Vibrations of Structures

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Lecture No. # 18

d'Alembert's Solution - II

In the last lecture, we have initiated some discussions on the wave propagation solution of one dimension continuous systems governed by the wave equation; and the reason why we are interested in the wave propagation solution is that we observed in our previous discussions that there are many systems in which we do not have eigen frequencies, we do not have Eigen functions, for example, systems with infinite dimensions or the spatial dimension infinity, so, infinite system that I want to say, so for such systems and also systems with special conditions, for example, a bar with boundary damping, we have observed that for such systems, we did not have Eigen frequencies or Eigen values.

So for such systems, we need to study this wave propagation solution. There are other examples that we have observed and discussed in our previous lectures. For example, we would like to understand, how transient solution evolves? For example, we have seen that for the initial value problem with velocity initial condition, we have a propagating front which interacts with the boundary of the system, and then there is reflection, and there can be other things that we are going to discuss in the course of this lecture; and then we are also interested in the wave propagation solution, because wave propagation is a very nice way of evaluating material properties or characterizing materials in terms of for example, wave speeds, in terms of presence of discontinuity or defects. So, from all these considerations, we would, we are motivated to study this wave propagation solution.

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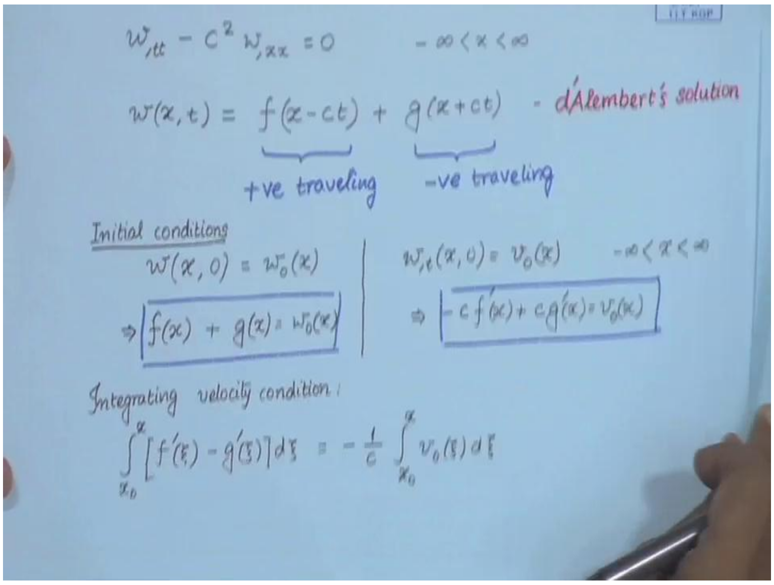

Now, in the previous lecture, we have seen that for a system governed by the wave equation, there are no boundary conditions, because the system is of infinite size. So, the spatial dimensions are infinity. When the disturbance is… we can consider, we can drop the boundary conditions even in cases where the disturbance has not reached the boundary and we like to study the transients in the system. So, for such situation also, the system is governed by the wave equation and there are no boundary conditions, we do not consider the boundaries. Then we observed that the general solution of the wave equation can be which is known as the d'Alembert's solution. Now, so the d'Alembert's solution tells us that the solution of the general solution of this wave equation in terms positive traveling wave and negative traveling wave; and as we will see that this solution will allow us to understand the transient behavior of systems that are governed by this wave equation.

Now, today what we are going to look at is, now given this system, and now suppose we are given some initial conditions. Suppose, the initial conditions are specified like this that the system is given an initial shape defined like this and the initial velocity distributed in this form, where to the domain of this problem from minus infinity to plus infinity. So, this is therefore the statement of our initial value problem for our system. So, let us then substitute this solution form, the general solution form, in the initial condition expressions and we obtain… from here and from here, we have… So, we have these two conditions. Now, let us first look at this velocity condition. So, if we integrate this velocity condition, what we obtain is… So, we are integrating over from x_0 to x, and I will put this -1/c on the right hand side. Now, I can integrate this to write. So, f evaluated at x_0. So, this is what we are going to get.

