Vibrations of Structures

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Lecture No. #16

Axially Translating Strings

Till now in this course, we have been looking at continuous systems which are stationary, which are fixed to the ground; but there exists a class of continuous systems with very unusual dynamic characteristics, which is due to its translation. So, we will consider today in this lecture, the dynamics of translating continua, translating strings. So, where do we find a translating continua? So, the common example is the travelling thread line in looms. In a travelling thread line, you have a string which is translating and it can have transverse vibration. So, this is one example of translating continua. The second example comes from cable car. So, in a cable car, there is a rope which carries the cable car or the gondola. So, the cable or the rope is travelling and the gondola or the cable car is attached to the strings. So, this forms translating continua or translating string. The third example is observed in rolling mills. So, in a rolling mill, material is being rolled and the material is constantly getting elongated and it moves at very high speeds and it can move between two rolls and or one roll; and then the dynamics of this material, which is being rolled, it may be a bar or a rail; so, that has to be studied with that motion, the axial motion of translation; that is imparted in this rolling mill.

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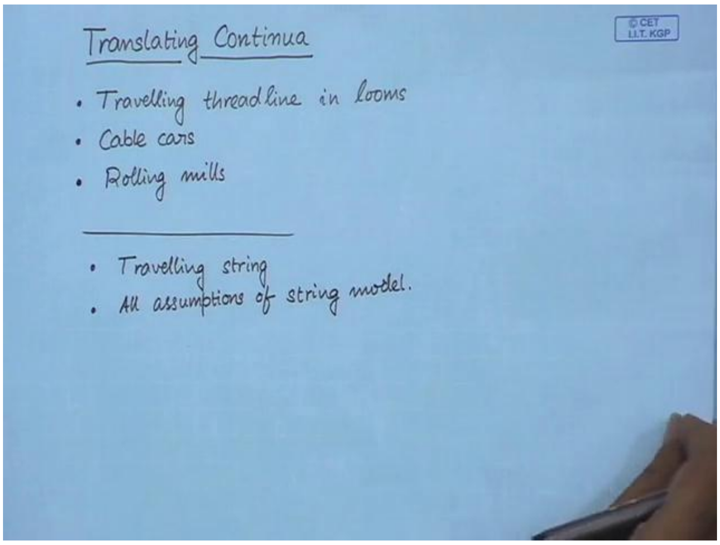

Now, these are variety of situations, where we find translating continua; but we are going to look at very simple situation. We are going to look at a travelling string. So, we are going to study the dynamics of a travelling string. So, when we are going to study this, we will make all the assumptions that we had made for the string model. So, we are going to study the dynamics of travelling strings with the assumptions that we had made for studying the dynamics of normal strings.

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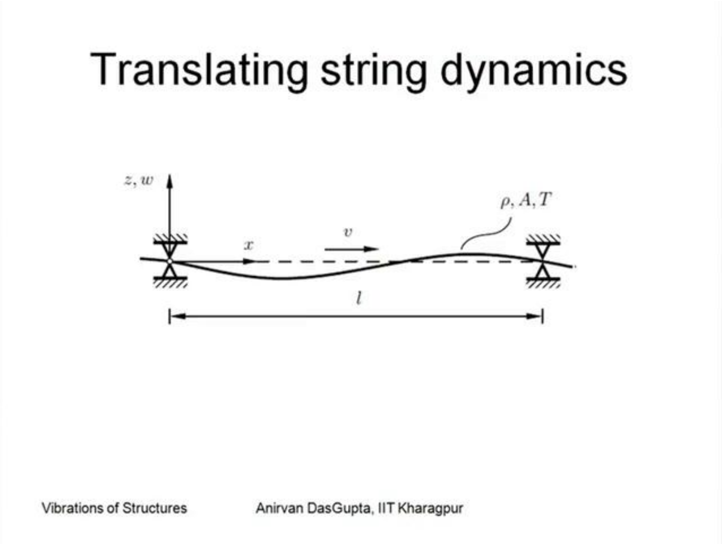

So, let us look at this picture. Here, I have shown a string that is in translation; so this is a string; it is travelling at a constant speed v; it is passing through two inlets; and the length of the string between these two inlets is l. So, we are considering a one dimensional continuum, string which is translating.

