

Vibrations of Structures
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Lecture No. # 14
Forced Vibration Analysis - III

So, we are being looking at forced vibration analysis of one-dimensional structures in the positive lectures. Today, we are going to look at; we are going to continue that our discussion on forced vibration. In the previous lectures, we have seen harmonic forcing, systems with harmonic forcing, which we have separated in space and time. There is a distribution function for the force, and then there is a multiplying harmonic function of time. Today, we are going to look at general forcing, which means forcing which need not necessarily be separable in space and time.

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General forcing

$$\mu(x) w_{,tt} + K[w] = 0$$

$$w(0,t) = 0 \quad w(l,t) = 0 \quad w(x,0) = w_0(x) \quad w_{,t}(x,0) = v_0(x)$$

Dirac delta fn.

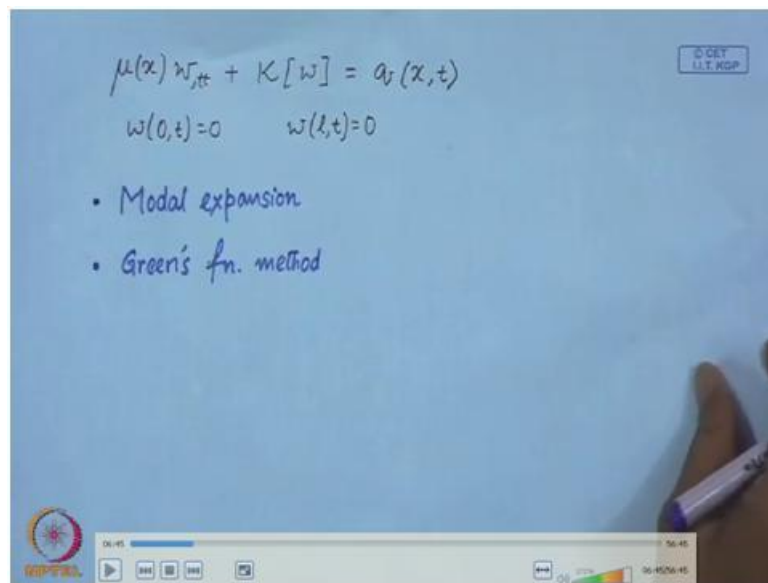
$$\mu w_{,tt} + K[w] = v_0(x) \delta(t) + w_0(x) \delta'(t)$$

$$w(0,t) = 0 \quad w(l,t) = 0 \quad w(x,0) = 0 \quad w_{,t}(x,0) = 0$$

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So, we are going to look at general forcing in today's lecture. Now, let us consider some examples of this general forcing. In one of the previous lectures, you have already seen that an initial value problem with certain boundary conditions and initial conditions; so an initial value problem can equivalently stated as... So, these two statements are equivalent. So, essentially we are converted the initial value problem to forced vibration problem. Here, this is the Dirac delta function and this is the time derivative of the Dirac delta function at t equal

to zero. This problem is the forced vibration problem, which is equivalent to this initial value problem. In this case, as you can see this kind of forcing is very special in form, and this is not harmonic forcing. Forcing of this type are when a force is travelling on a string for example, on a bridge, you have car travelling at a certain speed so you have the travelling load problem or travelling mass problem. Such forces load the structure, and they are not harmonic forcing. In fact, in such problems space and time are coupled, they are not separable as we will presently see in this lecture. Problems of this type can be considered to be of general forcing. I mean, these are the problem with general forcing.



Let us look at, how do we solve general forcing problem. So, we consider for example, let me state the problem. We have the equation, the differential equation of motion with of course certain boundary conditions.

This $q(x,t)$ is the is the general form of the force, which we are now considering. Now to solve or approach such problems, we have various methods. What we are going to consider here today are the modal expansion methods. We are going to concentrate on this modal expansion method and towards the end, we are also going to look at the Green's functions method. Essentially today, we are going to look at these two methods. Let us start with this modal expansion method.

