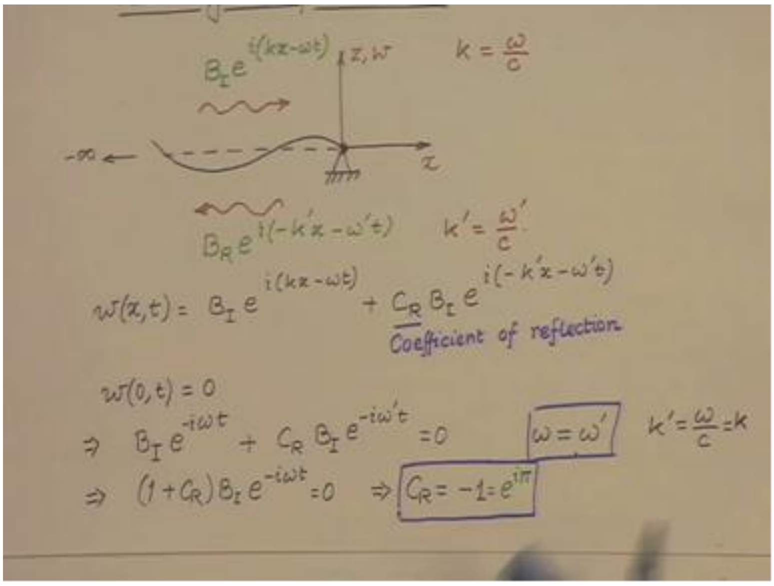

So, let us start with scattering at a fixed boundary. So, we will study this problem with the help of the example of a string - a semi infinite string. So, here we have a string which is fixed at this boundary, and we consider that harmonic wave is incident from minus infinity. So, positive traveling harmonic wave is incident at this boundary; and then this wave comes and hits this boundary, there must be a reflection, because the material ends here. So, there must be a reflection.

Now, we are going to consider an incident wave which is represented by… So, this is the amplitude of the incident wave and this is the space time evolution. So, this the positive travelling wave. Now, when this wave reflects, let us consider that the reflected wave is represented by… Now, here I have used k and omega for the incident wave, the wave number and the circular frequency of the wave, whereas for the reflected wave I am using k prime and omega prime. Now, this medium is the same.

So, we must have as we have seen in our previous lecture that this k must be equal to omega over c, and similarly, k prime must be equal to omega prime over c. Since the medium is the same the wave speed in this direction or this direction must be the same. So, the wave speeds must be the same. So, k prime and omega prime are related by this relation. So, the total wave field in this string plus the reflected wave. Now, what I am going to do is, represent this amplitude of the reflected wave which is B_R as a coefficient of reflection times B_I and there is this exponential power. So, here I am introducing, this is the coefficient of reflection. So, here we have this coefficient of reflection.

Now, the boundary condition at this fixed and is of course, this must be zero. So, when I substitute this total wave field in this boundary conditions condition, this gives me… Now, this relation has to be satisfied for all time for all time t and that is possible if and only if… So, this condition is required, so which means that the frequency of the incident wave must be equal to the frequency of the reflected wave. Now, immediately using this, here we find that… Therefore, k prime must be omega by c and that is equal to k. So, which means that the wave number of the reflected wave must be same as the wave number of the incident wave, since the frequencies of these two waves must be the same. So, once we put this condition here what we have is… That implies C_R is minus 1, because the other part is not equal to zero. Now, this can also be written as… So, minus 1 - the reflection coefficient minus 1 can also represent as exponential i pi which tells us that there is a phase inversion; so, the phase change of pi. So, the reflected wave suffers a phase change of pi. So, the C_R equal to minus 1 implies that the wave undergoes a phase change of pi.

