Vibrations of Structures

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Lecture No. # 19

Harmonic Waves and Energetics of Wave Motion

So, we have been discussing about the wave propagation solution of systems in one dimension governed by the wave equation.

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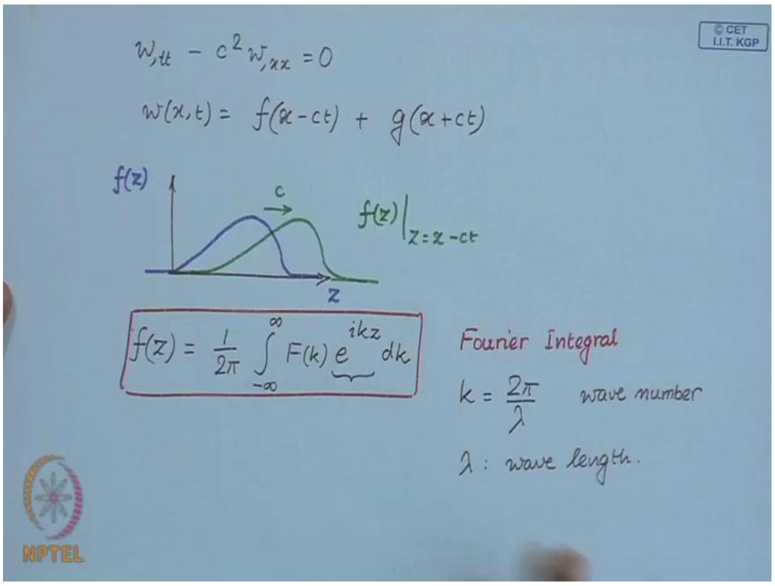

So, we have looked in one of the previous lectures that the solution of the wave equation can be written in terms of… the general solution of the wave equation can be written in terms of these propagating waves, one in the positive x direction, other in the negative x direction. So, let us say that we have a positive traveling pulse. So positive traveling pulse is represented by… So, this pulse travels as time progress; this pulse travels in the positive direction at a speed c. So, let us look at this form of the pulse f(z). Now, we know from the theory of Fourier transform that a pulse of this form may be represented in terms of the Fourier integral. So, this we called the Fourier integral. So, a pulse like this may be represented in terms of… this integral you can see, you can interpret this integral as a super position of harmonic pulses, this can be written in terms of cosine and sine of k z. So, therefore a pulse like this is a super position of sines and cosines and represented in terms of this integral, where this k is known as the wave number and lambda is the wavelength.

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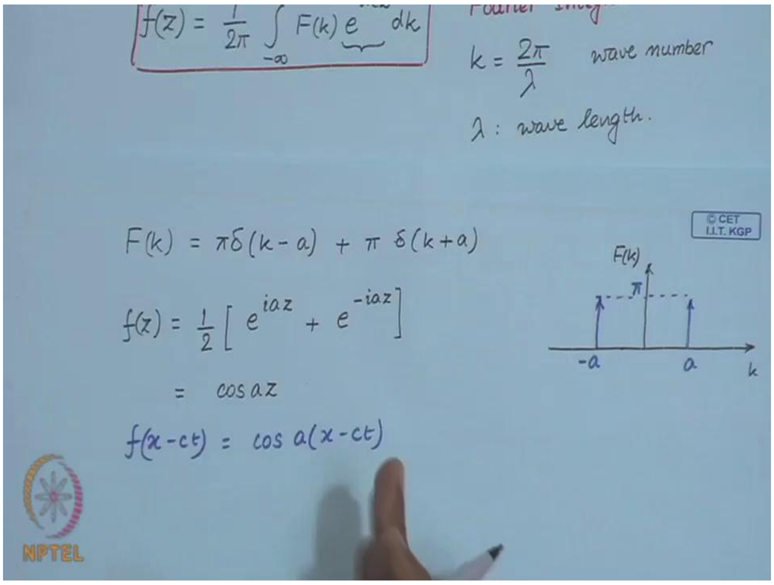

So, if you consider an example, simple example; suppose F(k) is nothing but say pi times delta… So, if you consider it like this and this let us say you have plus… So, F(k) is the function of the wave number. So, these are Dirac’s delta functions or distributions represented with this arrow. So, if you substitute in here, now when you have the Dirac’s delta at a, so k gets replaced by a, and this pi therefore, what you have, here it gets replaced by minus of a, and this is nothing but cosine a z. So, if you want to have a propagating cosine pulse or cosine wave, so, this is the representation; and by so this is you can say that the spectrum of the cosine function in the Fourier space. Like this you can represent different kinds of functions; and therefore, you can correspondingly find their propagating representation, by replacing this variable by (x-ct) or (x+ct) as the case might be.

