Vibrations of Structures

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Lecture No # 17

d'Alembert's Solution - I

With this lecture today, we are going to start the second module of this course. So in this module, we are going to study the wave solution for continuous systems in one - dimensional. Now, why should we study this wave solution or what is commonly known as d'Alembert's solution? So, to look at the reason, let us make quick recapitalization of what we did in modal analysis of continuous systems.

So, let us consider the case of the string. So, what we had was this equation of the motion and the boundary conditions. Now, we search for a solution, which has a very special form.

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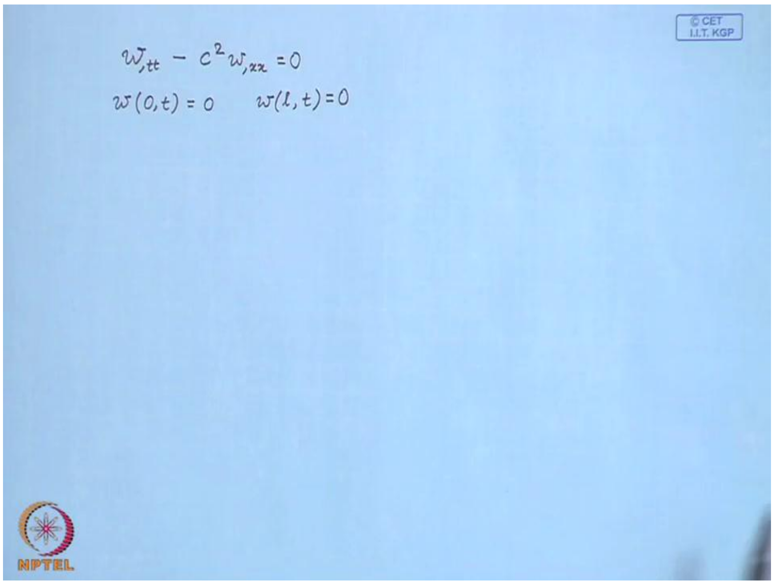

So, consider this equation of the string along with the boundary conditions, fixed-fixed boundary conditions. So, what we have done; we have considered special solutions of this equation and the boundary conditions. So, here I show this equation of the string and the corresponding boundary conditions of the two ends. Now, we assume a solution which had the structure like this. So, we had a space dependent amplitude function multiplied by a harmonic function; this is in the complex form; and we can take either the real part or the imaginary part or a combination of these two parts as we have already discussed. For the problem of the string, for example, this is a self-adjoint problem as we have discussed. So, w, the amplitude function is real. So, we have a solution which we wrote as a combination of the real and imaginary parts of exponent of i omega. Now, we substitute this solution form in the equation of motion and what we obtained was the Eigen value problem, so, the Eigen value problem consisting of the differential equation as shown here and the corresponding boundary conditions. Now, for this Eigen value problem, we have a general solution which has a structure like this. Now, with this solution we are going to, we are going to put this solution in the boundary conditions. So when we substitute this solution in the boundary conditions, we obtain this vector equation. We have a matrix multiplied with this coefficient {D H} and that must be equal to a zero vector. Now for non-trivial solution for D and H, we put the determinant of this matrix to zero; and that gives us the characteristic equation which we can solve for the Eigen values; and putting back the Eigen values in this vector equation, we solve for D and H, we have the Eigen functions. So, the point to note in the modal analysis is that we obtain the Eigen values and Eigen functions when we have the boundary conditions, this and this. So, in our original equations, these are the boundary conditions. So, if we do not have boundary conditions, then what happens? So there can be many situations when we may not have the boundary conditions or the boundaries are so far away that we can consider, for short time interval, the system is infinite. So for the systems with infinite extend, therefore, we do not have these boundary conditions; and therefore we do not have the modal solutions. So, which means we do not have Eigen values and Eigen functions as we have discussed in the modal analysis; and you can remember that in most of the approaches we have used till now, we have used this modal expansion technique to solve the system, for initial value problem, for forced vibration problem. So for various kinds of problems, we have used the modal expansion for solving the problem. Now, once we do not have these modes, the Eigen functions in the conventional sense, how do we go about solving the system? What kind of solutions should we expect for such systems? Now in the course of our discussion, we also came across a problem in which we did not find Eigen values. So, this is a system that we have discussed before in which is a bar, a fixed bar and the free end of the bar is connected to a damper which is fixed to the support. Now, we have this equation of motion and the boundary conditions as shown; and for this problem remember we have obtained a characteristic equation. So, this is the characteristic equation where this gamma is related to the Eigen value of the problem s and a is a constant which is defined in terms of the speed of propagation of axial waves, the damping, and the material properties and geometry of the bar. Now you see that when a equals to one there is no value of gamma for which this characteristic equation can be satisfied. It must be minus infinity; so there is no finite value of gamma. We will say that there is no existence of an Eigen value when a equals 1 and a equals 1 when this this damping coefficient d takes on this very special value EA over c. So, for this problem, we did not have Eigen values. So, this is another problem, where the... and it is very similar to the problem of an infinite system or semi infinite system for which we do not have this Eigen values and Eigen functions. So, what we find is that there are situations, where you may not have or the modal solution does not exist, because either system is infinite or because of very special situations or parameters in the system, the Eigen values should not exist.