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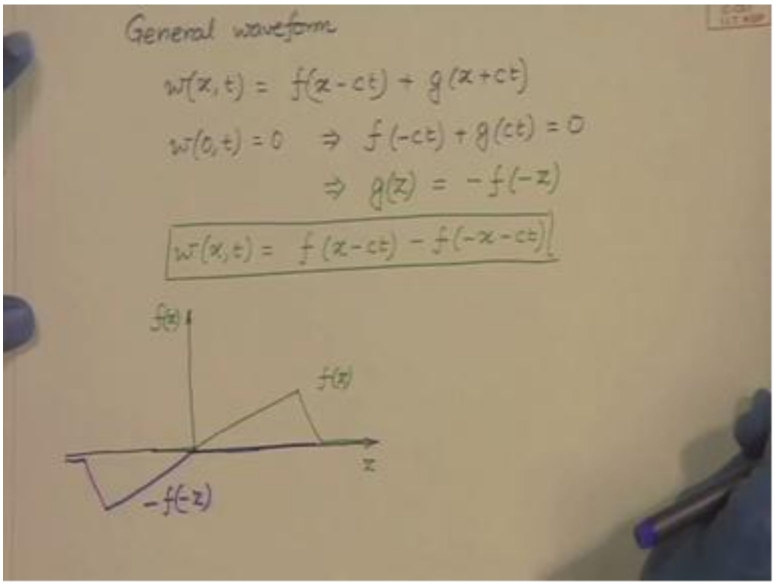

Now, if you consider a general wave form for the same reflection process from a fixed boundary, then we have the field - the wave field represented like this as we have done before. Now, the boundary condition again is… and that implies… So, this is plus. So, this must be zero. So, what we therefore have is g of z, let me write c t as z is equal to minus of f of minus of z. So, which means that our solution, the total wave field becomes… Now, let us see, what is this, minus of f of minus z. So, suppose this is the f of z, then f… So, minus of f minus z would be… So, if this f of z, then this is minus of f of minus z. So, the total wave field is superposition of these two traveling waves now, one in the positive direction and the other in the negative direction. So, let us see this.

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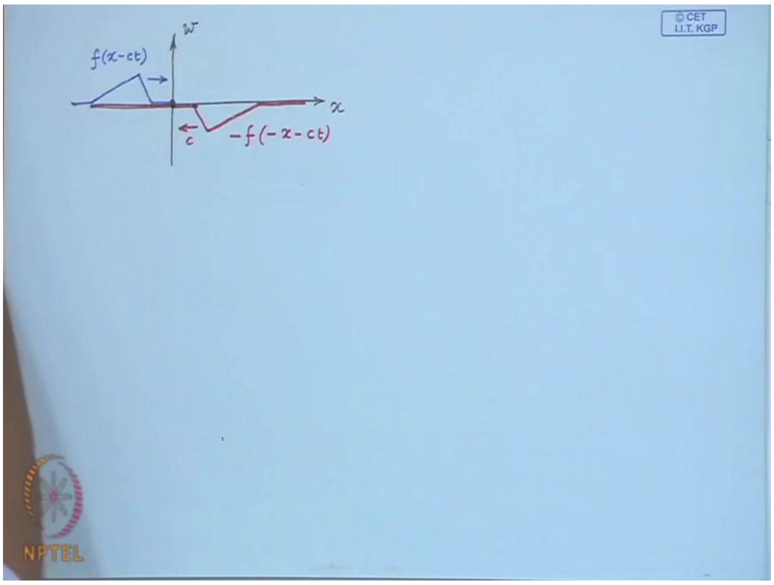

So, consider that there is this incident wave at this boundary. So, this is f of x minus c t. So, this is moving towards the boundary, and imagine that there is a wave which is minus of f of minus x minus c t. So, this is traveling in this direction, the same wave speed c. So, superposition of these two is going to give us the net wave field in the string.