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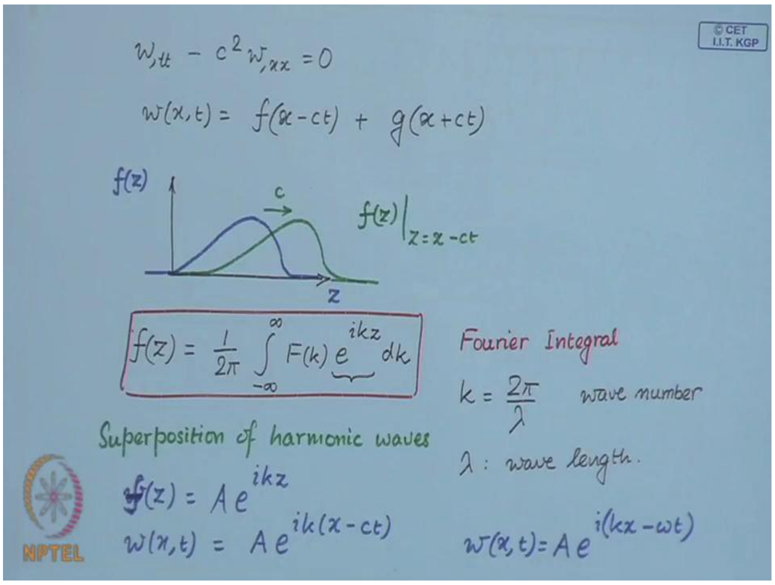

So, finally what we understand by this integral is that we can consider any wave pulse as a super position of harmonic waves. So, these are superposition of harmonic waves. Now since we are dealing with linear systems, a super position of harmonic waves if that represents a pulse, therefore if we study the propagation of the harmonic wave, then we can equivalently construct our the propagation of our pulse by just summing up or superposing all these harmonic individual harmonic waves. So, our problem is somewhat simplified that we do not have to deal with complex wave forms, if we can study the propagation of a wave of this form… So, w… so this is may be this I will call f, so this is we will actually represent this as… where omega is the frequency. Now we will see how this comes about.

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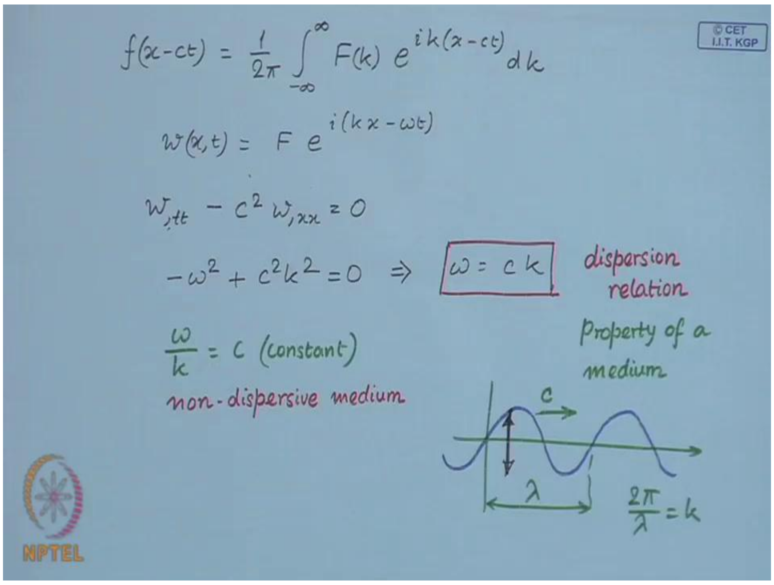

