Vibrations of Structures

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Lecture No. # 13

Forced Vibration Analysis – II

So, in the previous lecture we had initiated discussions on forced vibration analysis of one-dimensional continuous systems. So, in our previous lecture, we started with the case of harmonic forcing which as we discussed is a separable forcing in space and time. So, just to reiterate let me recapitulate the what we started.

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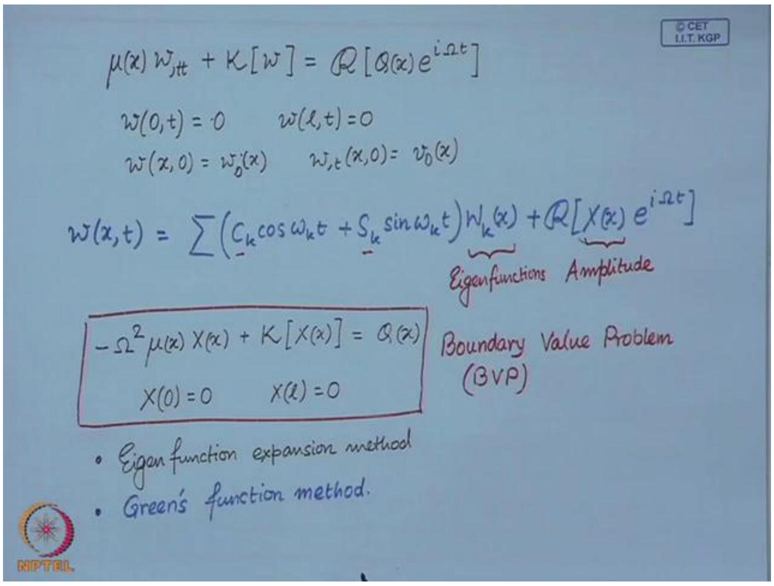

So, we were looking at the forced vibration analysis of a system governed by this equation of motion, along with boundary conditions, say for example, let say zero, and some initial conditions. So, this was our problem. To this problem, we considered, we observed that the solution can be written out as the homogeneous solution, which we have expanded in terms of the Eigen functions of the unforced problem; so, this is the homogeneous solution and along with that we have the particular solution, where this X is the amplitude function of the particular solution and these are the Eigen functions. So, this is the solution form that we have for a system like this; and when we substitute this solution form in the equation of motion what we obtain is… so, this homogeneous solution actually goes to zero, once we substitute it here. The contribution from the particular solution which can be written upon simplification; and along with this the amplitude function of the particular solution must also respect the boundary conditions.

So, for these boundary conditions correspondingly, we have these as the boundary conditions on the amplitude function. Now, this we define as the boundary value problem. So, this is the boundary value problem which we have to solve in order to calculate the amplitude function of the particular solution. So, this is the unknown here; and of course, we have unknowns C_k and S_k which must be determined from the initial conditions after we have solved this amplitude function of the particular solution.

So, we solve this boundary value problem, determine X, substitute it here and then apply the initial conditions to solve for C_k and $S_{k;}$ and the solution of C_k and S_k we have seen in one of our previous lectures. So, today we are going to focus on the solution of this boundary value problem. Now in the previous lecture also we discussed about the solution in terms of Eigen function expansion method, using the Eigen function expansion.

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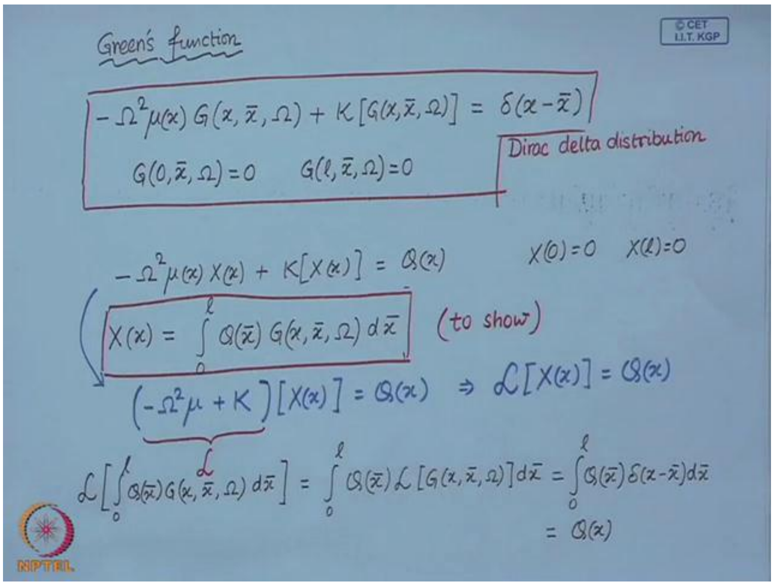