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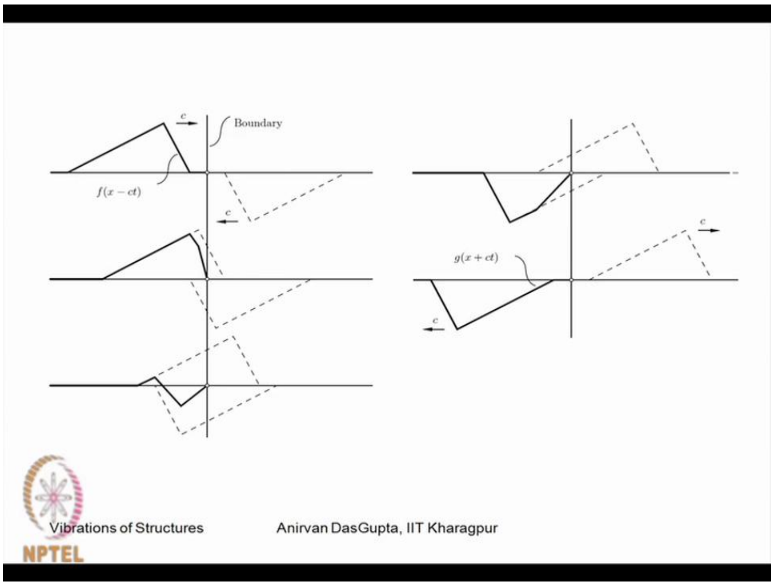

So, here I have shown some snapshots of this reflection phenomenal at a fix boundary. So, this is the incident wave and this is an imaginary wave which is propagating towards the left. Now, the superposition of these two waves… So, this solid line shows the actual configuration of the string. So, as time progresses, so, you finally have this is the reflected wave which is traveling in the negative x direction. So, here I show an animation. So, the wave pulse appearing from traveling towards positive x direction from minus infinity hits this fixed boundary at x equal to zero and gets reflected. So, the figure here - the above figure, I have shown what is actually observed. So, that is how you will observe the string to behave. Here, I have shown this superposition of the two waves which is giving the same solution. So, the second figure below shows this superposition of the two waves coming from two directions. So, you will find that these two superposing waves do not satisfy the boundary condition at x equal to zero. But after superposition the sum of these two waves satisfies the boundary condition. The individual waves do not. So, that is how this reflection process takes place.

Now, in a previous lecture, we have discussed this initial condition on a string - initial velocity condition on a string and we have observed that when we provide an initial velocity condition distribution, then we find that this is the front, that is propagating and at the boundary again it reflects back. So, let me let me show this, what happens around this boundary. So, in this next animation here, what we observed is, there is a wave front which is appearing; the front is traveling towards the positive x direction, and it hits the boundary and gets reflected. Now, this reflection process is shown in the figure below. So, here we find that there is a front propagating which is this red dashed curve, and there is a negative propagating inverted wave which is because of the reflection, as we have discussed just now.

So, the superposition of these two waves is creating the reflected waves. So, the front comes hits the boundary and is reflected back. So, what we observe is that as the wave comes, it hits and then there is this the reflected wave is the superposition of these two waves that we have just discussed. Now, one is the incident wave, the other one was the reflected wave. The reflected wave is inverted as well as it there is the mirror image created. So, we have minus of f of minus z. So, f minus z is the mirror and minus f gives us the inversion. So, there is an inversion as well as the reflection. So, that is what we have observed.

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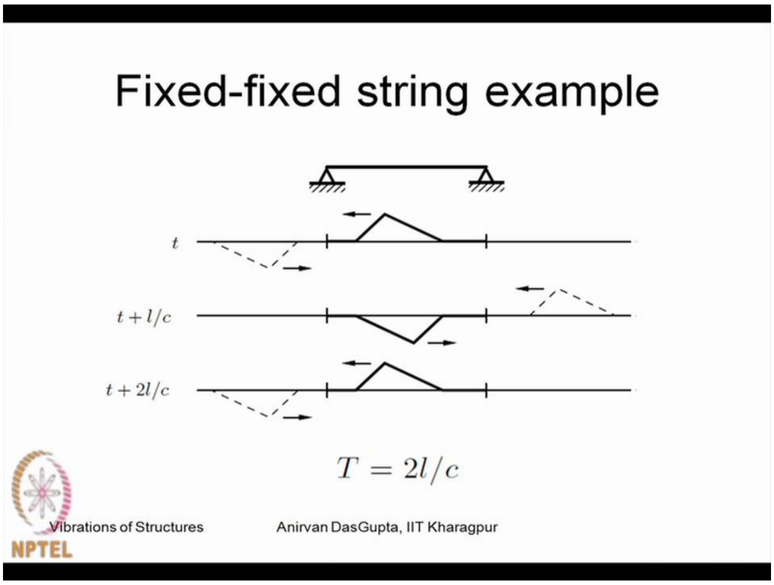