So, we can write… so this is the superposition of propagating harmonic waves and what we are going to do is as I just now indicated, we are going to write it like this. So here k is the wave number, omega is the circular frequency. So, this is k x, and k c - I am writing as omega. So, how do I come to this form? So, if you substitute this solution form in the wave equation, then you see that this choice of k c equal to omega satisfies the wave equation. So… and if I substitute this solution form, then I will have…

So, this is double derivative with respect to time, so this is minus omega square, and here this is double derivative with respect to space that will be minus k square. So, this will be plus c square k square. So, that will imply… So the relation between the circular frequency of the wave and the wave number is given by this relation. This relation is known as the dispersion relation; this is known as the dispersion relation. So, it is the relation between the circular frequency of the wave and the wave number. Now, this dispersion relation is the property of the medium. Say for example, in the case of the string, this is the dispersion relation, but later on as when we will discuss beams, we will find the dispersion relation is different. So, this dispersion relation characterizes a medium or a material.

So, a dispersion relation which is such that omega over k is a constant, is not a function of k or omega, is known as a non dispersive medium. So, what is this, I mean physical significance of this dispersion relation; so, let us understand briefly. So, what we have is an infinite system of infinite extent let us a string or a bar, which is of infinite extent, and there is a propagation of harmonic wave in the string or the bar, in the medium as we will commonly call this. So, when harmonic wave propagates, a harmonic wave is associated with the wave number and a frequency. So, wave number is related to the length of the wave, the wavelength.

So, let us say, suppose this is the wave that is propagating; this is the wave that is propagating. So, there is a definite length lambda, and 2 pi over lambda is the wave number, and there is a definite frequency associated. So, if you look at a particular point of the medium, let us say this point; then the point of the medium is oscillating at this location as the wave moves. So, there is a definite frequency associated with this oscillation.

So, these are the two things that are associated with the propagating harmonic wave. Now what the dispersion relation tells us that these two cannot be independent, these two cannot be independent. So, the wave number or the wavelength and the frequency of the wave, they are related and that relation is a property of the medium. So, the medium decides what should be the relation between the wave number and frequency of a traveling harmonic wave. Now if this relation is such that this omega over k as you see here is a constant; it is the not a function of either omega or k, as you have here, then the medium is known as the non dispersive medium.

Now let us understand what is dispersion briefly; so, you see this c as you know, is the speed of a propagation of this harmonic wave or any wave pulse in the kind of the medium that we are discussing, which are governed by the wave equation. So, the media which are governed by the wave equation are all non dispersive medium; now in that c represents the speed of the waves. So, if I look at this crest the speed at which this crest moves. Now if that is independent of the wavelength or the frequency, then something interesting happens. So, what happens is all waves that means, waves of all wavelength as I have discussed that general pulse can be caught off as a super position of harmonic waves of different wave number, this is superposition of harmonic waves of the different wave number, which means it is a super position of harmonic waves of different frequency as well, because frequency and wave number are related for a medium.

So, all these waves for of different wave numbers, they are traveling at the same speed in independent of the wave number or the frequency. So, they are all traveling at the same speed. Therefore, what happens is a wave pulse that has a certain shape at this point as it moves, it retains its shape. So, as time progresses and as the pulse moves, it retains its shape, because all waves that are superposing to produce this pulse, they all moving together, but in a non dispersive medium, this does not happen. Some waves are certain wavelength or wave number they travel at different speeds than harmonic waves of another wave number or another wave length. So, what happens is this pulse loses its shapes. So, it possibly flattens out as it moves. So, in a non dispersive medium, the pulse or the wave form it retains its shape, as it moves in a in a dispersive medium, it will lose its shape.

Now, air is the non dispersive medium. So, what I am speaking the sound that I am producing is reaching you with no distortion. Had it been a dispersive medium, then it would have been very different. So, that is dispersion. So, what we will be discussing or restricting ourselves too is non dispersion medium for the moment. Now we have this speed c, which is the ratio of the frequency of the harmonic wave and the wave number is the wave speed as we have seen, we qualified little further, we call it the phase speed and we will come across another speed as we progress in our discussions, when we come to dispersion medium. So, this is the phase velocity or phase speed.