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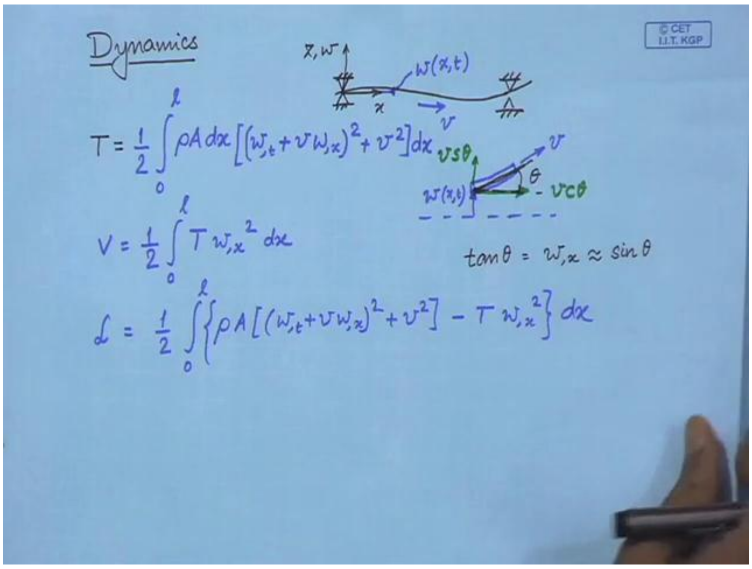

So, let us look at the dynamics. So, we will derive the equation of motion of this translating string using the Lagrangian equation. So, let us write down, therefore, the kinetic energy of the string. So, if you consider the transverse displacement of the string at any location x as w(x,t), then if I draw this little element at location x; so this element has a displacement w(x,t) and it is moving at a speed v. Now, this angle, suppose this angle is theta; then we know that tan of theta is del w/ del x, is approximately equal to is sin of that angle. Now, I can write this kinetic energy as half rho times A times dx, is the total mass of this little element; and now, it has got velocity not only in the transverse direction, but also in the horizontal direction; and the transverse direction velocity is a combination of this del w/del t, which is the local time derivative and there is a there is a convective part, because of it is velocity. So, the transverse component, because of this velocity is v. So, if I write down, this is v sine of theta and this is v cosine of theta. So, therefore… plus v sine of theta, now sine of theta is del w/del x; so, this square plus v cosine theta whole square; now, with the assumption that del w/del x is small, v cosine theta can be approximated as this; and therefore, if I integrate this over the total length of the string, I have the total kinetic energy of the string. The potential energy is same as that of normal string. Therefore the Lagrangian is given by this expression.

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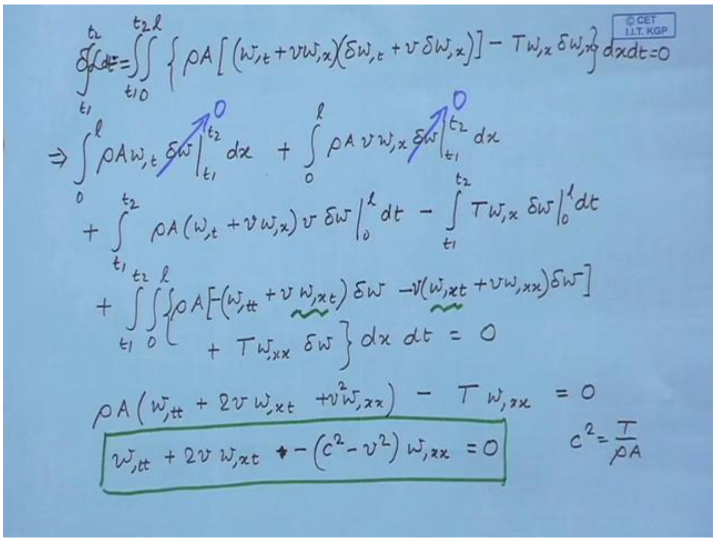