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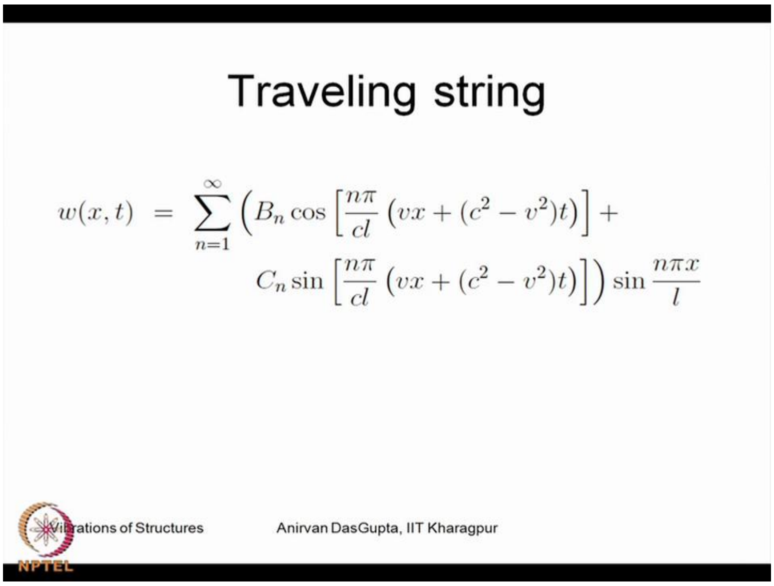

Then we also discussed the case of a travelling string. So, if you look at the travelling string, then we had the solution of... I mean general solution that we obtained in our previous discussions. We obtained a general solution which look like this, what I have put on this slide. So, here you can see that the space part and, the space and the time they are not separable as we have seen in the modal solution.

So, even if you take one of the modes, say for example n equals to 1, then also you will see that this is not separable; the solution not separable, which is not the case for, say the static string. So, here we say that this problem is non self adjoin and the solutions are of very special type.