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Modal Expansion Method

$$\mu(x) \ddot{w}_{,tt} + K[w] = q_f(x,t) \quad w(0,t)=0 \quad w(l,t)=0$$

$$w(x,t) = \sum_{k=1}^{\infty} p_k(t) \underbrace{W_k(x)}_{\text{Eigenfunctions}}$$

Eigenvalue problem

$$-\mu \omega_k^2 W_k + K[W_k] = 0 \quad W_k(0)=0 \quad W_k(l)=0$$

$$\sum \mu(x) \ddot{p}_k W_k + K[\sum p_k W_k] = q_f(x,t)$$

$$\sum (\mu(x) \ddot{p}_k W_k + \underline{K[W_k]} p_k) = q_f(x,t)$$

$$\Rightarrow \sum_k (\ddot{p}_k + \omega_k^2 p_k) \mu(x) W_k(x) = q_f(x,t)$$

We have discussed in the previous lecture how or why this modal expansion technique works? It is because the modal space forms or the Eigen functions of the Eigen value problem of a system form a complete basis, which means any shape of any configuration of the system can be represented by these Eigen functions. This is the basis of this modal expansion technique. So, we are going to look at this problem with certain boundary condition. As we have discussed the solution, we will represent in this form. So, these are the Eigen functions of the corresponding Eigen value problem. When we substitute this in the equation of motion... So, these Eigen functions already satisfy the boundary conditions; so we need not worry about the boundary conditions. So here, we now have to determine these coefficients, so that this expansion satisfies the equation of motion. We have substituted this expansion in equation of motion and now here this K is a linear differential operator; and therefore what we can do is we can commute this summation and the operator and we can rewrite this equation in this form. Now recall, that the Eigen value problem for the system. So, this is the statement of the Eigen value problem; so here, the differential equation of the Eigen value problem. From here we can see this expression of the stiffness operator operating on the Eigen function; and then what we obtain... this. Now if I use the orthogonality condition, so I take the inner product. So what I do is, I multiply both sides of this equation with w_j and integrate over the domain of the problem. Then we know that from orthogonality of this Eigen functions, we already known that, if j is not equal to k then this is zero. So, what this is going to do is filter out the j^{th} coefficient in the expansion. So, we are going to get... So, we are going to get the j^{th} equation and therefore, I can write this in this form where this force

term f_j is given by... and this I can take for different W_j and I can generate all these equations corresponding to the coordinates of the individual modes. So, I have these equations governing the coordinates of the system in the configuration space or modal space and that is a forced dynamics problem where the forcing in the j^{th} mode is expressed in this form. So, once we have the applied external force, we can easily compute this from this external force. Now, this is a second order ordinary differential equation. So, we will need two initial conditions for this problem to solve the problem completely. So what we need are the initial conditions. Now, we are given these initial conditions on the field variables. So, we once again have this expansion... and similarly from the velocity initial condition, we get this equation. So, we have to solve, we have to solve this equation with initial conditions on p at zero and \dot{p} at zero, which we are going to now solve from these two equations.

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The image shows a handwritten derivation on a blue background. At the top, it states:
$$\Rightarrow \sum_k (\ddot{p}_k + \omega_k^2 p_k) \mu(x) W_k(x) = q_r(x, t)$$
 Below this, it says "Inner product with $W_j(x)$ " and notes the orthogonality condition: $\langle W_j, \mu W_k \rangle = 0 \quad j \neq k$. The next step is:
$$(\ddot{p}_j + \omega_j^2 p_j) \int_0^l \mu W_j W_j dx = \int_0^l q_r(x, t) W_j(x) dx$$
 This is then boxed to give:
$$\Rightarrow \boxed{\ddot{p}_j + \omega_j^2 p_j = f_j(t)}$$
 Finally, the coefficient $f_j(t)$ is defined as:
$$f_j(t) = \frac{\int_0^l q_r(x, t) W_j(x) dx}{\int_0^l \mu(x) W_j(x) dx}$$
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Once again if we use the inner product; now here remember that this is the orthogonality condition. Therefore, what we must do to solve these coefficients in a straight forward manner., so what we do is we multiply both sides with μ and W_j and integrate over the domain of the problem. Once we do that, it is going to filter out the j^{th} term in the expansion; and therefore that is how we can solve $p_j(0)$ and we can take, we can obtain all this infinitely many initial condition on the modal coordinate. Similarly, we can also find out the initial velocity condition. So, once we have all this initial conditions p and \dot{p} , then we can solve this problem. Once we have these initial conditions then we can very easily solve, we already know the solution of this kind of forced dynamics. Here one thing, do note that what we have

achieved in this process is the discretized equations of motion. We have discretized the force vibration problem. Now, let us look at some examples.