Now, let us look at this case of a fixed-fixed string. Now, when you have a pulse which is traveling to the left let us say, then it hits this fixed boundary, and you have a wave that is the reflected wave is like this. This reflected wave goes this positive traveling wave now, goes and hits the right boundary and there is the negative traveling reflected wave. So that then completes the full cycle. So, the time taken, here I have written out the time taken by this wave to travel this whole part. So, if we mean that the time period is 2l over c. So, let us see the time period is 2l over c. It takes this much of time for the pulse to once again repeat itself.

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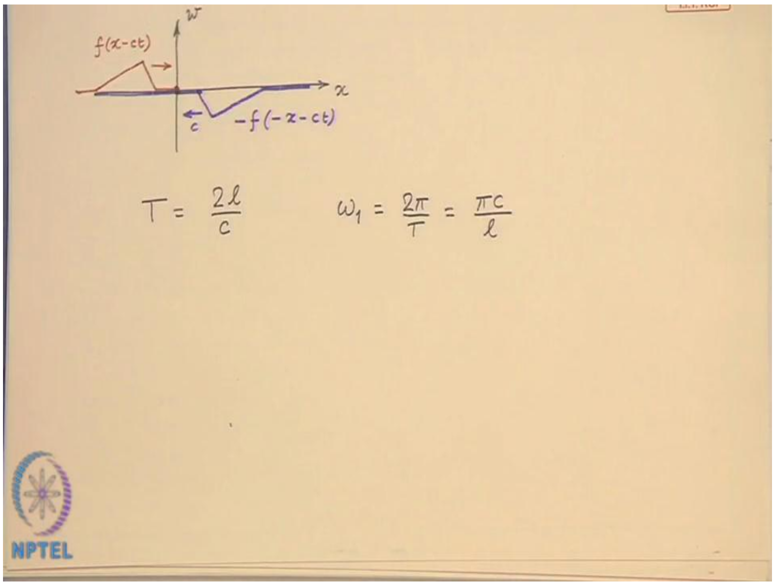

So, this from here, we can estimate the fundamental frequency - the circular frequency which is 2 pi over T which is pi c over l; and this is the fundamental frequency or fundamental circular frequency of fixed-fixed string. So, the frequency of a finite system can be estimated or calculated based on this wave reflection process that we have just studied. So for the fixed-fixed string, we have been, we will do calculate the fundamental frequency.

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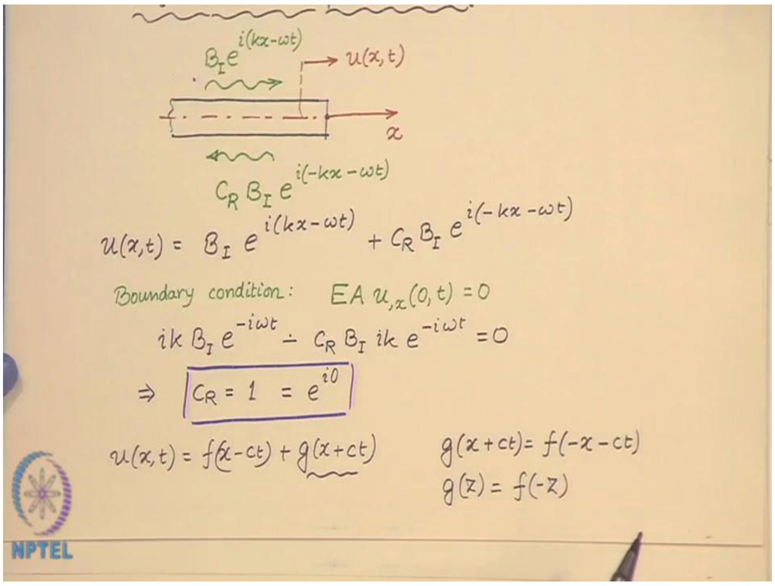

Now, let us look at this example of reflection at a free boundary. So, for this we will consider a uniform bar. At any point x of the bar, we have this actual displacement. So, this is the field variable u of x and t. So, again we have this incident wave at… So, this is the free end now. This is the free end. So, this is an incident wave given by this and since the material ends here, there must be a reflection. So, this we are going to once again represent as reflection coefficient times - B_I times a negative traveling harmonic wave; and what we learned from the previous example, these k and omega must be the same as we have seen in the previous example. So, the total wave field in the bar is given by this expression.

