Vibrations of Structures

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Lecture No. #02
Transverse Vibrations of Strings-II

In the second part of our discussions on Transverse Vibrations of Strings, we are going to look at strings with interaction.

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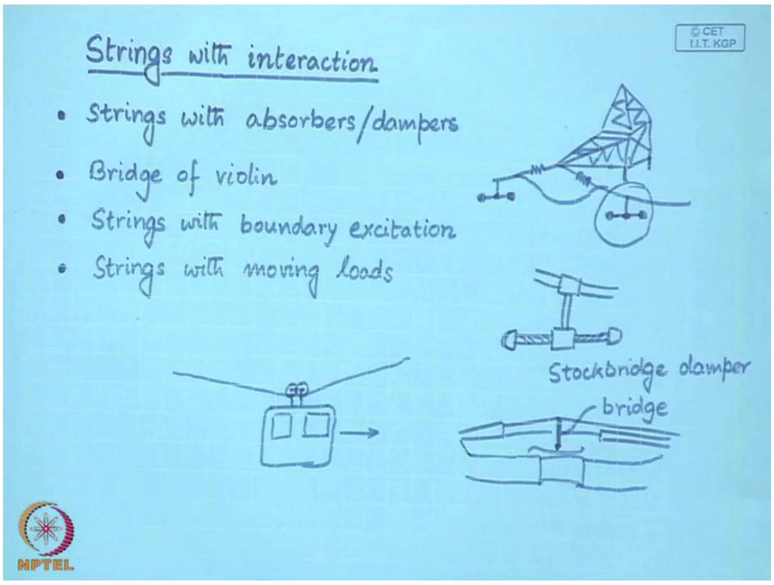

So, this term interaction, by the term interaction I would mean that the string is in contact or is connected to another system, or an actuator, or it could be a passive system like, an absorber or a damper. So, let us look at some examples of strings with interaction. So, as I mentioned just now, we may have strings interacting with absorbers or dampers. So, such strings are found in for example, in high tension cables, so this roughly looks like this. So, here is the pylon and from so, you must have seen such structures in high tension, I mean carrying high tension cables.

So, if you look carefully, at around near the support point, you may be able to see something hanging like this. So, this is a string, this is a span of a string, and near the support point, you have this kind of attachments. So, if you look at these devices closely, they are mounted on the on this cable; and they look like this; here are some masses, and this is the cable, and here there is a multi strand cable, which acts like a spring and a damper, this is called a stock bridge damper. So this is an example of a string interacting with a damper. Such strings are also found in, for example, the piano. In the piano, you have dampers, which stock or damp out the vibration of strings. So in order to understand the efficacy of this stock bridge damper, which is essentially put to damp out the vibrations, wind induced vibrations of high tension cables; so to understand the effect of such external devices, so you have to analyze the strings along with these absorbers or dampers. Then, interaction of another system is also found in, for example, a bridge of a violin. So, the use of the bridge is to transmit the vibrations of the strings to the soundboard or the main body of the violin, so that it can be amplified. So, if see a violin; so this is the bridge, which transmits it to the body of the violin. Then you can have strings which are connected to an actuator, which may excite within the span of the string or may be at the boundary, or it could be another system, which is vibrating and is exciting the string at the boundary. Then you can have strings with moving loads. So, typical example would be a cable car, something like this. So, here this cable car will be moving on a taut string. So we see that there are various examples, in which a string may have to interact with an external discrete or even sometimes a continuous system, another system. So we need to model and analyze such strings.

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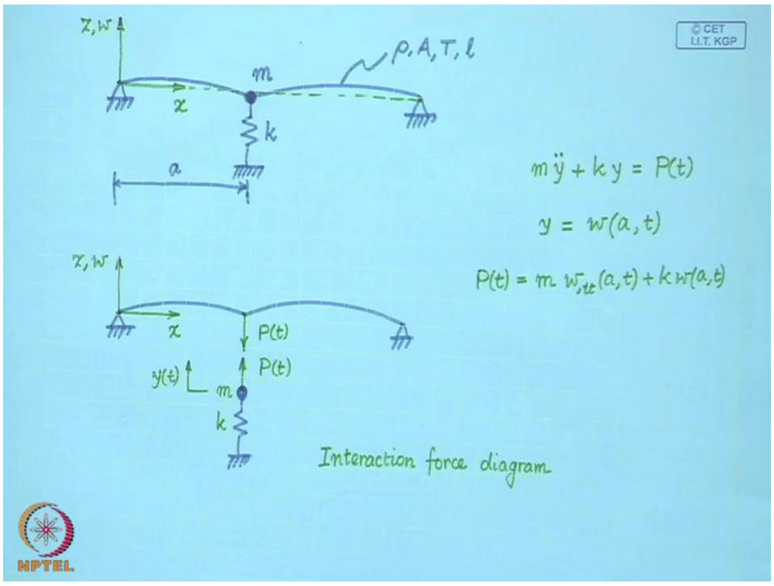