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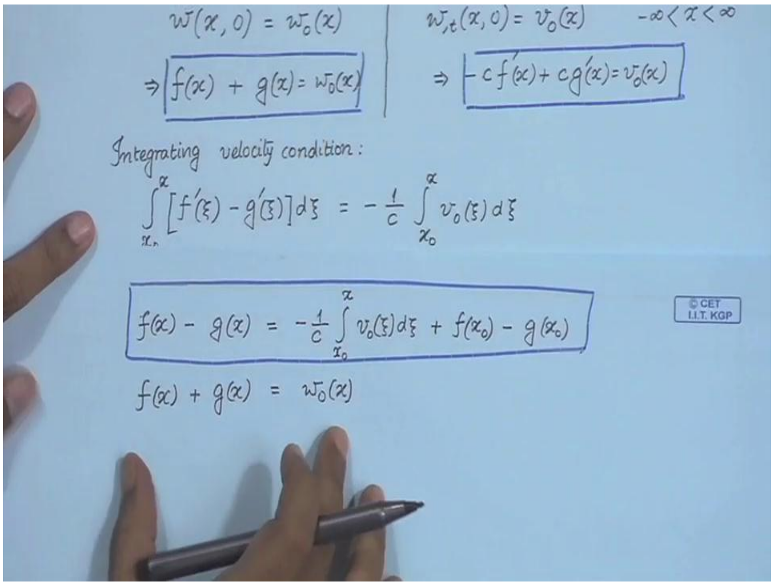

Now, this along with what we have obtained here, so these are the two equations we can solve simultaneously to determine f(x) and g(x).

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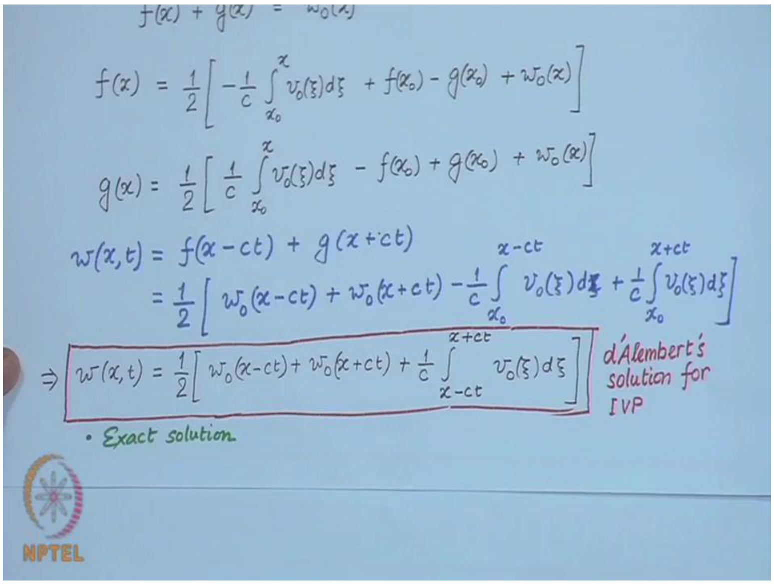

So, for example, if you just add these two, we obtain f(x), and similarly, if I subtract the first from the second… So, we have these expressions of f and g. Therefore, we can now write the general solution. So, we have to substitute in case of x, here (x-ct), here we have to substitute (x+ct), and we have to add these two. So, these are independent of x. So, when we add these two they will get canceled. So, what we will obtain… So, let me write this w_0 term and we have these integrals.

Now, therefore, I can write this as you see, if you look at these two integrals then the limits are from x_0 to (x-ct) with a negative sign, and from here it is x_0 to (x+ct) with a positive sign, and the integrants are the same. So, I can combine these two integrals, and write them as… So, here I reverse the order. So, this becomes plus (x-ct) to x_0 and this is x_0 to (x+ct). So, this becomes (x-ct) to (x+ct). So, this… So, this is the d'Alembert's solution for the initial value problem. So, given the initial shape of the system, the configuration of the system, and the initial velocity distribution, using this solution, we can determine the solution at any space point at any time instant.