Today we are going to look at this boundary value problem using what is known as the Green's function method. Now, what is this Green's function? Now Green's function is the solution of the boundary value problem. So, we write it like this. We represent the Green's function using G, capital G such that… So, this was the differential equation of the boundary value problem as you can see here. So, the Green's function is the solution of this differential equation with very special form of Q(x) which is the Dirac delta applied at x equal to x bar. So, this is the Dirac delta distribution; and of course, the Green's function also satisfies the boundary conditions of the problem. So, this boundary value problem defines the Green's function. So, you realize that the Green's function is the solution with a special forcing in terms of Dirac delta applied at x equal to x bar. So, which means that Green's function is the solution of one dimensional continuous, what we have discussing at present, so, one-dimensional continuous system when you apply harmonic concentrated forcing of frequency capital omega applied at the location x equal to x bar. So, the solution of that system is the Green's function. So therefore, the Green's function is the solution, is the amplitude at, because of harmonic forcing of frequency capital omega applied at x bar. So, that is the significance of this Green's function. Now when we talk about green's function we always use homogenous boundary conditions. So, the boundary conditions must be homogenous like this. Now if it is non-homogenous then we have discussed one of our previous lectures that how to homogenize the boundary conditions. So, we are going to discuss the Green's function in the context of only homogenous boundary conditions. Now how is this solution of the Green's function going to help us in solving a problem with an arbitrary force distribution Q(x).

So, let us look at the problem. So, let me write again. So, this is the problem of course, with this boundary condition. This is what we want to solve. We claim that the solution of this, So, the amplitude function corresponding to an arbitrary forcing capital Q(x) is given by integrating over the domain of the problem performing this integral. So, we integrate over x bar; we integrate over x bar from 0 to l; x bar goes from 0 to l; and we are left with a function of x which is the solution of this problem. So this is what we have to show. Now let me rewrite this differential equation as… This is only a way of representing this in a short form. So, I will write this as some L, some operator L acting on X gives Q. So, this L is… So, this is L. So, this is only a short way of, abbreviated way of writing this whole thing. Now if this is the solution, let us see what happens. If this is the solution, I substitute this in here. So L, so this is the linear differential operator. So, I can assume that this can commute with this integral, because this integral is over x bar; L is an operator on x. So, here I can write, I can interchange the integral and the operator; and since this acts on functions of x, so, it will act only on G. Now L acting on G is what we already know. So, this is nothing but L acting on G. So, therefore, this integral actually reduces to delta of x minus x bar integrated over x bar and that is nothing but Q(x) which is the right hand side. So, which means that this must be a solution; this must be a solution to this problem. So, we need to solve for this Green's function which is for a very special form of force distribution and which might possibly be quite simple as well we hope. And once this is done, then we can, given any force distribution we should be able to continue it on this integration being performed or can be is doable, then we can easily solve for the amplitude function for an arbitrary forcing, in which case the solution we can write, our solution of the original problem as the homogenous solution plus the real part of the of this argument. So, we can easily find out the total solution and from here the unknowns can be found by using the initial conditions. Now the problem remains that how to solve for this G. So, let us take this example of a string once again. So, we have a concentrated harmonic forcing at x equal to x bar. So, this is the problem we are going to solve essentially.

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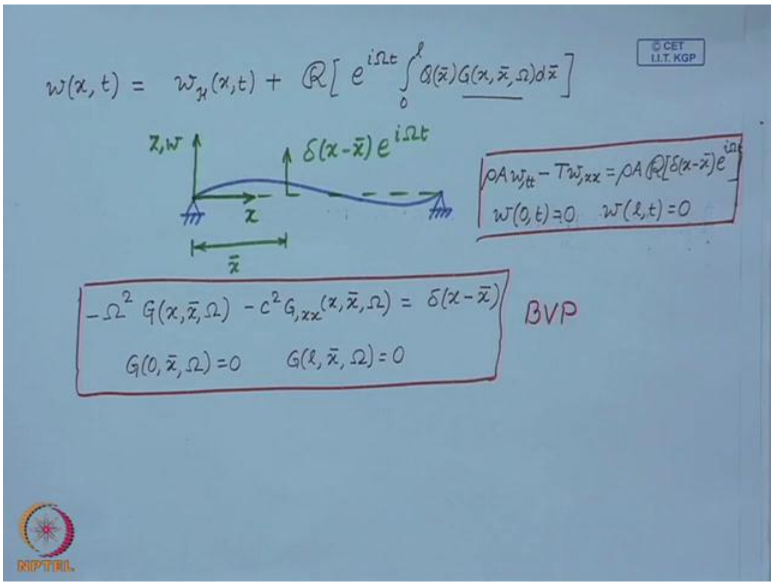

So, finding out the Green's function is essentially solving this problem; and if you write down the equation of motion and you substitute the solution form that we have been considering, we get the boundary value problem. So, we are going to solve; Let me write down the equations. Let us solve the problem like this. So, this with the boundary conditions, so this is the problem we want to solve. The boundary value problem for this can be written as… So, this is our boundary value problem. Now to solve this, what we do is we are going to look at two regimes of this string. So, one is… let me call this as the left regime and this as the right regime. So, from 0 to x bar is the left L and from x bar to l is the right regime R.