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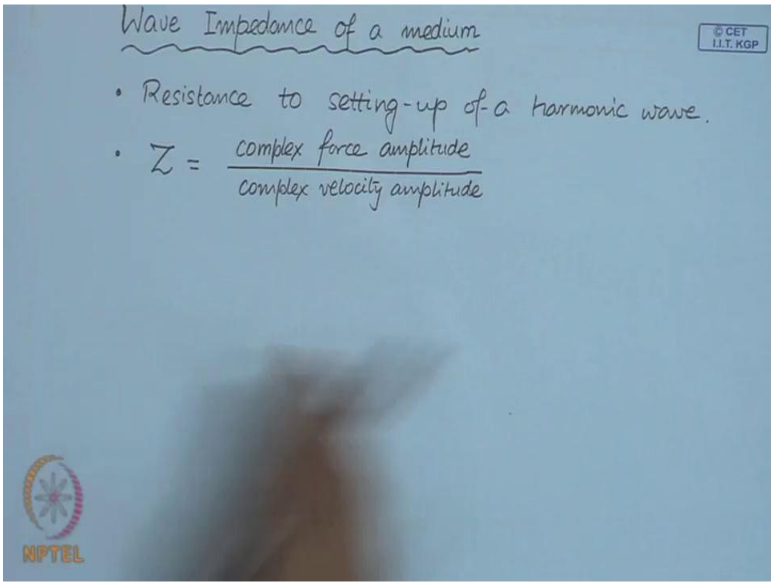

Now the next important thing that we are going to discuss is wave impedance of a medium. So, when we want to set up the harmonic wave, so now, we will be always talking in terms of harmonic waves; when we want to set up the harmonic wave in a medium, let us say a string, we have a taut string and you want to produce a disturbance or wave in that string, the harmonic wave in that string. So, you will take one end of the string, which may be connected at distance, the other end is connected at distance, and you produce a wave; you will be met with a resistance, the medium on the string will resist this disturbance. Why it will resist? Because you are going to pump in kinetic and potential energy; so you must do work; so there must be a resistances to this motion.

So, this resistance is resistance to setting up of this harmonic wave, these we roughly term as impedance. We will make the exact definition as we proceed. So, impedance is the resistance offered by a medium to setting up of harmonic waves in a medium, so that, we will called as the wave impedance of a medium. So, this is the resistance to setting up of a harmonic wave, the mathematically this is written in terms of… So, impedance is denoted as Z is the ratio of the complex force amplitude part divided by the complex velocity. So, remember that we are discussing everything in terms of complex representations. So, we will represent this everything, so force velocity everything in terms of complex numbers.

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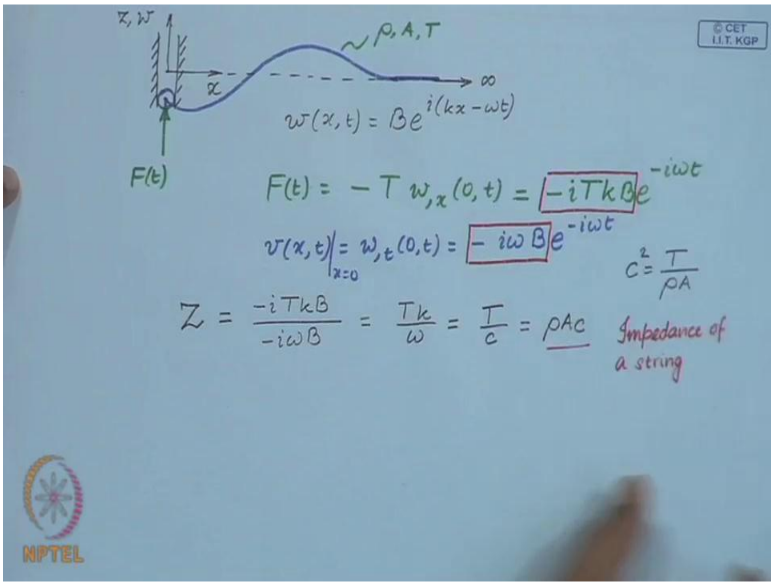