Now, the boundary condition, so, this is the free boundary. So, for a free boundary, we know that is a force free or a natural boundary condition. So, once you substitute this wave field in the boundary condition… and since… So, this… This time function this will not be zero. So, this has to be satisfied for all time. So, this would imply upon simplification… So, C_R - the reflection coefficient this time is one and one is nothing but exponential, you can write it like this. So, this tells us the reflected wave suffers no phase change. So, it is reflected as it is. Now, you can perform the same analysis, what we have done in the previous example for a general wave front. So, if you consider… and use this boundary condition, then you will find that the reflected wave can be represented by… So, which means that g of z is actually f of minus z. Now, if you recall that for a fixed boundary g of z was minus of f of minus z and as we have seen f of minus z is just the mirror reflection of f of z, f of minus z is mirror of mirror reflection of z. So, there is no inversion of the wave as we have seen, there is no phase change.

Now, here in this animation, we see a wave pulse traveling from minus infinity towards the positive x direction; so, positive traveling wave pulse getting reflected at a free boundary. So, once again the figure above is what we observed and the figure below shows this superposition of these two traveling waves. So, you can see that the reflected wave is nothing but a mirror reflection of the incident wave; and these two waves superpose to give us the motion of the bar and it could be a bar, it could be a string with free end. If it is a bar, remember that this value of the displacement is actually the axial motion of the bar. So, we have observed here for the free boundary case, there is only a mirror reflection at the boundary.

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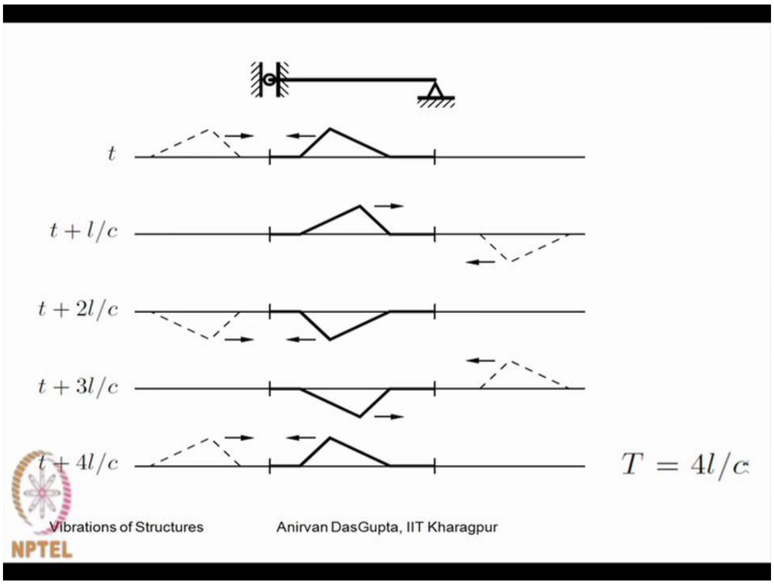

Now, this shows another example where we have applied what we have just now seen. So, this is the free and fixed string. So, this is the sliding end and this is the fixed end of the string. Now, consider once again the pulse traveling to the left and from this free end it is being reflected like this. So, this is the reflected wave that goes to the fixed and it gets reflected. So, there is the mirror reflection as well as the inversion. So, that wave comes in and once again gets reflected like this which again gets reflected from the fixed boundary; and now we come back to the initial state where we started with. So now if you calculate which is given here, the total time that it takes; so, in this case, you have a time period of 4 l over c; so which means that the frequency - the circular frequency is 2 pi over the time period, so, which gives us… So, pi c over 2 l; and this is nothing but the fundamental circular frequency of a fixed free string or fixed free bar.