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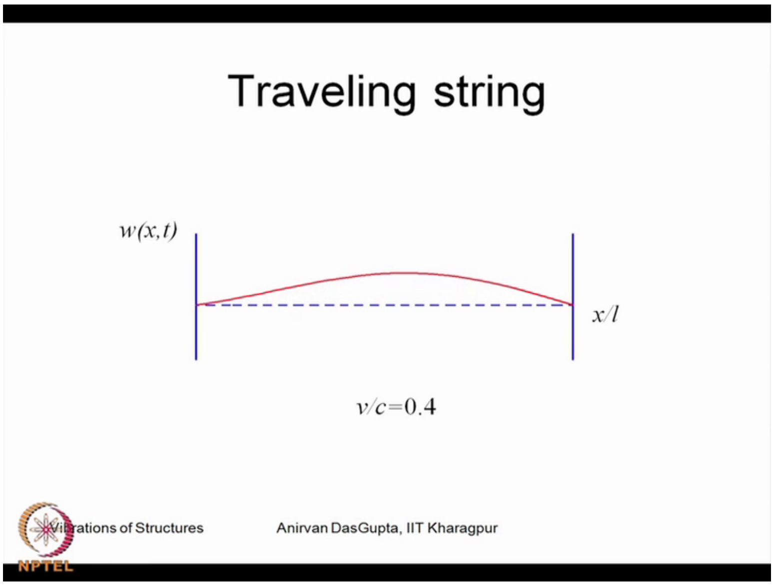

So, let us look at the solution for n equals to 1, I have put this solution for n equal to 1 and v over c as 0.4. So, you can see that the characteristics of the modal solution, that we have all discussed where in the previous lectures; so, that is not there. So, you cannot... for example, the string does not pass through the equilibrium point; all points of the string do not pass through the equilibrium point at the same time.

So, there is a propagation of this; there is a propagation of disturbance that you can possibly see in this. Then two points of the string are neither and they neither have phase difference of 0 or pi.

So, that was another characteristic of the modal solution that is missing in this solution. So, this is another example. So, this is a finite system, this system is finite. Yet we see that our solution does not satisfy, even for this single mode, the solution does not satisfy the properties of the modal solution. Then we looked at in the forced vibration problem.

If you recall, we looked at the string with travelling force. So, there we observed in the solutions very strange kind of behaviour.

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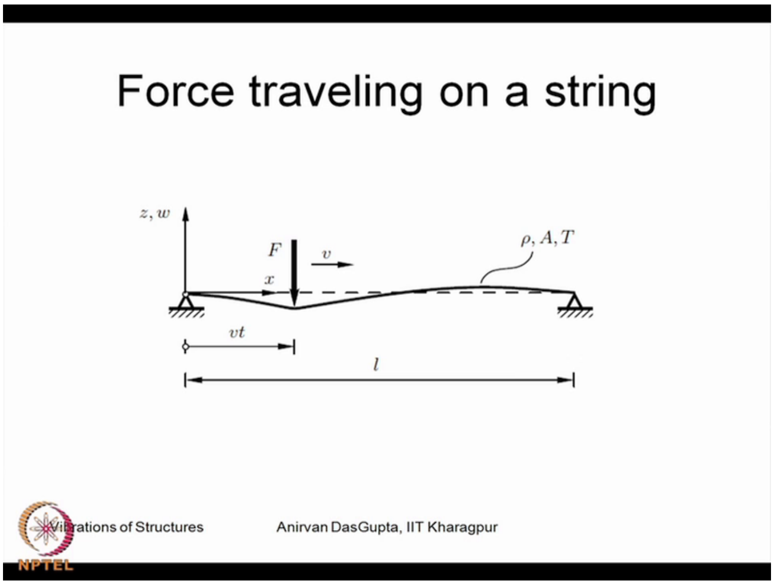

So, let us look at it once again. So, this is the string on which a force is travelling at a speed v.