So, to take an example let us consider a string. So, suppose you have semi-infinite string. So, these other boundary is far off, which we idealized as infinity. So, the other boundary where it is phased is at infinity. There is tension in the string, and this end which is the frictionless sliding end is being forced with harmonic force let us say. So, this force must be equal to written in terms of the tension in the string; so this force must be equal to negative of tension times the slope at zero. So, this is x and if you considered that the harmonic wave propagating with amplitude B, propagating in the… So, what you set up, is the positive travelling wave; then this is obvious, because of as you disturb, the disturbance is going to move to the right.

So, you set up a positive propagating wave, and since there is no other boundary and no other disturbances source at infinity, at anywhere in the string other than this point that will not be any negative traveling waves. So, there is only a positive traveling harmonic wave set up in this string, and this will be true if of course, there are no defects, no intrusion, no mass elements etc. in the string. So, this string is uniform. So, this therefore, if you substitute this this solution, what you obtain is so I have put is equal to zero; so that is the force. Now the velocity at x equal to zero is given by… So, the complex force amplitude is this part and the complex velocity amplitude is this. So, we define impedance as... Now omega over k is the phase speed, the speed of the harmonic wave in the string and using the definition of the speed once again. So, you see c is or c square is T over rho A for this string. So, I will replace this tension in terms of density and area of cross section of the string. So, that will give us… So, that is the impedance of a string.

Now, similarly if you consider a bar, a semi-infinite bar to which you apply a force, a harmonic force. So, for the boundary condition… So, when you apply a harmonic force here, you assume that the harmonic waves are set up in the bar, which can be represented as this. So, this is the… So, we have positive traveling harmonic waves in the bar, and this turns out to be putting x equals to zero, and similarly you can write the velocity. Once again so here I have negative sign and here also have this sign. So, this is the complex force amplitude, this is the complex velocity amplitude. Therefore, once again I can define the impedance of a semi-infinite bar as EA… once again omega over k c and EA can be the speed of waves in a in a bar, can be written like this. So, what we have is… So I eliminate this E as rho times c square here and here I have the impedance of a bar. In the case of a bar, this is axial waves, and these are essentially pressure of stress waves, and you can define what is known as the specific impedance.

So, specific impedance is this impedance over area; so its impedance per unit area. So, this is the ratio of the complex stress amplitude divided by the velocity amplitude. So, this is in terms of stress. Now in the two case that we have studied this impedance appears as real number; these are real quantities, but in general impedance can be complex, is a complex number and dependent on. So, its complex, and it is can be a function of frequency. So, in general, it can be complex, and it can be a function of frequency. Now as I mentioned that when you disturb a string, so if you take semi-infinite string and the free end, which is the sliding end, you disturb it, you give the harmonic wave, what you will be setting up is the traveling harmonic wave.

Now as I said that you will feel a resistance, when you want to set up these harmonic waves. So, why do we have this resistance, because there was some undisturbed string we pumping in energy, we are pumping in energy into those parts of the strings. So, as I start moving as you have seen that for end of the string may be still undisturbed. So, and when the disturbance reaches this, this part the point up to which the disturbances has reached, up to that part thus the string has kinetic and potential energy. So, you see I am pumping in energy into the string; and this is energy is therefore, flowing as the disturbances propagates in the string. So, this wave, the harmonic wave must be carrying energy, a traveling harmonic wave must be carrying energy vector, in order to disturb the yet another disturb regions of portion of the string. So, let us try to calculate this flow of energy or the power that flows as you pump in or as you disturb a string.