So, let us start with a simple example. So let us consider here, string made of a material of density rho, area of cross section A, under a uniform tension and length l; we consider an oscillator connected to the string at a distance a from the origin. Let the mass of this oscillator be m and the stiffness be k. So, when I have to analyze this system, what I will do is, I will draw what is known as an interaction force diagram. So, this looks like this, in which, I have separated the oscillator from the string. So in place of this, I will introduce the interaction force, let say P, which will be a function of the time. So this, we will call the interaction force diagram. Now let us look at the oscillator; let us first look at the oscillator. So, I know that the equation of motion, so if I consider the coordinate of the mass point as y, then I can very easily write m times y double dot plus k y must be equal to the force; now this is the same force that acts on the string at a location a. So here, since the mass has to be coupled to the string, we have this y given by w at a at any time t. So therefore, if I make this substitution, then I can write… So, this is the force, the negative of this force, actually acts on the string.

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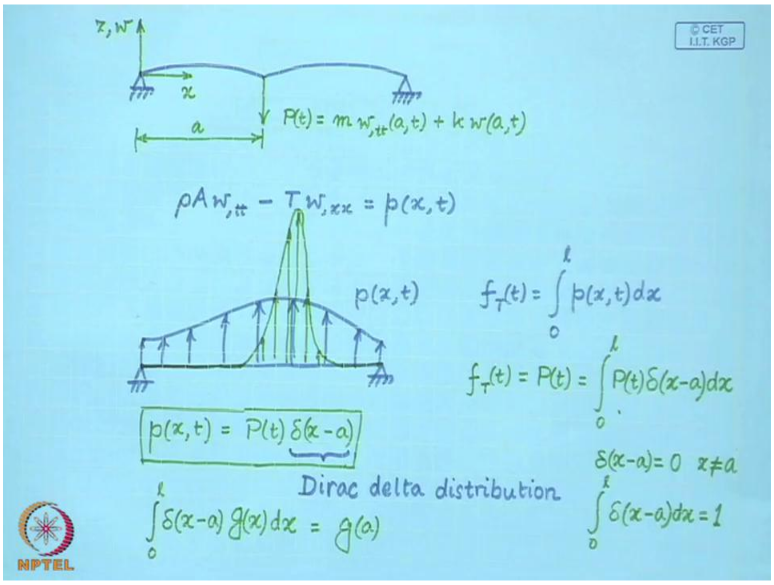

So this is our picture; so this is concentrated force acting at a location a. Now remember that when we wrote out the equation of motion, the transverse motion of a string; so this equation was… So, this is a uniform string with no axial force, so this was our equation of motion; where this small p (x, t) is the force distribution at the force per unit length. Now here, what we have shown is a concentrated force acting at x is goes to a. So, somehow we have to now, represent this concentrated force as a distribution. So, how do we represent a concentrated force as a distribution, this is what the question is. So, let us look at distributed forces. So suppose this is a distributed force on a string, let say.

Now what is the total force? So, this total force is obtained as by this integral over the domain of the string of the distributed force, because this is force per unit length. So, that multiplied by this infinitesimal length and integrated from 0 to l would give as the distributed, the total force acting on the string. Now if my distribution gradually becomes narrower and narrower, but retaining the same total force.

So the total force is fixed, we know this, in our example that we are considering at this point, we know the total force that is acting. Now this distribution happens to be 0 from here to here, and here to here, and it is non-zero only on a finite region of the string, not on the full domain, so its non-zero only on a finite region. So, you can imagine as this non-zero region shrinks to zero, as in the case of a concentrated force here, that I have shown, retaining the same total force, then this is going to shrink to 0, but its amplitude the magnitude here is going to blow up. But the integral under this is be will be finite and that will be given by this quantity.