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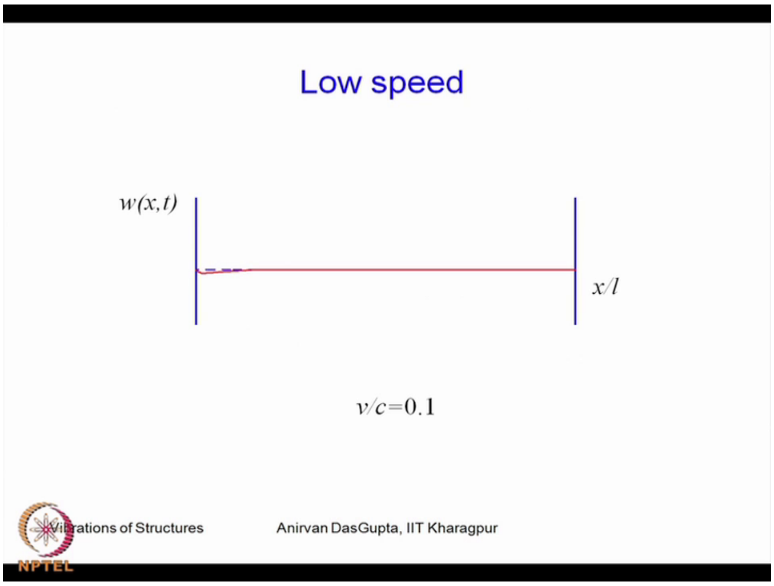

So, here I show you one of the solutions that we have discussed before. At a low speed, the force is travelling at a low speed, 10 percent the velocity of transverse waves in the string and you will find the string has very strange kind of motion; which looks like discrete. It remains almost static at a certain configuration for some times; certain portions of the string remains static at certain configuration unless a king, which brings in a new configuration changes the angle of the string or the solution of the deflection.

So, if we look carefully, then you will find that there is a there is a king, which is travelling back and forth in the string. So, these are some… so this is another situation where you find that the solution there is a transient, there is a propagation of information back and forth in the string. Another example, we have again discussed this example before in initial value problem, where we talked about initial velocity conditions.

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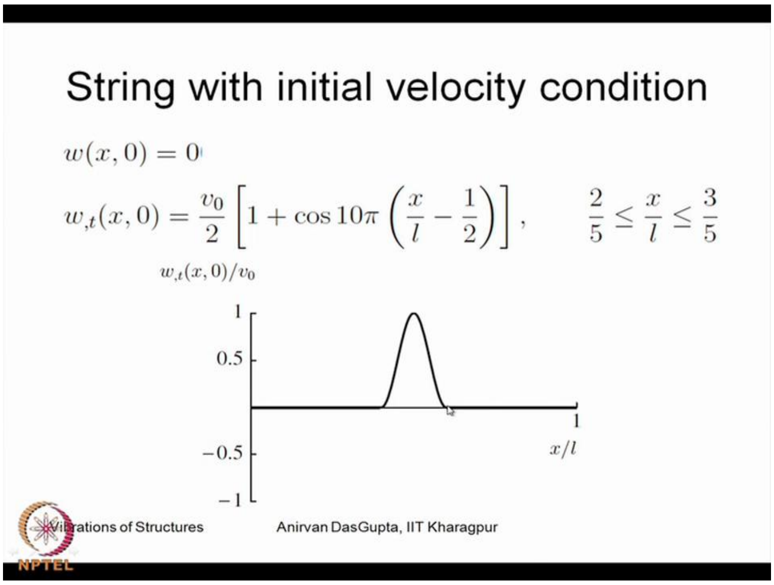

So, let us look at this once again. So, we have considered a string with initial conditions as shown here. So, initial displacement was zero, initial velocity was given a distribution like this. So, in the in the central one-fifth portion of the string, we have this velocity distribution like this. So, this is the velocity distribution displacement is still zero, but the velocity is distributed like this.

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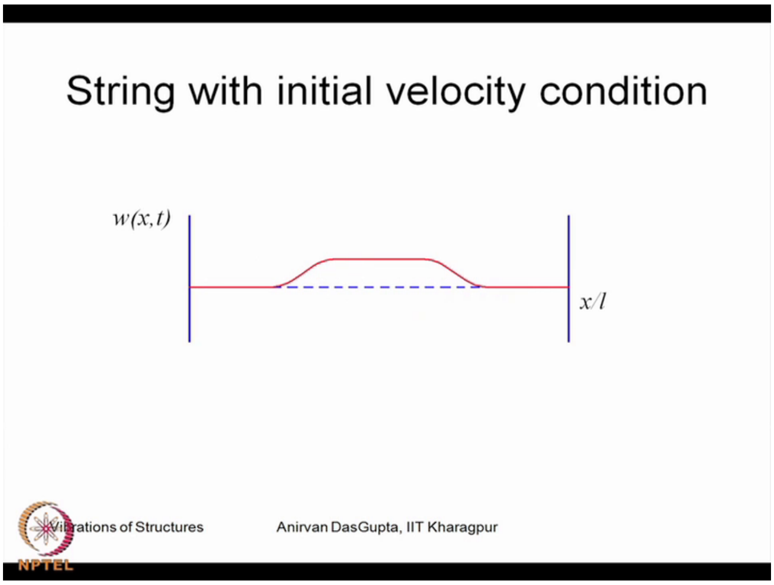