So, this solution is an exact solution. In that, what we have been discussing previously, we were, for normal strings with fixed boundaries, we were expanding in terms of the Eigen functions, and there are infinitely many Eigen functions. But when we do the computations, we have to some where truncate, because we either sum up the series somehow; but if we cannot sum up the series and have a closed form solutions, then we must truncate the series somewhere. Now, once we truncate the series, what we get are the approximate solutions of the equation, of the wave equation which we have already seen in the previous lectures. But this solution is in that way exact, because we have the solution in terms of the shapes of the initial configuration and initial velocity distribution over the string. The next thing to observe is the solution is dependent on the velocity distribution, since this is an integral. So, the initial velocity distribution is integrated from (x-ct) to (x+ct). So, therefore the velocity distribution in this range is of importance for the solution. So, let us understand this in the space time diagram what we have discussed in the previous lecture.

So, let us consider the space time diagram. We have seen in the last lecture that the solution at any point (x,t), let us say, it is a summation of a positive travelling wave and a negative travelling wave. Now, in the space time diagram, the positive travelling wave moves along the characteristic, the positive characteristic of the wave. So, let us say… So when the time has progressed from zero to t, the equation of this blue line is given by this where this is, let us say x_0 and for this green line, this is say x_1. So, therefore this x_0

point is nothing but (x-ct). So, the slope of this line is plus c, and the slope of this line is minus c. So, this x_0 is actually (x-ct) and this is (x+ct).

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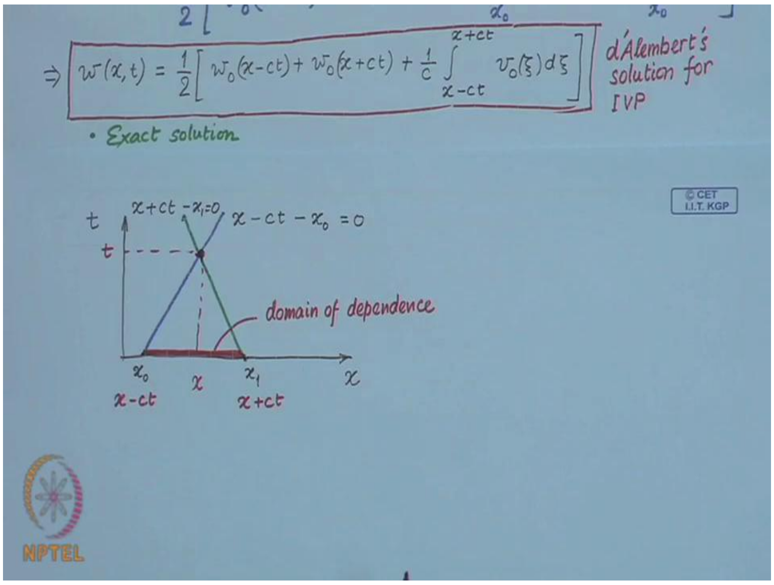

So, solution as you can see in this solution form, it depends on this whole interval, the integral of the velocity distribution over this interval. This is called domain of dependence of the solution. So, the solution at any space point x any time at any time t is dependent on the distribution of initial conditions in this domain. Now, if the initial velocity is zero for example, if the initial velocity is zero, then of course, this integral vanishes. So, then we find that the solution here is dependent only on the displacements at these two points at time t equal to 0. That gets summed up here, as we see here. So, let us consider these two cases separately.

So, let us consider the first case with zero initial velocity. So, which means that this is zero for all x. In that case, if you look at the solution, then… So, this is what is happening. Now, to understand this, let me rewrite this. So, whatever is the initial configuration or shape of the system, we half it. One half will travel in the positive direction, the other half will travel in the negative direction. So, let us see this.