So, how do you achieve such a thing. So, the total force must be equal to P(t) that we have here, but that must come by integrating out a distribution. So, let me write it like this that distribution written like this. So, this is our distribution; where I have introduced, this is now a distribution delta (x-a), which is known as the Dirac delta distribution. Now, I mean, we can very easily derive the properties of this Dirac delta distribution from, what we expect this distribution to do. First of all this distribution must be 0 everywhere on the string except at x equals to a.

So, this distribution must be 0 everywhere on the string except at x equal to a, and what happens at x equal to a, as you realize that this is going to blow up, so we would not write that value. But what we are going to do is we are going to write from this integral, a second property of this Dirac delta distribution which is this. So, the integral under this distribution over the domain of the string is 1; so, the area under this distribution is 1.

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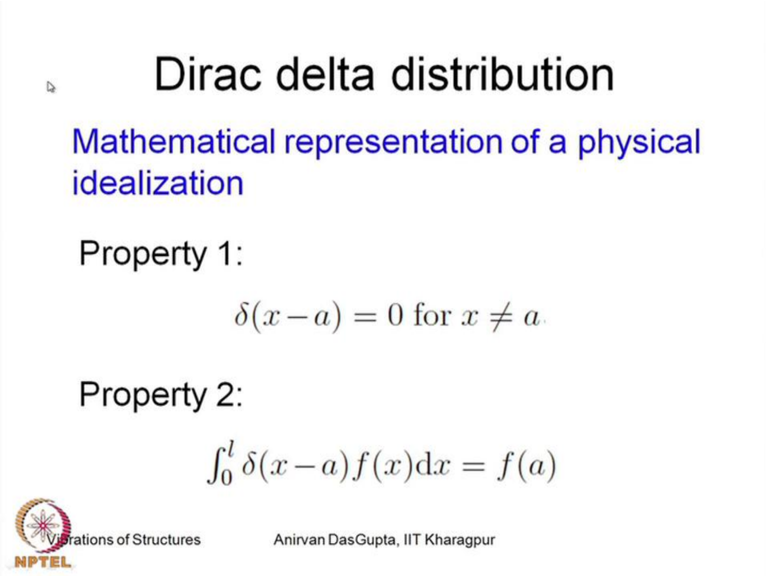

So, what this distribution achieves this Dirac delta distribution achieves is that it gives us mathematical representation of a physical idealization. So, what is the physical idealization? Now in macroscopic world, you do not have concentrated forces, that means forces acting over 0 area. You always have forces acting over distribution, over distributed area. So this, but then what happens is for example, in the violin string on the bridge, the portion on the bridge is much much smaller than the length of the string or the damper that damps out the vibration in a piano, the contact may be very small compared to the total length of the string. In such cases, we can have physical idealization that this is almost a point contact. So this Dirac delta distribution gives us the mathematical description of this idealization. So, further generalization of this integral property is…

So this is the further generalization of this integral property of Dirac delta distributions. So, here I have a put together these properties of the Dirac delta distribution that we just now discussed. Now using this, let us get back to our string.

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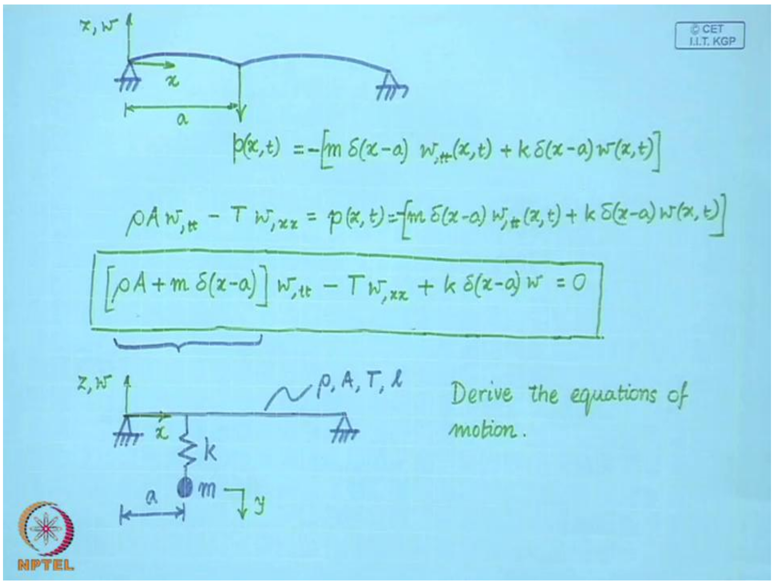