Now, when we take the variation of this… So, here I have of course time integral; so this must vanish. Now, I integrate by parts, this term with respect to time, this term with respect to space and again this term with respect to space. So, this implies… This term has to be integrated with respect to space, and this term also with respect to space. Now, this term get derivated with respect to time, and there appears negative sign. So, we have these terms. Now, here by our standard arguments, at time t equal to t_1 and t_2 there cannot be any variation of the configuration. So, these boundary terms must vanish, time boundary terms. Now, here we have the special boundary terms. Now once again, we use the same argument; and we say that we can vary the domain independent of the boundaries and therefore they must individually vanish. If that is done, then we finally have the equation of motion. So, here we have one v additionally. So, these two terms add up to give us this two v w(x,t); and this term comes from here; and we have this additional term from the potential energy. So, this must be zero. Now, this can be simplified by diving by rho A. I will write it like… where c square is… Now, this is the equation of motion of the travelling string. Now, here, let us identify a few terms. This term is because of the Coriolis acceleration; while this v square del square w/ del x square, this term is the result of the centripetal acceleration. So, if the velocity of translation goes to zero, then these two terms vanish. Then you recover back the equation of motion of a normal string. Now, the boundary conditions are obtained from the boundary terms here. Now, these are the boundary terms. Now, these two can be combined; and for our case that we are going to consider, the boundary conditions are simply… and that makes the boundary terms zero in this case. So, these are our boundary conditions so, I have derived the equation of motion and the boundary conditions of the string.

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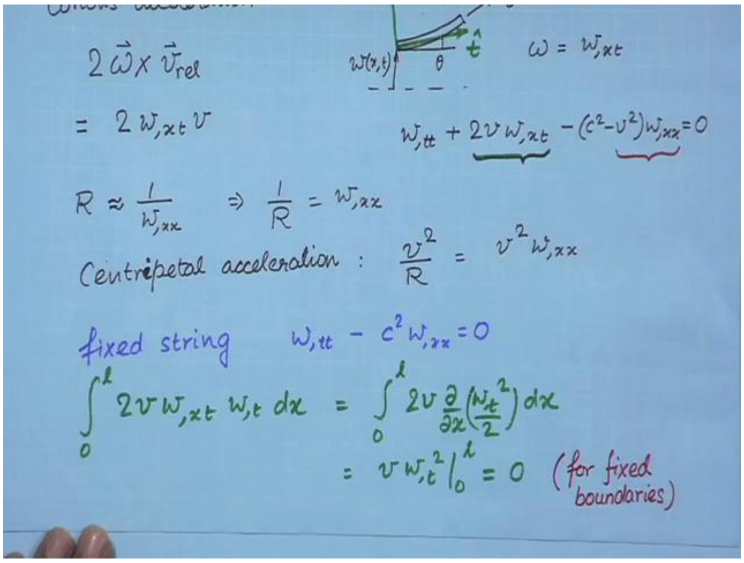

:Let us quickly look at the string, once again. So here, I am drawing an element of that string. Let us quickly understand the equation of motion that we have just now derived from the Newtonian approach. So, in the Newtonian approach, what we need to do is, we need to consider a frame with this element. So, this is the tangential direction to the string and this is normal. So, this frame rotates. So, this angle is theta. The angular velocity of this frame, let us say omega, can be written as, approximately $w_{,xt}$; so it is del square w/ del x t. So, del w/ del x is nothing but this angle, the rate of change of angle gives us the angular velocity. So, when we derive the equation of motion from the Newtonian approach, we must consider the motion of this element in a rotating frame. So, we must consider this Coriolis acceleration, normally we write two omega cross v_{rel}. So omega is a vector perpendicular to the plane of this paper, and v_{rel} the relative velocity with respect to this frame, which is at a location x. So, this frame is only giving the orientation, where it is located at x; it is not moving with the string. So, relative velocity is nothing, but v. So, if you calculate the Coriolis acceleration, then it is two… and this is the term, we find in the equation of motion. So, let me write down the equation of motion once again. So, this is the Coriolis acceleration term, and in addition to this we have the curvature of this element. So, this element is moving on a curved path; and the radius of curvature, so the radius of curvature can be written as approximately… so the curvature which is one over R is the double derivative of the space variable; this is well known; and the centripetal acceleration is given by v square over R; and that turns out to be… So, these two terms we find in our equation of motion. So, when we look at the Newtonian approach, we have to consider these two terms as now we have this element in a rotating frame. So, we have to consider these two terms. Rest of the Newtonian approach is same that we have followed for the string. Now, look at this equation. You see, in a fixed string we have the equation like this. So, this double derivative with respect to space also appears. This appears with c square minus v square. Now, there is a possibility that v might just equal c; in that case this term rubs out and remember, if this term rubs out; this term was responsible for the stiffness of the string as we have here. So, the double space derivative is responsible for the stiffness of the string. So, when the velocity, in the event when the velocity equals c, which is the speed of transverse waves in the string, then the string losses stiffness. So, this is unusual behavior that we see in a translating string. The second thing is this term, the Coriolis acceleration term, this has a single time derivative. Now, when we discussed damping, normally we had in the damping terms a single term derivative. So, the question might derives the whether this term is conservative or not. So, to see that will follow the procedure we had followed previously while discussing damping in continuous systems. We multiply the whole equation with the velocity and integrate it over the domain of the string; and this happens to be… This can be simplified and written like this. So, I can integrate over this spacial coordinate. So, we have this velocity square computed at l minus velocity's square computed at zero. Since we have considered this fixed-fixed boundary, this turns out to be zero. So, this tells us that for the fixed boundary case, the translating string is conservative; rest of it, as long as c is greater than v, the rest is of is a conservative system. Now, this term, which is also conservative; so this does not contribute for the energy at all. SO, for the fixed boundaries, this term is conservative. So, there is a possibility that if the boundaries are flexible, then this system might be non-conservative.