Suppose the initial shape is given like this. So, this is... So, at time t equal to zero, this is the shape or this is the configuration of system, suppose you consider this to be a string. Suppose this is a string, so the shape of the string is like this. Now, at later time… So, this initial shape let me first half it. So, this is one half of this function. So, one half will travel in the positive direction and one half will travel in the negative direction. So, let me try to… So, this has travelled in the negative direction and one half will travel… So, the net displacement of the string is given by the summation of these two. So, this is going to be… So, here there is a king. Now, as time progresses further, these two pulses are separated out. So, the net solution looks like this. So, this is… suppose this is t equal to t_1 and this is t equal to t_2 then this pulse is this solution and the other one is the negative travelling pulse. So, that is how the system is going to evolve as time progresses. So, given this initial shape, one half of that configuration or the shape of the function will move to the right, the positive direction and the other half will move to the left; and this is how it is going to look like. But, usually giving an initial shape is more difficult in practical situations. What we can give quiet easily is initial velocity condition. This is very important from practical considerations as well. Now, we have seen from one of the previous discussion that the initial value problem, let us say… So, an initial value problem of this form can be equivalently represented, this governing equation, there is no boundary conditions as we are considering infinite systems.

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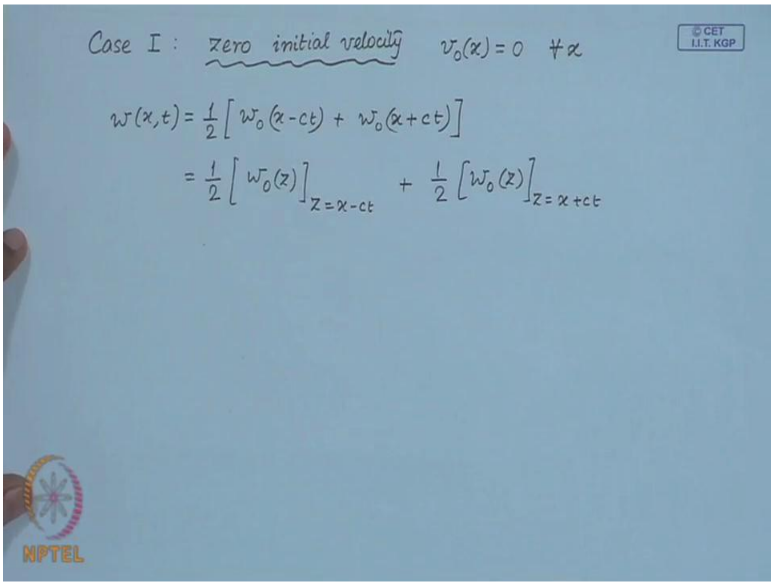

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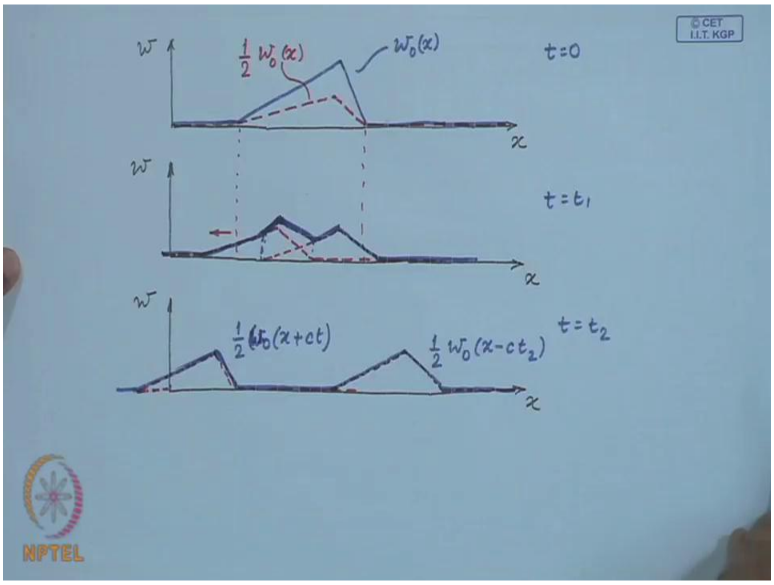