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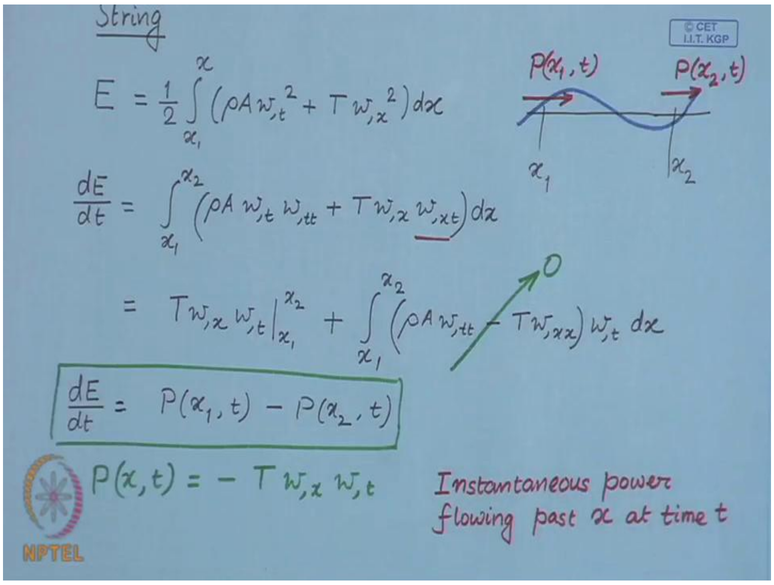

So, let us write down for a string or, let us take the example of a string. For a string, the total energy, the energy let us say; so, now this is the infinite string or semi-infinite string, let us calculate the energy of the string. So, you have this string; let us calculate the energy of the string in this region from x_1 to x_2. So, this consists of the kinetic energy and I have put this half common an outside, and this is the potential energy. So, the total mechanical energy in this region or in this portion the string is given by this E. Now let us try to calculate the rate of change of energy of this portion is string. So, once you take time derivative, so we have… Now to bring this expression in a convenient form, I will integrate by parts this term. So, I will integrate by parts taking this as the second function.

So, then I can write, so the first function integral, the spatial integral of the second function, and then this integral; now when we do the integral of this functions, spatial integrals, so, I will have del w/del t here and I also have del w/del t, which I can take outside and there will be a negative sign and spatial derivative of this function that is what we have. Now this being the equation of motion, so we assume that this is being governed by the wave equation of the equation of the motion of the string, so this must be satisfied, so this is zero. So, what you have left with this. I will write this as… where I have defined…. So, the rate of change of energy of this segment of the string is given by this. So, this must have dimension of power. So, time derivative of energy is power. So, this must be power. So, rate of change of energy of this portion of the string, this segment of the string must be equal to the power that comes at in through x_1 in to this region minus the power that leave this region through the point x_2. So, this is nothing but the balance of the energy for the string; there is no other loss like radiation or damping in this portion.

So, energy must be come in through this or power must be come in through this boundary at x_1 and it must leave this boundary at x_2. So, the rate of change of energy must be the amount of power that is coming in minus amount of power that is leaving this segment of the string. This therefore, is the instantaneous power, flowing. So at time t, if you look at the time instant t, then this is the expression of the instantaneous power that is flowing past the point x, any point x at that time instant t. So, this equation gives us the balance of energy. So, how the energy is changing of the segment, is changing with time; suppose energy is constant then this to must balance; that means, the power coming in must balance the power is going out; in that case the energy of this portion of the string is not going to change with time. Now let us do a little calculation for a wave.

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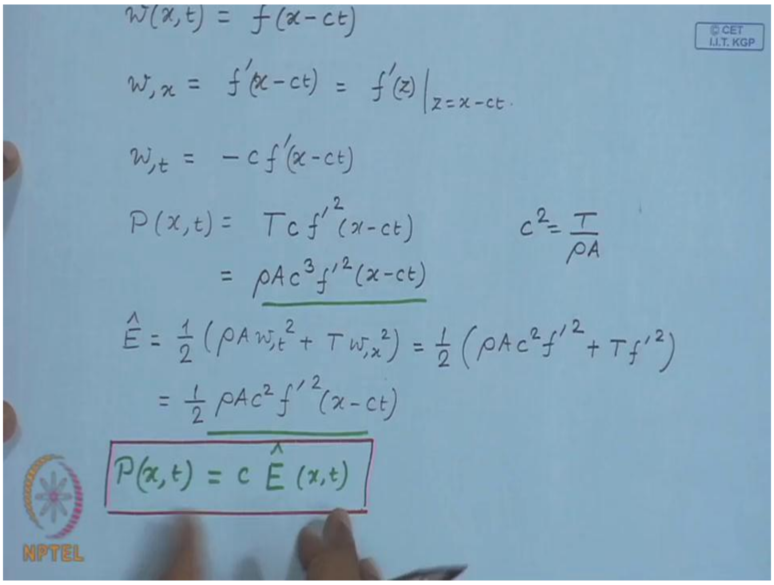