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Initial conditions

$$w(x,0) = w_0(x) \quad w_t(x,0) = v_0(x)$$

$$w(x,0) = \sum_{k=1}^{\infty} p_k(0) W_k(x) = w_0(x)$$

$$w_t(x,0) = \sum_{k=1}^{\infty} \dot{p}_k(0) W_k(x) = v_0(x)$$

Inner product with $\mu(x) W_j(x)$

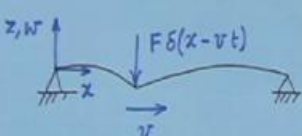
$$p_j(0) \int_0^l \mu W_j^2 dx = \int_0^l \mu(x) w_0(x) W_j(x) dx$$

$$p_j(0) = \frac{\int_0^l \mu(x) w_0(x) W_j(x) dx}{\int_0^l \mu W_j^2 dx}$$

$$\dot{p}_j(0) = \frac{\int_0^l \mu v_0 W_j dx}{\int_0^l \mu W_j^2 dx} \quad j = 1, 2, \dots, \infty$$

$\langle W_j, \mu W_k \rangle = 0$
 $j \neq k$

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z, w

$F \delta(x-vt)$

v

$$\rho A w_{tt} - T w_{xx} = -F \delta(x-vt)$$

$$w(0,t) = 0 \quad w(l,t) = 0$$

$$w(x,t) = \sum_{k=1}^{\infty} p_k(t) \sin \frac{k\pi x}{l}$$

eigenfunctions

$$\ddot{p}_j + \omega_j^2 p_j = -\frac{2F}{\rho A l} \sin \frac{j\pi vt}{l} \quad j = 1, 2, \dots, \infty$$

$$\omega_j = \frac{j\pi c}{l}$$

The first example that we are going to consider is that of a force travelling on a string. Let us look at this problem. So, we have a taut string on which there is a concentrated force which is travelling at a speed v . So, this F is a constant. So, let me write down the mathematical formulation of this problem. So, we have the string equation of motion with this forcing and of course the boundary conditions. These are the boundary conditions and there can be certain initial conditions. Now here, we will be looking at the forced problem using this expansion in

terms of the Eigen function of the taut string. Now, if you substitute this here and follow the procedure that we just now discussed then for the j^{th} mode the discretized equation of motion appears to be in this form, where this ω_j ... So, this is our equation of motion.

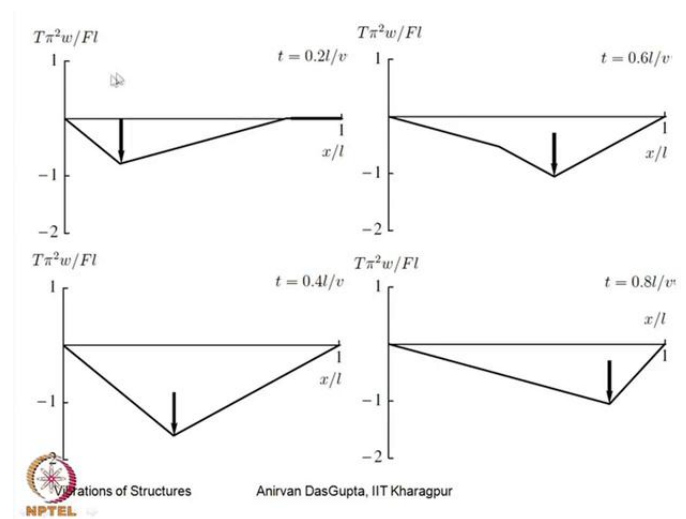
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The image shows a handwritten derivation on a blue background. At the top, it says "eigenfunctions" and shows the equation $\ddot{w}_j + \omega_j^2 w_j = -\frac{2F}{\rho A c} \sin \frac{j\pi vt}{l}$ for $j = 1, 2, \dots, \infty$. Below this, the particular solution is given as $w_j(t) = C_j \cos \omega_j t + S_j \sin \omega_j t - \frac{2F l}{\rho A c j^2 \pi^2 (c^2 - v^2)} \frac{\sin j\pi vt}{l}$. A note says "from i.c." and gives initial conditions $w(x,0) = 0$ and $w_t(x,0) = 0$, leading to $C_j = 0$ and $S_j = \frac{2F l v}{\rho A c j^2 \pi^2 (c^2 - v^2)}$. A boxed equation shows the final solution: $w(x,t) = -\frac{2F l}{\rho A c j^2 \pi^2 (c^2 - v^2)} \sum_{j=1}^{\infty} \frac{1}{j^2} \left[\sin \frac{j\pi vt}{l} - \frac{v}{c} \sin \frac{j\pi ct}{l} \right] \sin \frac{j\pi x}{l}$. At the bottom, it specifies $0 < t < \frac{l}{v}$. There is a small logo in the bottom left corner.