And let us look at the animation that we have seen before once this velocity condition is given. So, you see that seems to be a propagation of fronts in two directions, which go and hit the wall and then seem to come back and repeat itself. So, this is another example. So, all these are... So, in the time skill that we are looking at, these behaviours are transient behaviours. So, we have looked at various examples, in which we find that the motion of the system is in terms of propagation of waves in the system. Now, let us look at the modal solution of a normal taut string which looks like.

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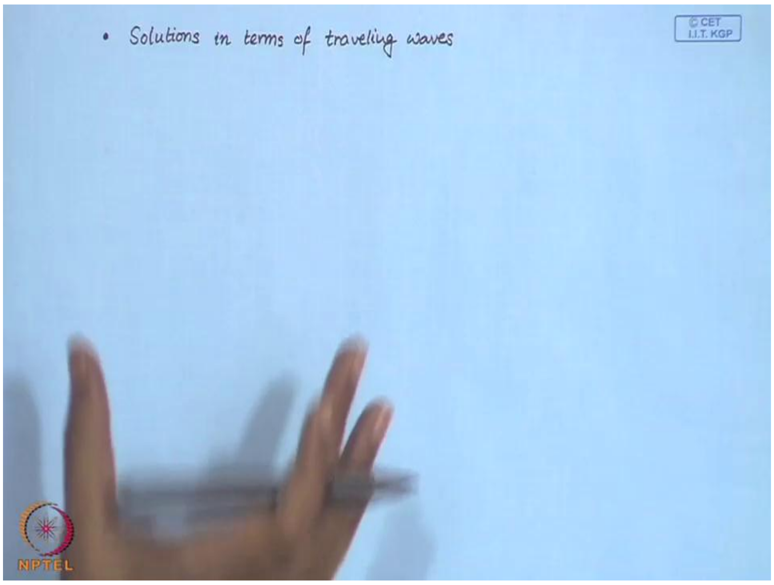

So, what we have seen is we have... So, we have all these examples; we can understand the behaviour of the system when we look at the solutions in terms of travelling waves. So, our motivation comes from these examples, in which we find that either the Eigen frequencies do not exist or we find that there are the solution is very strange in terms of travelling disturbance; and these solutions are also very important or this kind of behaviour is also very important in studying, let us say, defects in materials.

So, if there is a wave propagating in a structure or in a material, then if we understand how the wave interacts with the boundary or with internal defects, for example, cracks etc. or different phases in a material, then we can say many things about the material or the structure. So, from various all these points of you, we are motivated to study the wave propagation solution for continuous systems. So, we would like to study the solution of continuous systems in terms of propagating waves.