So, we have looked at these two examples of scattering at fixed boundary and scattering at a free boundary. Now, we are going to look at, we are going to generalize this, we are going to look at scattering at a boundary with finite impedance. Now, fixed boundary has infinite impedance, free boundary has zero impedance in a in a certain sense. So, what we are going to now look at is scattering at a boundary with finite impedance.

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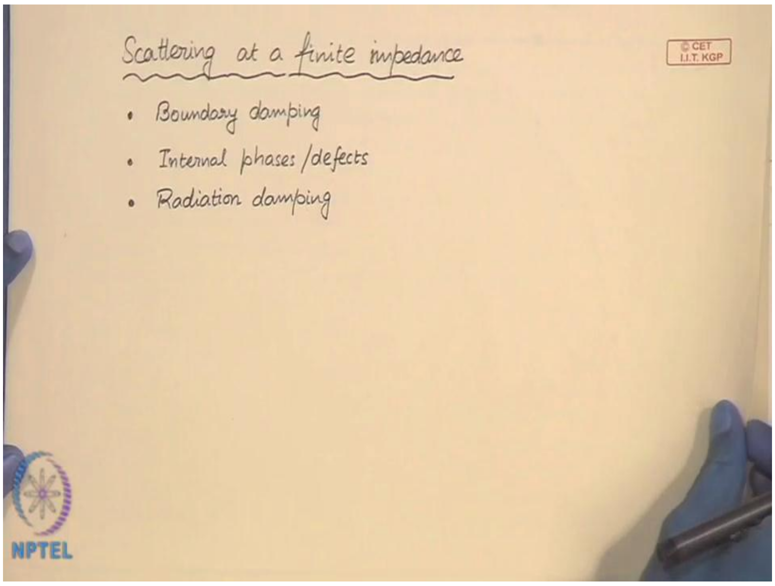

So, this case arises when you have, let us say boundary damping or you have internal phases in a material or even defects then you can have scattering at finite impedance. The third case occurs when you have radiation damping. So, we are going to look at, so these are the situations where you can scattering at at finite impedance.

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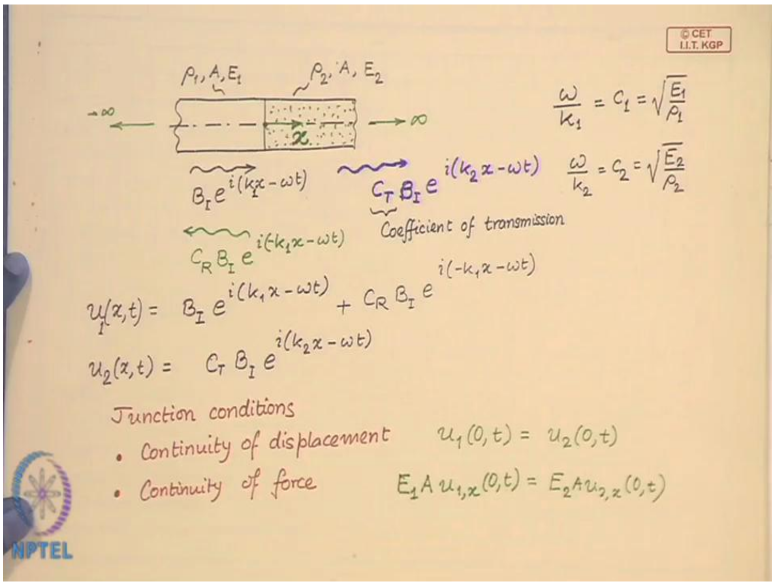

So, we are going to look at these situations with the example of a uniform bar which is made up of two materials. So, we have… So, this is the junction. This material 1 has material and geometric property like this. The material 2 is like this. Now, we consider that there is an incident wave from the left. So, the right propagating harmonic wave which we will… Now, here I am introducing the wave number k_1 in this material, in the material 1, I am introducing the wave number k_1, and then there is a reflected wave which will have the same wave number k_1, only thing is the negative traveling wave. And there is a transmitted wave that propagates the material 2 which we will indicate as… Now, this wave number is k_2, and observe that I have taken the same frequency as you know the frequency has to be the same what we observed from our previous calculations. Even if you consider them as different, you will come to the same conclusion that this frequency must be the same. This comes from the material continuity.