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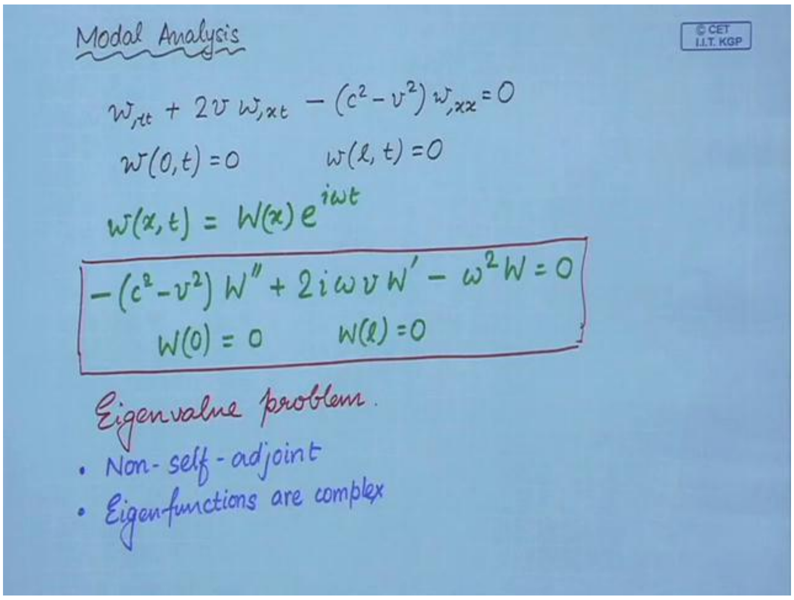

Now, let us look at the modal analysis of the translating strings. So, our equation of motion… the boundary conditions as usual. We consider, as we have done before, the solution; we search for a solution with this structure, as we do in case of modal analysis; and if we substitute in this equation and rearrange then this can be written as… So, this therefore, is our Eigen value problem. So, this is our Eigen value problem for our problem. Now the first thing that strikes us is there is an imaginary coefficient with; and there is an imaginary coefficient and there is a single special derivative of W. Now, it can be easily checked that this problem is non self adjoint. Till now whatever systems we have considered they are all self adjoint. This is the first system which is not self adjoint which makes the dynamics of this system very interesting. The second thing is these Eigen functions, the Eigen functions are complex which we can easily make out from here. So, the Eigen functions are going to be complex. So, let us try to solve this Eigen value problem.

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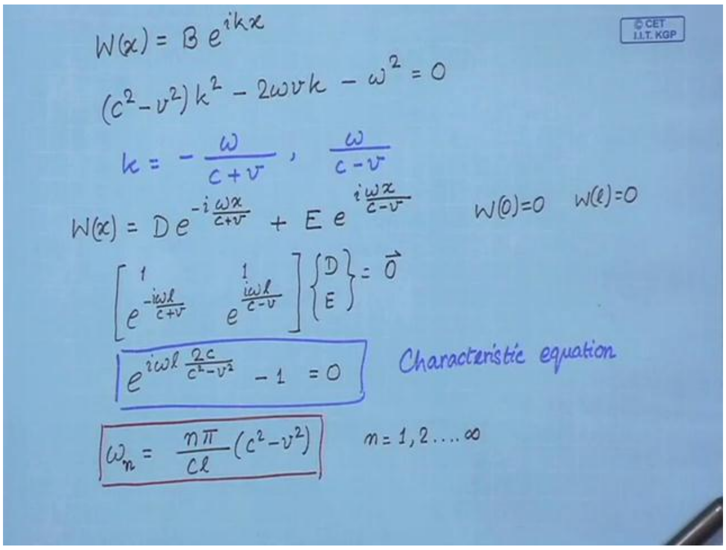