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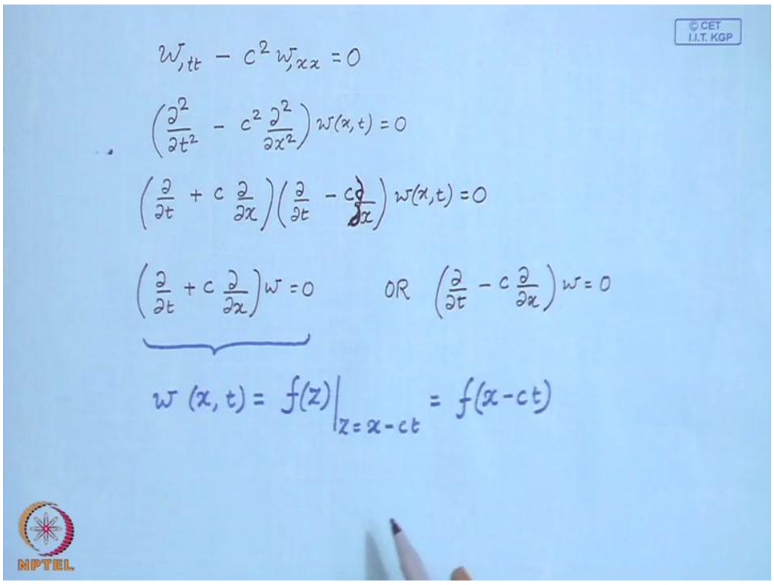

So, let us look at the example. So, let us consider the wave equation. So, what we are looking at is this wave equation and we want to find out or determine the solution of this wave equation in terms of propagating waves. So to do that, let us write this equation like this. So you see, this equation can be… So, let me write it first in terms of this operator. Now, this operator I can factor. So, I can factor this operator like this. Now, for a solution, you can easily see that this equation is satisfied; whenever this or this is satisfied. So, let us first look at this equation. Now, you will find that the solution of this equation can be written as… If w(x,t) is some function, any function f (z), z replaced by (x-ct), if we substitute this function, of course, it has to be differentiable; so, if we substitute such a function here, then you find that this satisfies this equation. This can be easily checked by substituting here. Now, what is this solution?

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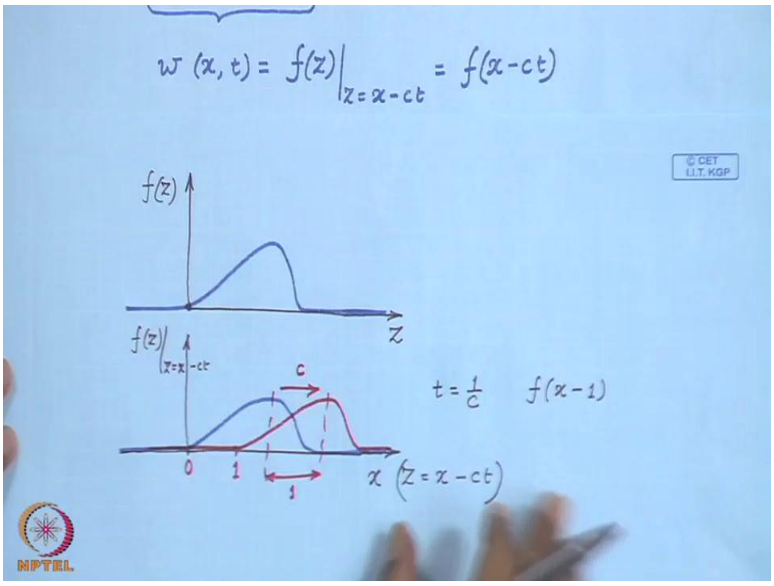