So now, I will represent this distribution, this distributed the force distribution on this string as the mass times the Dirac delta distribution at x equal to a. Now here, I have written the acceleration at x equal to a, but from the previous property that I just now discuss, I can write it like this. So, this is the force distribution, because of a concentrated force at x equal to a; and now I can write out my equation of motion, which on slight rearrangement, so this actually turns out to be… So, when I write the distribution on the string, so this comes with a negative sign, because of the convention that the upward is positive.

So this is, finally, the equation of motion of the string with an oscillator attached to it. Now you can see from this term, this term is nothing but the mass per unit length. So, rho A was because of the string, and this is because of the oscillator. Similarly now you have a stiffness coming into the system at x equal to a. So, this completes the description of a string interacting with an oscillator, which is coupled to it. As an exercise, you can try out as an example like this, in which a we have string made of material of density rho, area of cross section constant which is A, under a tension T and having the length l, and there is a an absorber, an absorber connected to the string, the absorber is the mass m and stiffness k, attached at x equal to a. So, derive the equations of motion. So, here you have to note that there will be two equations, one for the string, the other is for the absorber, because the absorber is now a separate coordinate the location of the mass of the absorber is the separate coordinate. So, you can try out deriving the equations of motion of this system.

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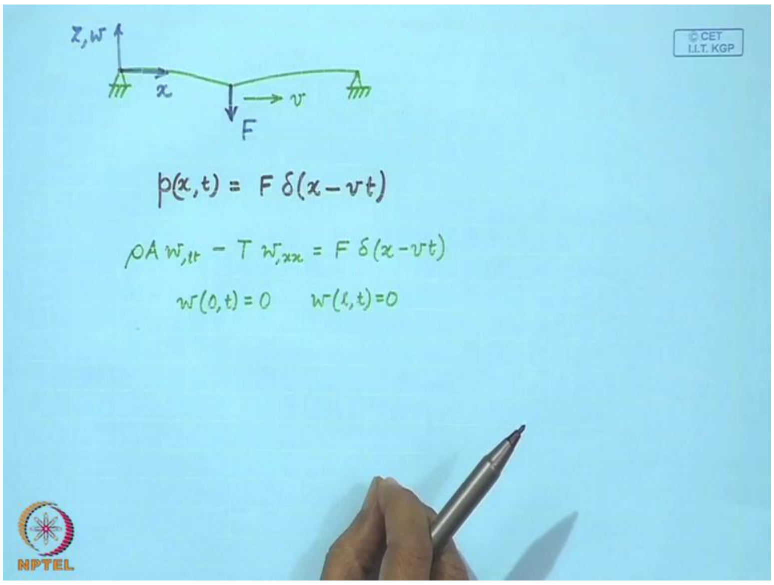

Now let us look at the cable car problem. So, in the cable car problem, what you have, in the first approximation, you may consider this to be a constant force, this is an approximation. So, this is not exactly the cable car problem, but something analogous to it; and this constant concentrated force is moving at a speed V on the string. So, if you have this situation, the force, concentrated force is moving at a constant speed V on the string. So, this force again has to be represented as a distribution. The force distribution, which is now using the Dirac delta function or distribution, you can very easily write this force distribution, now on the string is F, which is the magnitude of the force times the distribution, but this distribution is shifting.

So, I am assuming that a time t equal to 0 and the force was at the origin that means at x equal to 0. So, this is the representation of the force distribution on the string, because of a cable car moving on the string. So, therefore, we can write the equation of motion for the cable car problem in this form; and with every problem of course, comes the boundary conditions, so, as we have discussed, so geometric boundary conditions in this case; now you can generalize this, suppose you have a mass moving, a beat moving on a string; then you find out the corresponding interaction force, because now the beat will have the acceleration, which comes because of its interaction with the string, and that will produce a varying force because of inertia of the beat. So, you can generalize this to analyze more and more realistic situations.