Now the solution of this equation can be easily written as... So, this is the solution of this equation provided v is not equal to c ; where c is the speed of transverse waves in the string. So, the c is the speed of transverse waves in the string. So, as long as the velocity or the speed of travel of this force is different from c , is less than c for example, this is going to be the solution. So, you see when v equal c , and if you look at this ω_j then we will find, we realize that when v equal c , this becomes resonant forcing of the string in the j^{th} mode and therefore in all modes. So, all modes will be in resonance, if v equals c . So, that would be a resonant forcing of the string and the solution is, as you know already is in a different form. So, this solution is valid as long as v is not equal to c . You can also find out the resonant solution using the standard techniques that you have studied in discrete vibration problem, in a course on discrete vibration. Now, this is the solution from therefore, here these are unknown coefficients which form the homogeneous solution of the system, which have to be found out from the initial conditions. We will consider a stationary string before the force starts travelling on the string; so initial displacement is zero; initial velocity is zero; and if you solve for this coefficients then you will see that the coefficients turn out to be... and therefore, the final solution of our problem... So, this is our final solution; and this solution is

valid of course when v not equals to c ; not only that, but also time interval for which the force is on the string. As soon as the force crosses the string, this solution is no longer valid. We have to solve for the string with certain initial conditions which are the final conditions at t equal to l over v of this. So, we will get the final displacement of the string and the final velocity at t equals to l over v from this expression that goes in as the initial condition of the string in free vibration. So, that will be an initial value problem. So, after this we have to solve an initial value problem to understand how the string behaves after the force leaves the string. So, that is a initial value problem which we have to solve using the final conditions from this solution; and we have already seen how the initial value problems are solved. Let us look at the certain configurations of the string. Here, you will see some configurations of the string at certain time intervals for example at $0.2 l$ over v ; so this force has travelled one-fifth the total length of the string. This is how the string looks like; you see that this portion of the string is as yet undisturbed. But when t reaches $0.4 l$ over v , so forty percent of the distance has been covered, this information that there is a force travelling on the string has reached this end. Here at $0.6 l$ over v , you can see bend forming here. So actually the information is now reflecting back and forth on the string; and this shows $0.8 l$ over v . So, these are snapshots; here I have a simulation or the animation of the same thing.

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So, I have taken number of terms in our expansion as I just now discussed and I have plotted the phase of the string at different time instants and prepared this animation. Now, here you can see that the string appears to move in a very strange manner; it looks static, certain portion of the string looks static in certain time instants. So what is happening is this

disturbance which is created by this travelling force travels on the string and a wave reflects back and forth on the string from the boundaries. Unless the disturbance reaches the boundary of the string, the string does not know that a force is travelling; so it remains, it look like static. Remember two things in this animation. One thing is that this is a slow motion of the actual thing that is happening. It is very difficult to observe this kind of motion by naked eye. You have to record it and run it in slow motion. The second thing is, this animation does not take care of the motion of the string after the force has left. This is just a repeat of the animation when the force travelling on the string. So here, we do not see that initial value problem solution. Now this is at low speeds when v over c is 0.1; so it's 10 percent of the critical speed. In this animation, this is near resonance, so v over c is 0.95, 95 percent of the critical speed. Here you will find that those reflections or the kinks etc., they are not so easily observable at least; and the amplitude is growing as the force moves from left to right which is a typical behavior near resonance solution. At resonance as you know the amplitude grows linearly with time, so this is near resonance, slightly lower, so you can see the behavior of the string is distinctly different from the previous behavior. So. This motion, this is a transient motion and this has to be understood in terms of the propagation of waves in the string which we are going to take up later in this course.