So, these two systems are equivalent. So, here we have forcing in place of initial conditions. So, a system with initial conditions can be looked at as a system with forcing in this form, where this delta t, the direct delta function at t equal to zero and delta dot is the derivative of the direct delta function, which has certain properties that we are going to discuss. Now, when we have only initial velocity condition, there is no initial displacement, so, our equation of motion is given by this. So, this is an impulse applied at time t equal to zero with the distribution given by $v_0(x)$. Now, for very special, in very special situations, for example, when we strike the key of a piano, or we strike the string of a santoor, for example, with the hammer, the mallet, then what we are giving is, we are giving an impulsive force at a particular location. So, for this distribution function $v_0(x)$, it happens to be a direct delta function. Then we are representing such situations like striking the hammer of a piano or the hammer of a santoor on the string. So, this by various choices, you can produce or this model. This governing equation governs the dynamics of let us say piano string or a santoor string. So, I mean, this system is important from practical consideration. So, let us look at to the initial velocity conditions.

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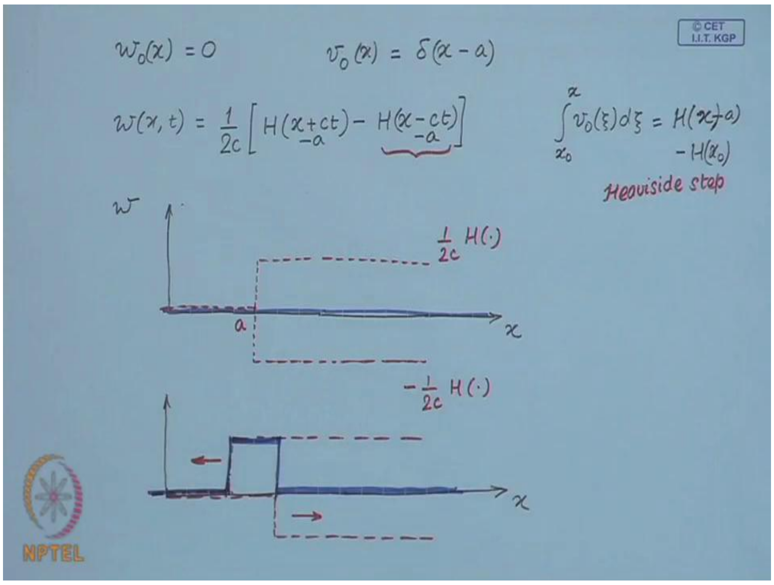

So, what we now have is, this is zero and $v_0(x)$ is something specified. Now, let us look at the situation that I just now discussed. Let us say, the initial condition, initial velocity distribution is the direct delta function applied at x equal to a. In that case, you see, thus if you look at the solution, let us look at the solution once again. So, these are zero; this is the direct delta function. Let me write down. So, if you substitute this expression in here, then we will find that… So, you have to perform this integral, you have to perform this integral.

Now, when this is direct delta function, then the integral of direct delta function is the Heaviside step function. So, if you take this, so, this is the Heaviside step function. So, using this Heaviside step and substituting it is here, you will find that the solution reduces to... So, if you substitute this in here. So, this H of at x_0 it cancels of, what remains is H(x+ct) minus H(x-ct) and divided by 2c. Now, this, let me once again draw the solution. So, initial shape of, let us say the string is zero. So, there is no displacement; and I have given this step, let say at x equal to a. So, this actually is… There is a minus a, because I have given the impulse at x equal to a. So, this can be represented… So, let us say, this is x equal to a. so this is the sum of two Heaviside step functions; one is this one, the other is minus… so, this is the lower one is the solution; and so, this is… and this is 1 over… So, the summation of these two is zero. Now, as time progresses, this is going to move in the negative direction and this is going to move in the positive direction. If that is the situation, then… Therefore, the total solution is the sum of these two. So, what we have is a rectangular pulse spreading, like this. Now, instead of an impulse, because impulse is little difficult to produce exactly, if you produce a distribution, then that pulse shape is going to change. And this we have seen is an example of a string with initial velocity distribution in one of the previous lectures. So, what happens is there is a hump which is spreading, spreading out in both directions, it is expanding; the hump is expanding. So, you can understand from this simple example that here there is a rectangular pulse which is expanding. So, this… and you can see that as time progresses, the string is going to take up a constant deflection which is given by the height of this. So, this happens to be 1 over 2c. So, this tells us about what happens when we have an initial velocity condition.