Now, for this kind of system, Eigen value problem, let us assume a solution with this structure. So, if you substitute in here, here I must point out this omega is circular Eigen frequency. So, this is quadratic in k and can be easily solved; and what we obtain… We obtain these two solutions of k. So, from this quadratic polynomial we obtain these two solutions. So, therefore our general solution of the Eigen function may be written as… in this form. So, this is the general solution of the Eigen function; and this must satisfy these two boundary conditions. If we substitute this solution in the boundary conditions, then we can write these two equations in a compact form and for non-trivial solutions, non-trivial solutions of D and E what we must have is the determinant of this matrix must vanish; and that gives us… So, this is our characteristic equation. This is our characteristic equation from which we are going to get the circular Eigen frequencies; and if you solve this the omega get indexed. So, these are the infinitely many circular frequencies of the string. So, from here you can easily see that if the translational velocity is zero then it reduces to the circular natural frequency of a normal string. Now, here we have c square minus v square. Again there is a possibility that this v might equal to c and you can well understand that when a system loses its stiffness, the natural frequency goes to zero. So, when v is equal to c, the natural frequencies go to zero; but what is interesting is all natural frequencies go to zero when v equals c.

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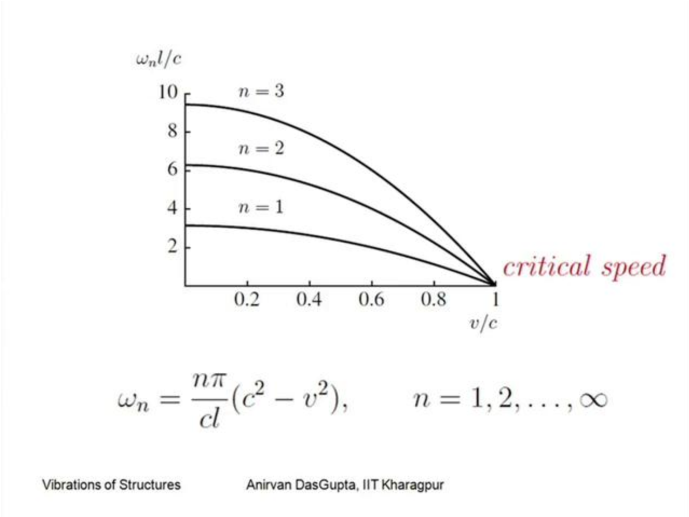

So, here in this figure, I have plotted the circular natural frequencies with v over c. So, these are all non-dimensionalized. So, as you can see that for all the modes when v by c equals to one which is v equals to c, all the natural frequencies, the circular natural frequencies, they got to zero; and the variation you can see from this figure. This speed v equals to c is known as the critical speed. So, at the critical speed, the string, the translating string loses stiffness. Now, we have till now calculated the circular Eigen frequencies. Now, let us look at the Eigen functions. So, these Eigen functions therefore also get indexed; and if you substitute these omega n in here; so when you substitute omega n here and solve for D and E; and you use those values of D and E in the Eigen function expression; so this is straight forward; and you can check finally the Eigen functions are obtained.

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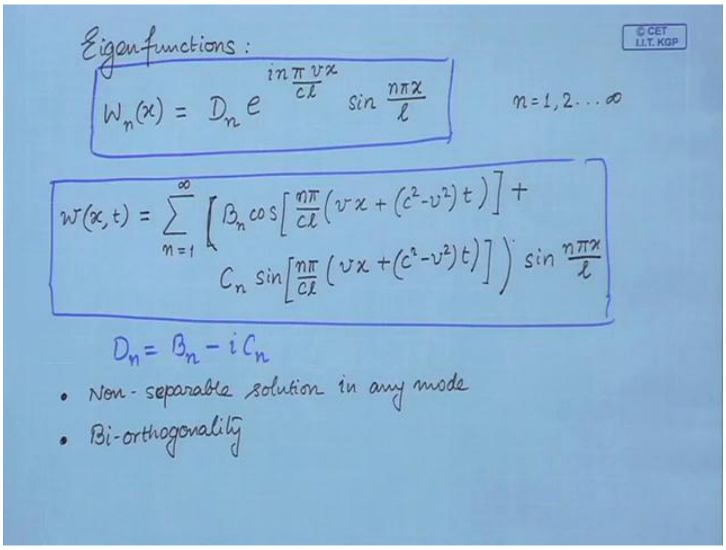