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Green's function method

$\mu(x) w_{,tt} + K[w] = \delta(x - \bar{x}) \delta(t - \bar{t})$

b.c. $w(0, t) = 0$ $w(l, t) = 0$ Dirac delta distribution

i.c. $w(x, 0) = 0$ $w_{,t}(x, 0) = 0$ Impulse at $x = \bar{x}$ & $t = \bar{t}$

$w(x, t) = G(x, \bar{x}, t, \bar{t})$ Green's function.

- Homogeneous b.c.
- zero i.c.

for $a_j(x, t)$

$$w(x, t) = \int_0^t \int_0^l G(x, \bar{x}, t, \bar{t}) a_j(\bar{x}, \bar{t}) d\bar{x} d\bar{t}$$

So, we have looked at the modal expansion solution. There is another method which we are now going to look at briefly. We have already come across this Green's function method in our previous lecture. So what we are going to solve is... So in the Green's function method, we solve the problems of this type where the forcing has a very special form; these are Dirac

delta functions, Dirac delta distribution. So the forcing is of a very special form like this. So what in effect this represents is, it represents the impulsive loading at the location \bar{x} and time t equals to \bar{t} . So we give an impulsive loading at the location x equals to \bar{x} of the system and at time t equals to \bar{t} ; and we are going to look at the solution of the system to such loading. And here the boundary conditions are homogeneous and the initial conditions are all zero. The solution of this system is known as the Green's function. We will represent this in this form. So, these points are important that we have homogeneous boundary conditions and zero initial conditions. So always this Green's function is calculated with these conditions. So, this will be classically assumed whenever Green's function is calculated; so, homogeneous boundary condition and zero initial condition. We know, we have seen in previous lecture, how non homogeneous boundary conditions can be converted to homogeneous boundary conditions. So, if required we have to do that and then calculate the Green's function. Why? Let us first look why this or how this Green's function method works? We frame that for any arbitrary forcing when we have forcing in the form of $q(x, t)$ a general forcing then the solution actually can be obtained in this form. Once we have this Green's function, the solution to arbitrary forcing can be determine like this. Let us see, why this happens? How this is happening?

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$$\begin{aligned}
 P[w(x,t)] &= q_f(x,t) & P[\cdot] &= \left(\mu \frac{\partial^2}{\partial t^2} + K\right)[\cdot] \\
 P[G(x, \bar{x}, t, \bar{t})] &= \delta(x-\bar{x})\delta(t-\bar{t}) \\
 P[w(x,t)] &= P\left[\int_0^t \int_0^l G(x, \bar{x}, t, \bar{t}) q_f(\bar{x}, \bar{t}) d\bar{x} d\bar{t}\right] \\
 &= \iint P[G] q_f(\bar{x}, \bar{t}) d\bar{x} d\bar{t} = \iint \delta(x-\bar{x})\delta(t-\bar{t}) q_f(\bar{x}, \bar{t}) d\bar{x} d\bar{t} \\
 &= q_f(x,t)
 \end{aligned}$$

Let me write this differential equation. Let me write this differential equation as... So, this is our problem where P ... I am representing this full operator with P . Now, what we do is we are actually solving this problem first with of course those homogeneous boundary conditions

and zero initial conditions. And then what I said was the solution was given by this. So, let me apply the operator P on this. Now this integral is over \bar{x} and \bar{t} . So, this operator can go inside and it will operate only on G, and we know that $P[G]$ is... and this is equal to therefore... So, therefore this is definitely a solution; so w given by this argument is definitely a solution of this system. So, this shows that we have the solution in terms of the Green's function.