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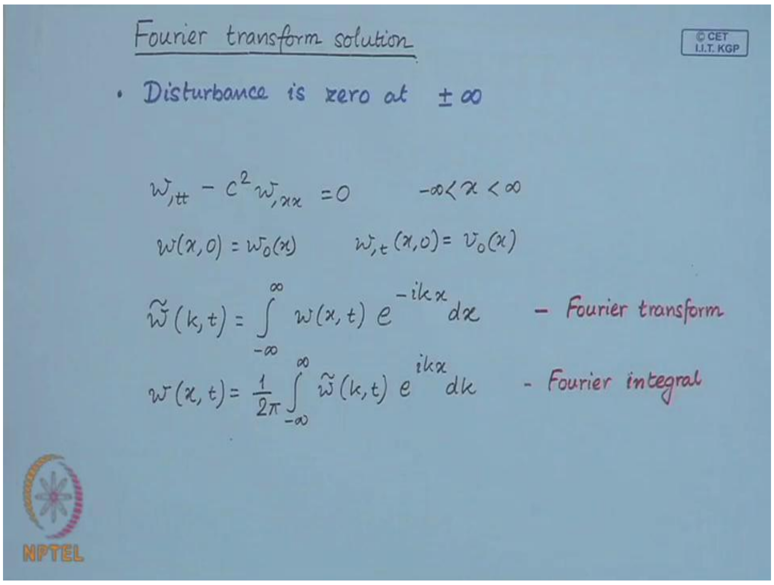

Now, we can solve this same problem using Fourier transform. Now, when we have systems of infinite extend, Fourier transform is a very powerful technique which can be used to solve the initial value problems. So, let us look at briefly how this is done. One thing we assumed, when we consider Fourier transform for initial value problem is that the disturbance, whatever initial conditions are given, is zero at infinity. This is, this assumption is made to ensure that the Fourier transform exists. So, we must satisfy the conditional existence of the Fourier transform otherwise this method would fail. So, with this assumption, so we must have the existence of this function. So, disturbance is such that or the initial condition is such that, its Fourier transform exists; and we are looking at finite time, so that the disturbance has not spread out from minus infinity to plus infinity. So, this is an assumption that you will make in this approach.

So, our equation of motion is given like this. There are initial conditions. We define the Fourier transform. So, this the definition of the Fourier transform, and we must also look at that the inverse Fourier transform or what is known as the Fourier integral. So, this is known as the Fourier integral.

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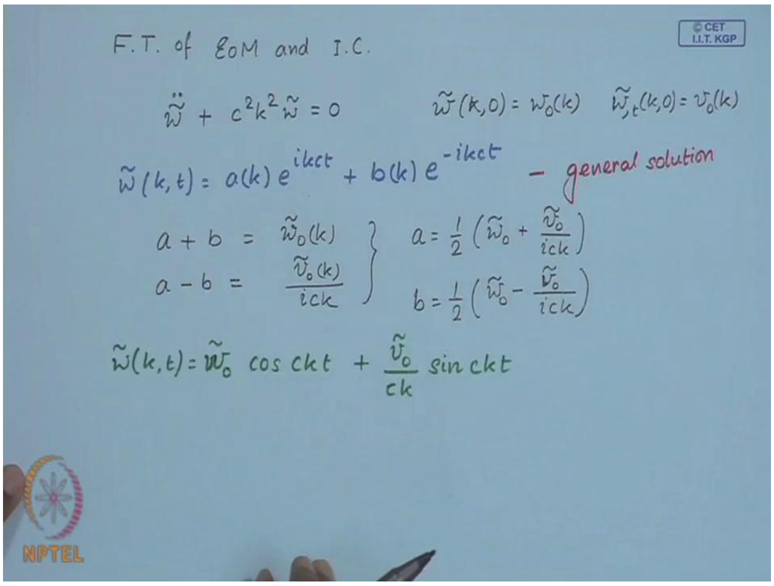