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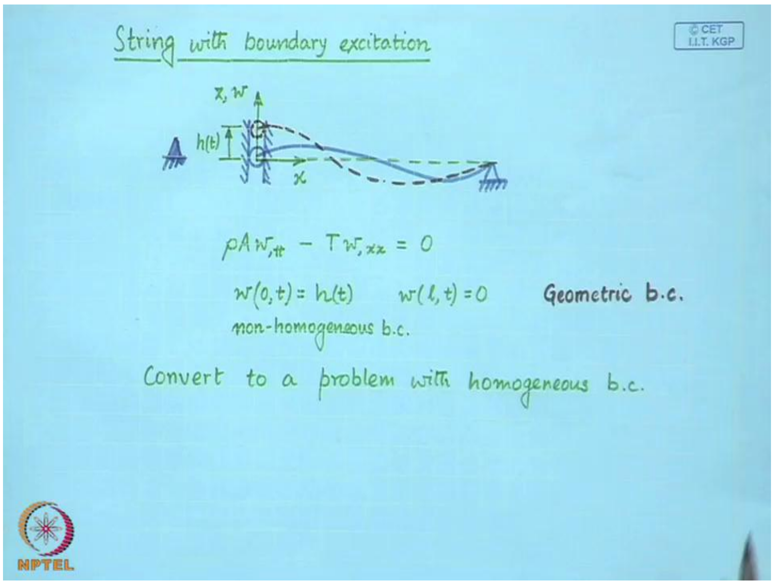

Next let us look at an example of a string with boundary excitation. So, we will put a string with which is a free to slide on this left boundary I will draw a displaced configuration of the string. So, we will assume that this string is being driven by a robust actuator, which actually gives the displacement condition on the boundary. So, this string is been being excited by a system or an actuator, which is robustly driving the left boundary. So, the equation of motion for the string, which is made of a material of density rho, area of cross section A, and a tension T, so, can be written in this manner. The boundary conditions may be written out at the left boundary at x equal to 0, the string is given a displacement, which is the function of time; and the right boundary, the transverse displacement is 0.

So, first of all in this case, both the boundary conditions, at both the boundaries, we have a condition on the displacement of the string. So, both these boundary conditions are geometric boundary conditions. Now, here we have a boundary which is being excited, I mean, it is the displacement at this boundary, this sliding boundary is a function of time. Now when we have such a situation, analysis may become a little difficult. So, what we want to see is whether this problem, which is a boundary excitation problem, a string with boundary excitation, whether this can be converted to homogenous boundary conditions? So, this boundary conditions for example, is non-homogenous.

So, this is a non-homogenous boundary condition. So, the question arises whether we can convert a problem with non-homogenous boundary condition like this, to a problem with homogenous boundary condition, because that in many ways simplifies the analysis of the problem. So, so we are interested to convert this problem to a problem with homogenous boundary conditions. So, there are many ways of actually converting this problem with non-homogenous boundary conditions to homogenous boundary condition. We will follow a method which relies on variable transformation.

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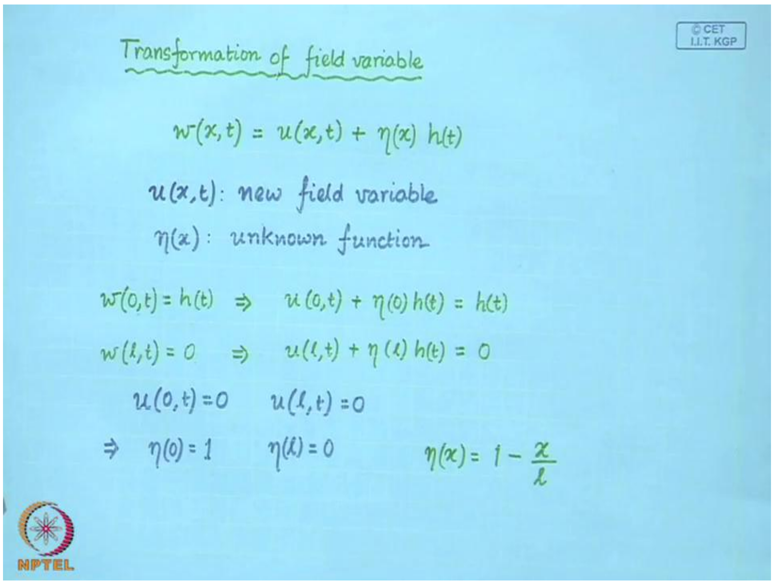

So, we want to find transformation of the field variable, which was w in this case, we want to transform it to some in to another variable; such that the problem can be converted to a problem with homogenous boundary conditions. So, let us look at this transformation. So, w (x, t) was our field variable. So, we introduce a new field variable and unknown function at this point eta(x) multiplied by h(t), this was the this was the motion of the boundary, given motion of the boundary. So, here u is the new field variable and eta is the unknown function, which we will find out.