Corresponding to the infinitely many circular natural frequencies the Eigen functions are obtained in this form. Now, here you can immediately see that this is complex. Here, D is an arbitrary constant which is used to normalize the Eigen functions. Here you can see that all the Eigen functions are complex. Therefore, we can write down our general solution using these Eigen functions and linearly combining; so we can write this as… So, this is our general solution where B_n and C_n are arbitrary constants which will be fixed from the initial conditions. So, here we can immediately observe that; first of all here we have used D_n as… So, D_n can in general be complex. So this is the general solution. This is finally our field variable. Here, you can see that the space and time, they are no longer separated. So, here they get combined here in these terms; and this immediately strikes us that this will make the orthogonality condition a little difficult. So, we have a strictly non-separable solution, even the modal solution, in a particular mode the solution is. So, in general the solution is non-separable even for a modal solution. So, this structure of the solution we will discuss later in this course. Here, the orthogonality of the solution as we have discussed till now in this course that does not hold. Here, it is replaced by something called Bi-orthogonality. Now, we have been using this modal solution, modal analysis for solving the initial value problem, forced vibration, damped vibration problem; and every one of these problems, we have to use orthogonality to solve the systems. Now, since the orthogonality is not trivial, it becomes little cumbersome to use the modal approach for solving various problems for travelling strings; but there is a convenient approach, which is using the Laplace transform technique. So, here I will briefly outline the Laplace transform technique for travelling strings. Now, in our forced vibration analysis, we had studied the Green's function technique for solving these forced vibration problems; and we have seen that if we can find out the Green's function of a system, then we can solve the forced vibration problem for harmonic forcing, for arbitrary forcing, general forcing as well as we can solve the initial value problem in a unified manner. So, therefore let us look briefly at this Laplace transform technique for travelling strings.

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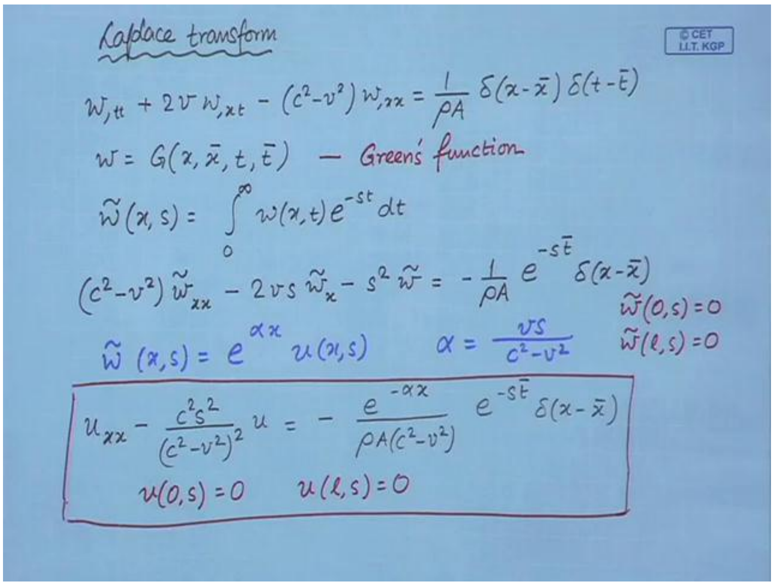

So, what we are going to do is, we are going to essentially solve the Green's function for this system; because as we have seen that once we know the Green's function of the system, then we can solve the forced vibration problem with general forcing; we can solve initial value problem etc. So, let us look at this problem. As we have discussed before with this kind of forcing what we solve is known as the Green's function. So, once we know this Green's function, we can solve various problems for the travelling string. The way to do this is, first we will take the Laplace transform of this equation. So, once you take the Laplace transform, you can easily check; so suppose I define this as a Laplace transform for the field variable W, then if I take the Laplace transform and simplify; this is straight forward. So, essentially we are giving an impulsive input at x equals to x bar and at time equals to t bar. So, this is the final equation in the Laplace domain where s is the Laplace variable. Now, you can solve this equation easily; but this can be further simplified if you use a transformation where alpha is… if you use this transformation in this equation, then this equation simplifies to… The boundary conditions, which are again obtained from here; you can easily see that the boundary conditions… for this in the transformed domain, the boundary conditions are the same.