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$\rho A w_{,tt} - T w_{,xx} = \delta(x-\bar{x})\delta(t-\bar{t})$
 $w(x,t) = G(x,\bar{x},t,\bar{t})$
 $\tilde{w}(x,s) = \int_0^\infty w(x,t) e^{-st} dt$
 $\mathcal{L}[w_{,t}(x,t)] = s\tilde{w}(x,s) - w(x,0)$
 $\mathcal{L}[w_{,tt}(x,t)] = s^2\tilde{w}(x,s)$ } with zero i.c.

$$\left. \begin{aligned} c^2 \tilde{w}_{,xx} - s^2 \tilde{w} &= -\frac{1}{\rho A} e^{-s\bar{t}} \delta(x-\bar{x}) \\ \tilde{w}(0,s) &= 0 \quad \tilde{w}(l,s) = 0 \end{aligned} \right\} \text{BVP}$$

$$\tilde{w}(x,s) = \sum_{n=1}^{\infty} \alpha_n(s) \sin \frac{n\pi x}{l}$$

Now, let us calculate the Green's function for the string. So, we are giving an impulsive loading at time t equals to \bar{t} at a location x equals to \bar{x} ; and the solution is the Green's function. We are considering homogeneous boundary condition and zero initial condition. First what I will do to determine this Green's function, we have to solve this Green's function known by solving this system, this problem. What we are going to do to simplify things is, we will take the Laplace transform of this equation. We defined the Laplace transforms as follows, and we know that the Laplace transform of... Here this is zero because the initial conditions are zero; similarly Laplace transform is given by this with zero initial conditions. Now, once you take the Laplace transform of this equation then and rearrange, we obtain this equation along with the boundary conditions like this. This is our boundary value problem that we need to solve now. Now, because of, the homogeneous, we have homogeneous boundary condition, therefore we can look for solutions in this form. So, substituting this in the boundary value problem and when we rearrange, take the inner product both sides with

$\sin n \pi x$ over l . So, we substitute and take inner product. So, we substitute this in the differential equation, and take inner product we $\sin n \pi x$ over l and both sides; and then it filters out the n^{th} coefficient, which I can write, this can be done very easily. So, we will obtain this coefficient; and finally the solution therefore is obtained. Now this is the solution in the Laplace domain. Now we have to take Inverse Laplace transform, which can be done by Residue theorem. And the final solution, these are simple steps, I will write the final solution.

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Substituting and taking inner product with $\sin \frac{n\pi x}{l}$

$$\alpha_n = \frac{2}{\rho A l} \frac{1}{s^2 + \omega_n^2} e^{-s\bar{t}} \sin \frac{n\pi \bar{x}}{l}$$

$$\tilde{w}(x, s) = \sum_{n=1}^{\infty} \frac{2}{\rho A l} \left(\frac{e^{-s\bar{t}}}{s^2 + \omega_n^2} \right) \sin \frac{n\pi \bar{x}}{l} \sin \frac{n\pi x}{l}$$

Inverse Laplace transform (Residue theorem)

$$G(x, \bar{x}, t, \bar{t}) = \mathcal{H}(t - \bar{t}) \sum_{n=1}^{\infty} \frac{2}{n\pi \rho A l} \sin \left[\frac{n\pi c}{l} (t - \bar{t}) \right] \sin \frac{n\pi \bar{x}}{l} \sin \frac{n\pi x}{l}$$

Heaviside step

$$w(x, t) = \int_0^t \int_0^l G(x, \bar{x}, t, \bar{t}) q_f(\bar{x}, \bar{t}) d\bar{x} d\bar{t}$$

Here this is the Heaviside step, which comes from the causality which means that the solution does not exist before the impulse is provided. So, t less than t bar is zero, the solution is zero. So, the solution is non zero only when t is greater than t bar. So, this is your solution of the Green's function and as I have discussed for any arbitrary forcing, you can determine the solution using this integral. This is the very powerful method of approaching forced vibration problems with arbitrary forcing. Now you can, we know how to solve the Green's function method. Here we have used the modal expansion technique to solve the boundary value problem. In a previous lecture we have seen how the Green's function was solved in a different method. This approach gives as a very powerful method of solving forced vibration problem of continuous systems. Now, in this lecture we have considered the general forcing of one-dimensional continuous system and we have looked at the modal expansion technique and the greens function technique for solving the forced vibration problem of one-dimensional systems. So, with that we conclude this lecture.

Keyword: general forcing, travelling force on string, modal expansion method, Green's function, Laplace transform.