Now if I substitute this transformation into the boundary conditions, so what were the boundary conditions? So, this is for the boundary condition at the left end. So that would then imply this condition; on the right boundary, the condition is w at (l, t) is 0. So, that would imply…

So, we have these two conditions now. Now we want to have homogeneous boundary conditions on the new field variable, which means we want to have and these two condition therefore, from here we will obtain conditions on eta, which are eta(0) must be 1, and eta(l) must be 0. Now there can be different choices for eta, which satisfy these conditions, the simplest choice would be… This could be the simplest choice; of course, you can have powers of this function. For example, (1-x/l)^2 can also satisfy this, but we will make one choice at this point, and we will discuss what are the implications.

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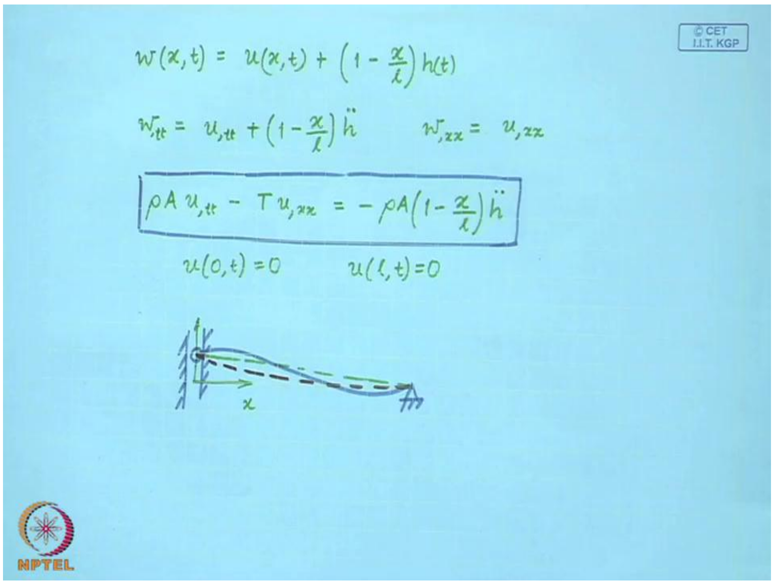

So, if you put this back into your transformation, you have this expression for our field variable w in terms of u, and if you substitute this in the equations of motions. So, we need to calculate the double derivative with time, so that will turn out to be…; and we also have double derivative with respect to space, which will be reduced only u,xx. So, therefore, our equation with this substitution becomes this along with the boundary conditions. So, we have a system with homogenous boundary conditions, but now our equation of motion has become non-homogeneous.

So, if you look carefully, this term is the result of the transformation and what this transformation does is actually to take us to a non-inertial frame. So, if you, with the choice of eta as I have done, we have shifted to a frame, which is indicated by this green line. So, this is moving this is all, I mean if h is periodic, then or even otherwise, this line would be moving, and this will be a non-inertial frame, in which we are expressing the equation of motion. So, as soon as we go to a non-inertial frame we have the inertial force, which is essentially coming on the right hand side of this equation of motion and making it non-homogeneous.

Now if you have a different form of eta, then what happens is, for example, if you take (1-x/l)^2, then is nothing but the transformation which will take to this kind of a frame, which is quadratic, but is still becomes non-homogeneous, I mean still remains non-homogeneous. So, your equation of motion becomes non-homogeneous, because of the inertial force and your boundary conditions become homogeneous. So, with this we come to the end of this discussion on transverse vibrations of strings. So, we have discussed today strings with interaction with external oscillators and external systems, and we have seen how to represent concentrated forces, how to represent the interaction and how to convert a problem with non-homogeneous boundary conditions to a problem with homogeneous boundary conditions.

Keyword: string with concentrated force, string with moving load, string with boundary excitation, non-homogeneous boundary conditions.