So, you must have this frequency is same, but the wave numbers may be different. Now, as you know these two different materials, the wave number and the frequency and the wave speed in these two materials are related like this. So, C_1 is the speed of axial waves in material 1, which is given by… and similarly for material 2, we have this relation. And here we have this as the coefficient of transmission. So, C_T is the coefficient of transmission, C_R is the coefficient of reflection. Now therefore, the total wave field in the in the in material 1 is given by… So, this is the total wave field in material 1. Similarly, the total wave field in material 2 is nothing but is transmitted wave.

Now, at this junction we must have some junction conditions. So, the first junction condition comes from the continuity of the material. So, this junction should not break. So, the material continuity has to be maintained at the junction. So, there are two junction conditions. So, the first one is material continuity or continuity of displacement, and the second one comes from Newton's third law which says that the force must be continuous equal and opposite of this junction, so continuity of force. So, let us write down these junction conditions. So, displacement junction condition tells us… So, the displacement of material 1 must be equal to the displacement of material 2, and the force continuity can be written as… So, this is the force on the material 1 that must be equal to…

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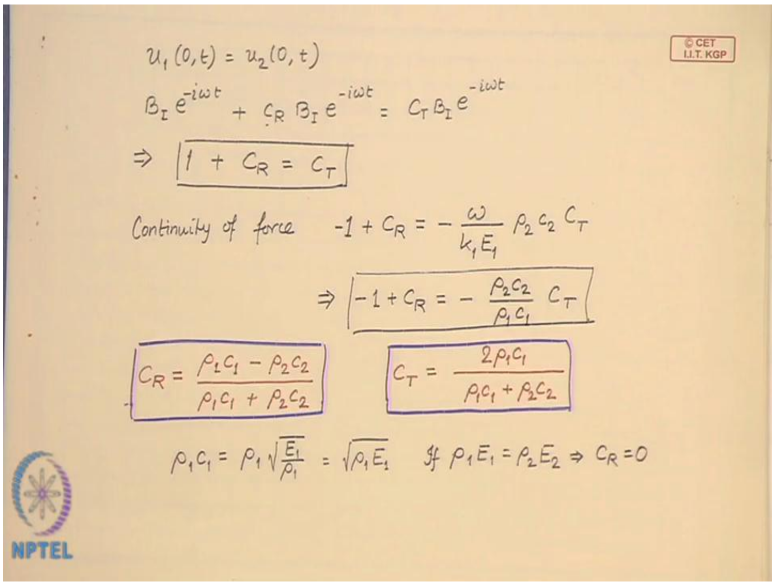

So, these two conditions when we substitute our solution or the wave field in these two junction conditions; so, let us see first the displacement condition. So, if you substitute in u_1. So now, all these frequencies have to be same as we have already assumed. So, this implies… So, this is the condition that we get from the continuity of displacement. Similarly, from the continuity of force, if we do the same calculation then… you can easily obtain… So, this which can be written like this, where we have used this omega over k as C_1, omega over k_1 as C_1 and E as rho1 C square, E as rho1 C square. So, one C square comes extend the denominator. So, this becomes rho1 C_1. So, this is the condition that we obtain from continuity of force. Now, using this two we can where easily calculate the reflection coefficient. So, this is the reflection coefficient and similarly you can calculate the transmission coefficient.

So, we have calculated the reflection and transmission coefficients. Now, let us make some observations based on this. You will find that you see this... So, this transmission coefficient is non-zero. So, which means some part of the incident energy is always going to go in to material 2. Now, this reflection coefficient can be zero, only when rho1 C_1 equals rho2 C_2. Now, this can happen in various ways, one is there is a single material, the material is the same. So, there is no reflection that is very obvious. Now, this rho1 C_1 can be written as… So, if we have two materials for which if rho1 E1 is rho2 E2 then the reflection is zero. But this is hypothetical situation, a theoretical possibility that this product is same for two materials, but that may not occur in nature. So, when we have this junction of two materials, there is some transmission, there is some reflection. Now, this case is an important example for certain applications. For example, when we do ultrasonic testing of materials, then we have ultrasonic transducer which we put on samples; and we expect that this ultrasonic wave will propagate within our sample completely; I mean there should not be any reflection at the transducer sample interface. So, let us look at this. So, this is same. So, we have this transducer sample interface and we do not want or we want that all the waves that we generate here should enter in to the sample. There should not be any reflection right from the interface. So, what we usually do in this case is we put thin film of the third material. So, the scheme is like this, here we have a third material at this interface. So, this is material 1, this is material 2 and in this we have the third material, and we can adjust the thickness of the material. So that, there is only transmission and no reflection; we can adjust the thickness, so that there is no reflection. So, there is complete transmission. So, this is very useful for such applications.