So, let us visualize this solution. Suppose you have the function f(z) of this form let us say. Now, what happens, when I replace z by (x-ct); and let me plot with x rather than… so here, I am replacing z with(x-ct) and I am plotting with x, the function at different time. Let us say at time t equals to zero, I have f(x) which is same as this. At a certain time, what will happen, suppose t equal to let's say 1 over c. So what we have at t equal to 1 over c is f(x-1). So, now I have to plot f(x-1). So therefore you find that the value at z equals to zero now occurs at z equal to 1. So therefore this pulse has moved a certain distance towards the right in time 1 over c; and it has moved a distance of one unit, so let us say this peak, if you track the peak it will be same as this, so this is one unit. So it has travelled a distance one unit in time 1 over c. So, the speed of propagation is 1 over 1 over c which is c. So, the speed of propagation of this pulse is c. Thus, we observe that this solution represents a travelling pulse or a travelling wave in the positive x direction at a speed c. This is the reason why this c in the wave equation is known as the speed of propagation of the wave. So, this is travelling in the positive x direction at speed c.; and is a solution of this differential equation. Similarly, if you look at this differential equation, then the solution by the same argument, you can find is given by g(x+ct). So, in this case, the function or the wave pulse, which is represented by g(z), now travels in the negative x direction. So, these are the two solutions of these two differential equations. Now, it can be shown that the general solution of the wave equation can be written as f(x-ct)+g(x+ct). So, we have, so the general solution is therefore a superposition of two opposite travelling waves and both of these waves are travelling at same speed c. Now, let us look at this situation. So, this is, this represents x, this represents time t and this is the displacement or the field variable. Now, at time t equals to zero, we have this solution. At a certain time instant, let us say if this is positive travelling wave which means if this is moving in the positive x direction, then the situation is like this. At yet another time instant, the pulse has travelled forward. So, if you now track… So, this is the line on which, let us say the crest of this pulse is travelling. Similarly, this is the line at which this boundary of this pulse system. So, we can draw what is known as the space time diagram; and in this diagram, suppose this point was the maximum amplitude point here so x_0 let us say. Next, this line tracks the peak of the wave and the equation of this line is given by this. So, this line is known as the characteristic of the wave equation. Similarly, it will have another characteristic for the negative travelling wave. So, this is another characteristic of the wave equation. So, we have both these two components in the solution, we have a positive travelling wave and a negative travelling wave, and the net solution is the superposition of these two solutions, so we can… This diagram is known as the space time diagram and the solution at certain point, therefore, depends on how these two waves, two opposite propagating waves, starting from t equals to zero reach this point; so this point at a certain t and at a certain position; so this is known as an event. So, the solution here will depend upon… So, these are the two characteristics; one is for the negative travelling wave and other is for the positive travelling wave. So, we will see very soon that the solution at the space time point depends on the initial conditions; so at t equals to zero what we specify are the initial conditions. So, the solution at this point in the space time diagram will depend upon the initial conditions given in this interval. So, this is x_0 then this is x_0-ct and this is x_0+ct. So, when we determine, the initial value, when we solve the initial value problem in terms of the propagating wave, then we find the solution at this point is dependent upon the initial conditions specified in this interval, not what is outside this interval. So, in this space time diagram we can visualise various things. Now, this solution, we have written out, so this solution is known as the d'Alembert's solution. So, this is the d'Alembert's solution of the wave equation. So, this structure tells us or in clearly shows that we can think about any solution of the wave equation, in terms of positive and negative travelling waves.

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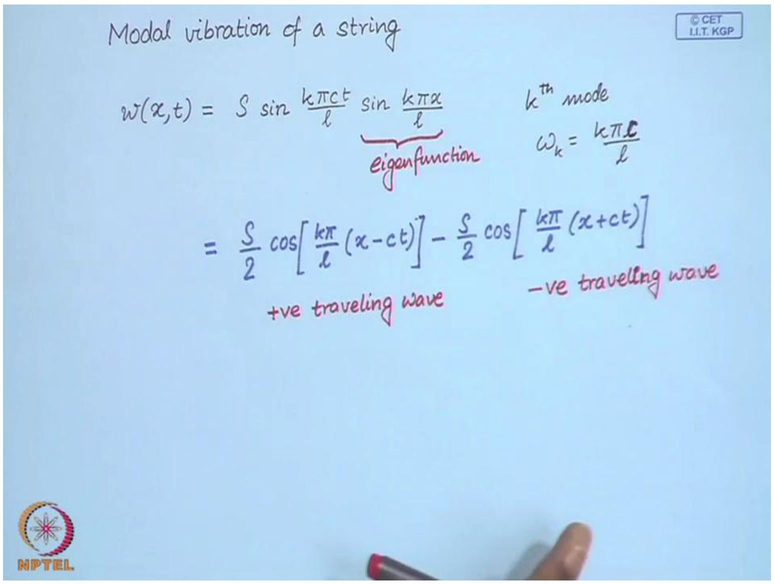