Now, what we do is, we take the Fourier transform of the equation of motion. So, if we do that, and of course also of the initial conditions. So, if you take the Fourier transform and simplify… So, this and along with that we have… Now, we know the general solution of an equation of this form. So, this is the general solution. Now, we use the conditions, the initial conditions and we find that a plus b must be equal to $w_{0}(k)$, and a minus b must be equal to… and from here, we therefore can solve a and b; and once we have solved for a and b, our solution can be written like this which can be easily obtained by substituting this in the general solution and simplifying. Now, once we have this, we use the inverse Fourier transform or the Fourier integral to write down… Once you substitute this here and perform the Fourier integral which is a straight forward, so, you can get to the solution which we have obtained before. So, we substitute this here and use the definitions of this w_0 tilde. So, when you put this in the inverse Fourier, then you will get w_0, and this has to be evaluated at (x+ct) and(x-ct). So, this is more or less straight forward, and you can obtain this solution.

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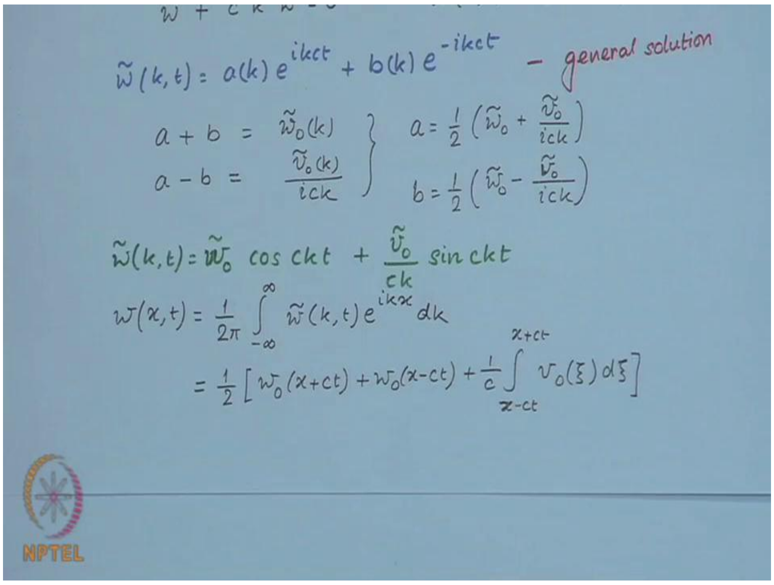

So, to summarize what we have looked at in today's lecture, we started with initial value problem, we have looked at some examples of initial conditions and we have obtain the corresponding solution using the d'Alembert's solution for the wave equation in terms of propagating waves; and we have understood, how for example, given an initial displacement condition or an initial velocity condition, how the solution, we can understand a solution, how we can see the evolution of the solution has time progresses, and these are in terms of propagating fronts. So, we have seen for example, in case of initial velocity condition, which is more practical, for example, in piano, the santoor, where we give impulsive loading on the string, how the disturbance front propagates as time progresses. And towards the end, we are briefly looked at the Fourier transform solution of an initial value problem and obtain the same solution as we have obtain earlier in this lecture. So, with that we conclude this lecture.

Keywords: wave equation, initial value problem, travelling wave fronts, Fourier transform.