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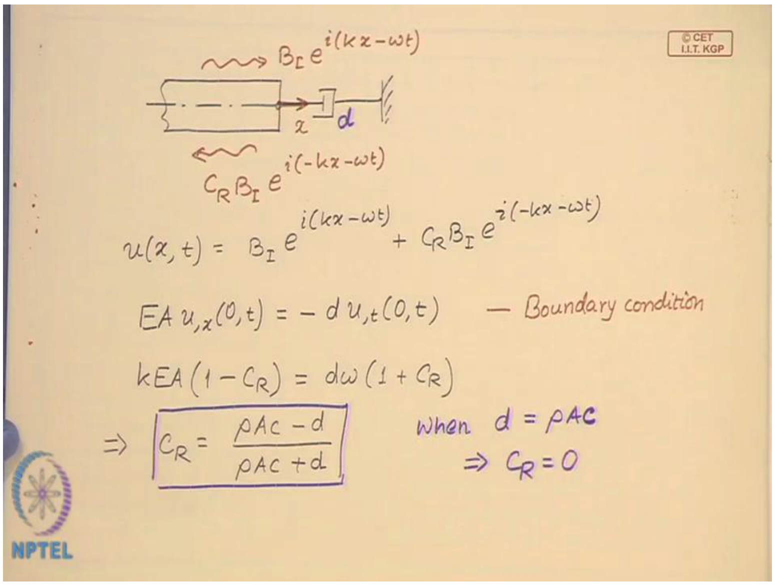

Now, let us consider a second example of scattering an impedance. This example we have seen before. So, this is the bar - a semi-infinite bar with boundary damping. So, once again we have an incident wave, reflected wave and we consider that this damper has damping coefficient d. So, the total wave field is given like this. Now the boundary condition is… So, this is our boundary condition at x is equal to zero. Now, when we put this expression the wave field in the bar in this boundary condition, you can easily arrive at this expression. So, from here we can calculate the reflection coefficient like this.

Now, we observe some interesting things. When d, the damping coefficient of this dashpot equals rho A C, the reflection coefficient goes to zero, which means there is no reflection in the bar. So, the wave comes and gets completely observed in the damper. Now, what is this value - this special value of d? As you if you re-call then the impedance of a semi infinite bar is also the rho A C. So, what this wave feels is that there is semi infinite bar attached at this point. So, as it this bar has been extended up to infinity. So, the whole wave goes into that. So, it gets completely observed at this boundary for this very special value of the damping coefficient. So, this is the reason, why when we did this modal analysis or when we solve the Eigen value problem for bar with boundary damping, we found that for very special value of this dashpot, there are no Eigen frequencies or Eigen values. So, now we understand that the Eigen frequencies or Eigen values, the concept of modes, this occurs only when we are finite system. But now, this special value of damping has almost made the bar as infinite in positive x direction.

So, the concept of Eigen frequencies do not exist and the kind of solution that we assume for modal analysis does not hold. So, in such particular special values of this damper, we have a perfectly absorbing boundary, and which is sometimes very important in application. For example, if we want to observe sound in a room then you must design your the wall dampers or the observing material on the wall. So, that it becomes perfect observer.

So, what we have looked at in this lecture, we discussed about scattering of waves at fixed at fixed boundary, at a free boundary and at interfaces with finite impedance, and we looked at some interesting results, and made some interesting observation based on what we discuss today. So, with that we conclude this lecture.

Keywords: traveling waves, wave reflection and transmission, boundary damping.