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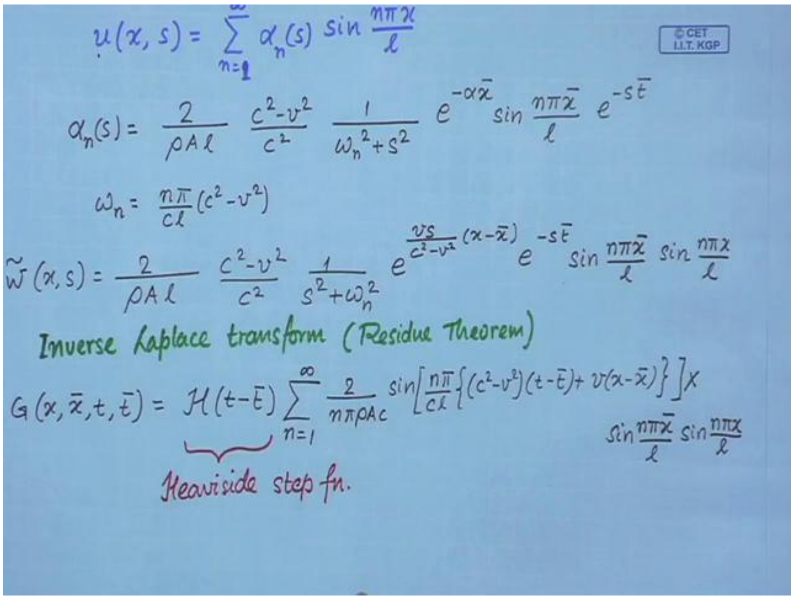

So, this is the boundary value problem that now we have to solve; and now with these boundary conditions; now we have this boundary value problem and with these boundary conditions we can attempt the solution. So, because the boundary conditions are like this, we attempt the solution with this expansion using the modes of a normal string. So, if you substitute this expansion here of course; if you substitute this expansion in the equation and finally solve for alpha n… These are straight forward steps which can be done very easily; we have been following the steps as before. So, if you substitute, then take inner product with this Eigen function and finally you solve for alpha n. Here, omega n are the natural frequencies of the travelling string. So, once you have alpha n, you can substitute back here, and you have u. So, if you have u, then you can substitute that expansion here to get W. So, finally this is the expression that you get. Now, here I have substituted the expression of alpha to finally obtain this; and if you take the inverse Laplace transform of this, which you can do very easily using residue theorem, if you do that then this Green's function, so the solution is the Green's functions; we can write this as… so, use residue theorem to inverse the Laplace transform. Here, it has simple poles at plus or minus i omega n. So, once you do the Laplace transform, we can easily obtain this. Here, this is the Heaviside step, which appears from the causality condition that you are considering the Laplace, this complex integral over contour, which closes the left half space when you do the inverse Laplace transform. So, you have to close on the left half space. So, this indicates that from the causality condition. So, we have this Green's function, which is a solution of the travelling string with an impulse impulsive force at x equal to x bar and t equal to t bar; this gives a solution at x t; and finally, for any arbitrary forcing q, so if you have any arbitrary forcing q(x,t) on the right hand side of the equation of the travelling string, then you can find out the response of the travelling string using this integral.

So, let us see what we have discussed in this lecture. So, we have discussed the dynamics of travelling strings. We have performed the modal analysis of strings and we have found that these Eigen functions of the travelling string are complex and the problem is non self-adjoint. This makes the dynamics actually very interesting though it is theoretically little cumbersome to handle; because now the orthogonality is replace by bi-orthogonality. But then we have this Laplace transform technique and we have derived the Green's function for the travelling string. Now, once we have this, then we can solve the initial value problem, the forced vibration problem etc. in unified manner. So, this gives us a very powerful way of approaching the dynamics of travelling strings. So, with that we conclude this lecture.

Keywords: translating string vibration, critical speed, non self-adjoint problem, complex modes, Green's function.