So, let us see, then the modal vibration of a string. Now, we consider a normal taut string with fixed-fixed boundaries and we have obtained the solution, let us say in this form. So here so this is what is known as the k^{th} mode; and we recall that the circular natural frequency of the k^{th} mode is give by k pi c over l and this is the corresponding Eigen function. So, this product can be easily written using trigonometric identities. So, this modal solution for the k^{th} mode can be written, can be expressed as a super position of these two solutions. Now, here immediately we can recognize that this is the positive travelling wave and this is the negative travelling wave. So, we can decompose this stationary wave solution, in terms of travelling waves as you see here.

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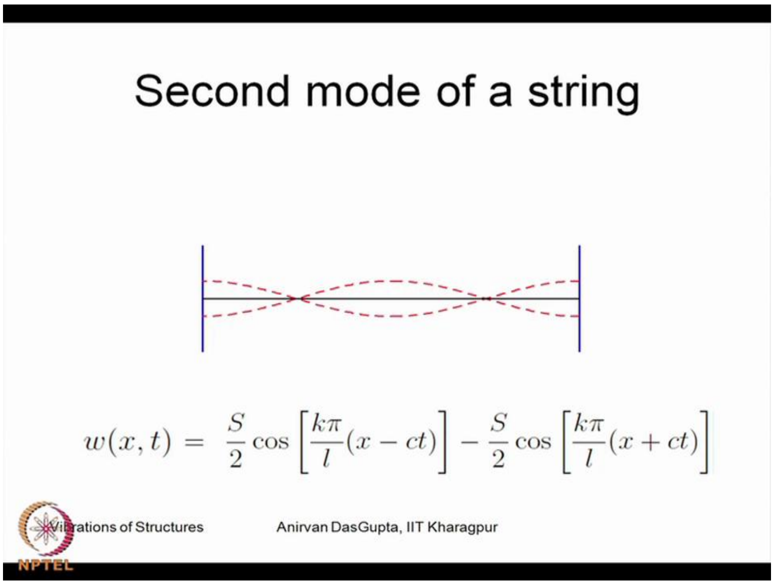

So, in this in this animation here you can see this decomposition. So, you find that... So, this… Here I have shown the second mode of fixed-fixed string and I have also shown the propagating waves, which are return below; and the black curve shows the actual configuration of the string. So, you can see for example, so this if you follow this crest, then this is the negative travelling wave and there is positive travelling wave, this is the positive travelling wave.

Now, these two waves do not satisfy the boundary conditions as you can see, but their super position does. So, super position is shown in black. So, like this you can understand various modes of the string as super position of two travelling waves, one in the positive direction, the other in the negative direction; and these two waves do not satisfy the boundary condition as you can see. Now, how at these waves... these travelling waves what we are decomposing? How do we understand this the existence of these travelling waves?

So, for that we have to look at the solution in detail and which we will do very shortly. So, let us recapitulate what we have studied today. We have looked at the travelling wave solution of the wave equation and we have looked at why we are studying these travelling wave solutions? So, what we have mentioned is that there are various possibilities or systems in which the Eigen values do not exists. For example, for infinite system or system in which the disturbance are not reached the boundary, may be there is an obstacle or there is the crack or discontinuity in the material, at which the propagating wave will have certain interesting characteristics or it will show certain interesting characteristics. For example, from boundary as we have seen in the case of an initial value problem that the wave front propagates and which is the boundary and then there is a reflection back. So, if we understand these interactions of the disturbance with boundaries with inclusions or cracks or discontinuities in the material, then we can evaluate the structure of the material. So, this is one reason why we are studying these waves' solutions for of continuous systems. So, with and this provides with a very strong motivation for studying the wave solution. So, we are going to look at the initial value problem in terms of the wave solution in the next lecture. So, we conclude this lecture here.

Keywords: wave equation, characteristics, travelling waves, initial value problem