So, let the solution of the motion of the string be given by… So, we have a positive traveling wave in the string. So, del w/del x is given by… Where this prime indicate derivative with respect to the argument, the prime represents the derivative with respect to the argument. So, it is nothing but this… Similarly, del w/del t is given by this expression and therefore, this instantaneous power, the expression of instantaneous power is... and this can be rewritten as writing T, so, T can be written as rho A c square; now, so that is the instantaneous power flowing past the location x at time t. Now let us calculate the energy density of the string.

So, energy density is nothing but so the total energy of the string is given by this. So, this we integrated over the segment just a moment ago. So, this is the energy density and if you calculate this let us see; so it is del w/del t whole square plus this is del w/del x whole square, this therefore, can be simplified using again this expression as… Now if you compare this and this, you can write the instantaneous power, flowing passed location x at time t is c times the energy density at that location. So, what this… So, this is an important conclusion that we draw, what we can say is that the energy or the energy contained in the string in term in terms of kinetic and potential energy of the string at any location x and time t is flowing at the phase speed, that means, speed of the wave in the string. So, that gives us the power.

So, the energy is flowing at a speed of the wave, the phase speed of the wave. Now this is true for all dispersive media. So, in all dispersion media the energy propagates or spreads at a rate c. So, all in non dispersive media, in all non dispersive media this is the relation between the energy density and the power solution. So, the speed at which energy spreads or the energy density in the string spreads is same as the speed of propagation of waves. This might be seen intuitively very clear, obvious, but the situation changes when you go to dispersive media. So, we will have some opportunity to discuss that later on in this course. Now we have been discussing about the instantaneous power flow at any location, at any time, but what is more physically or what is more useful or relevant in various situation is the average power .

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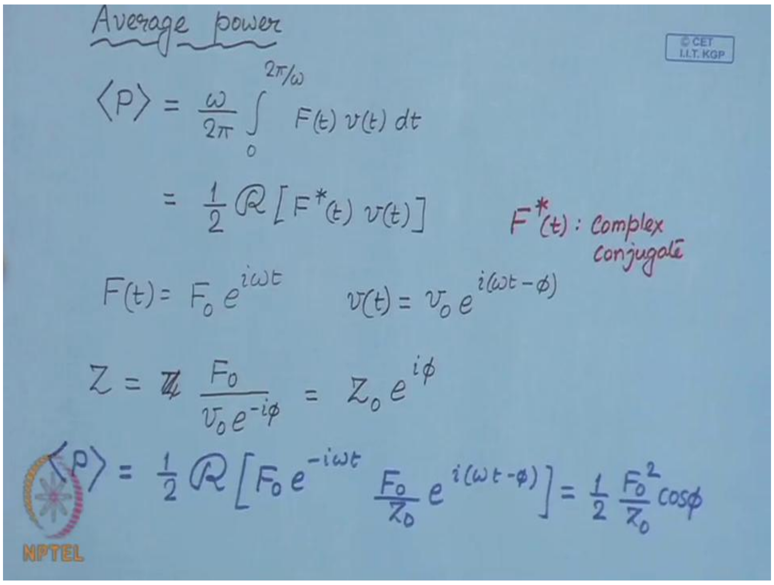

So, this average power, if you represent the force and the velocity, normally in vibration courses, you define like this, where this force and velocity they are real functions. So, if this is the situation, then this is the integral you perform in order to access the average power that is flowing, and this since we are using complex notations to denote force and velocity, this integral can be easily actually performed, if you take the complex notations of force and velocity, take the complex conjugate. So, this is the complex conjugate, of the complex is forced representation. So, normally we represent this, if it is a harmonic force and we know that from our study of discrete systems, vibration of discrete sense systems that in general, the velocity will have the phase difference with the force, the velocity or displacement both will have the phase difference with the force in general in case of undamped system, in the displacement is goes to zero and the velocity pi by 2.

So, therefore, now we have already defined our impedance. The impedance as I have mentioned is a complex number. So, and the ratio of the force and the velocity is the impedance. So, the amplitude is F_0, and here it is v_0 exponential minus i phi… So, this can be represented as… So, the impedance can be represented like this. So, I can calculate the average power in terms of that is the impedance. So, F star is F_0 and the velocity I can write in terms of the impedance as… and that turns out to be… which you can very easily determine, we have to take the real part of this, so that is turns out be cosine of phi.

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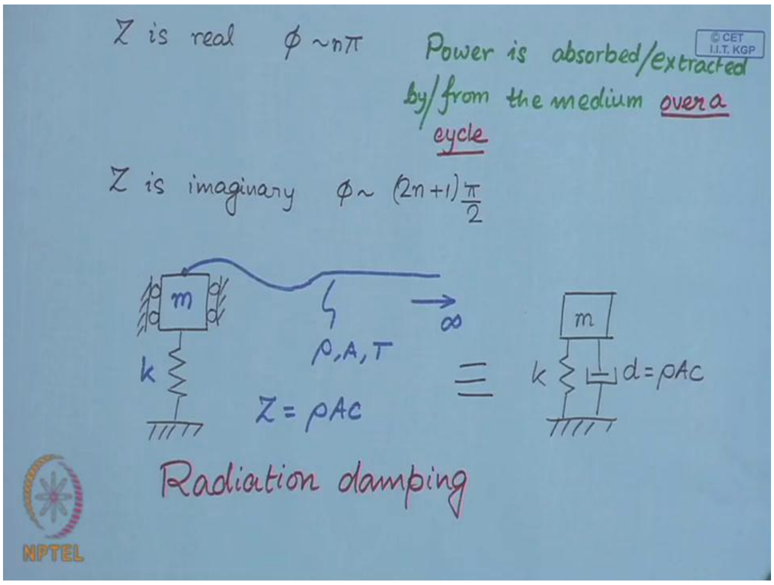

So if Z is real, then phi it is like n pi, it can be zero, it can be pi etc., so depending on the sign now, if it is zero, then this impedance is positive as you can see from this expression. If it is pi, then it is negative. So depending on that sign, power is absorbed or the expression of average power, we can find out from here, the expression of average power so cos phi, power is absorbed or extracted. So, when it is positive, the power is absorbed by the medium; when it is pi, the power is extracted from the medium. This is on an average over the cycle, this is important over a cycle. So, in parts of a cycle it may be absorbed, but on other part it may be extracted, but the overall thing is given by here. So, this is over a cycle, now if Z is imaginary, which means phi is… Power cannot be absorbed or extracted from the medium. So, this is called reactive medium. So, as you can see this will be zero. So, all in average, power is neither absorbed nor extracted from the medium.

So, now let us look that an example. Suppose you have a mass and string; and this is connected to a semi-infinite string under the tension T. Now as the mass oscillates, as we have calculated the impedance of this semi infinite string is rho A c. So, that is the impedance of the string. So, once you connect this oscillator to the string and since the impedance is real and positive, power is always absorbed in the medium by the string and therefore, I can think of this system as a damped oscillator with the damping coefficient rho A c, which is the impedance of the string. So, impedance is force divided by velocity. So, this is the damping.

So, here you find that energy is been radiated of these, this is an equivalent model of this system. Normally when we draw dashpot like this, it represents conversion of mechanical energy to thermal energy, it heats up the fluid inside; but here it is not conversion to a thermal to a thermal energy. It is converted to vibration of a string or disturbances propagating in the string. So, here the medium is observing the energy and so this system is actually radiating out to the energy in to the medium which is the string. So, this is an example of radiation damping. So, this example illustrates what is radiation damping; and we will discuss thus phenomena later on as well. So, today we have looked at harmonic waves and propagation of energy in medium due to waves. So, with that we conclude this lecture.

Keywords: harmonic waves, impedance, energy transport, radiation damping.