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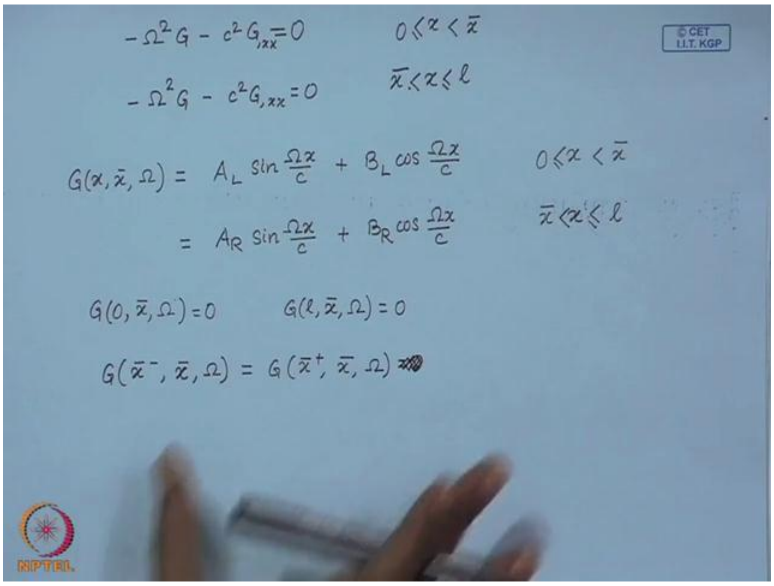

So, in these two regimes, we can rewrite this differential equation of the boundary value problem as… So, this differential equation of the boundary value problem over these two regions of the string can be written like this. So, only at x equal to x bar, there is this Dirac delta function acting; in the other regions there is no forcing. Now, we can easily write the general solution for the Green’s function in these two regions. So, these solutions can be written as… So, over the two regions we have these two solutions. Now, we also have the boundary conditions. Now, we also have the continuity conditions at this junction x equal to x bar. So, see the solution of the boundary value problem, the differential equation of the boundary value problem. So, the general solution is given by this. Now, we have these four unknown coefficients A_L, B_L, A_R, and B_R which are to be solved in order to determine the Green’s function. So, we will need four conditions for solving these four unknowns. Now, two conditions are obtained directly from the boundary conditions. One further condition has been obtained from the continuity of the string at x equal to x bar. The fourth condition must of course come from the force condition. So, we have the force balance at x equal to x bar. Now, this force condition for the string can be easily found by directly integrating the differential equation of the boundary value problem.

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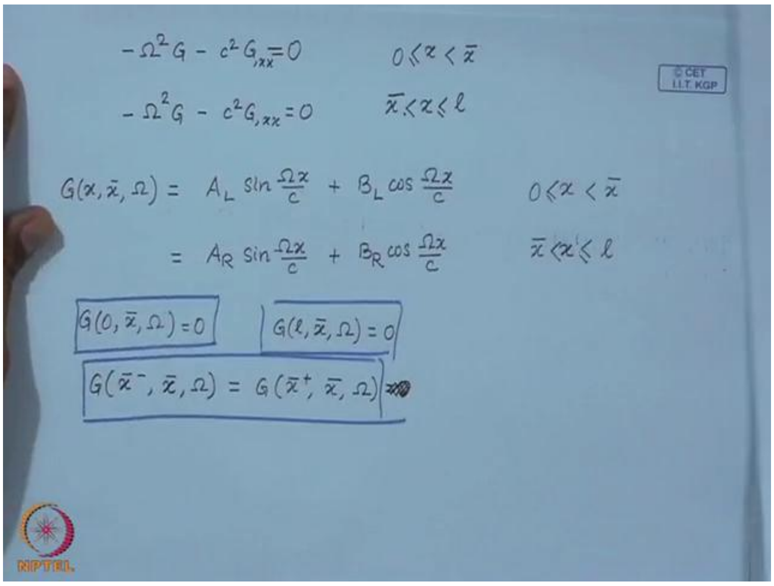

So, our differential equation of the boundary value problem is given by this. So, if we integrate this, both sides of this equation over the domain of the string, then we can easily obtain the fourth condition which is nothing but the force balance condition at the junction x equal to x bar. Now here we can… So this therefore implies… Now over this interval 0 to l, in almost, at almost all points this integrant is 0 except at x equal to x bar. So, this we can write as… So, from 0 to x bar minus epsilon this integrant is zero; from x bar plus epsilon to l this integrant is again zero. So, is only small region around x equal to x bar we, I mean this integrant is non zero. So, that is what we are interested in finding now. So, this term when integrated over such a narrow region from x bar minus epsilon to x bar plus epsilon, since G is continuous function as we have already put the condition of continuity on G, for a continuous function G the integral over diminishing domain goes to zero; because G is continuous and it does not have any short discontinuity etc. Now this term can be integrated once and what we have. So, this term, this is not going to contribute in this integral, this term is going to; because the first integral of this will be the slope, will represent the slope of the string; it will del G/del x that represents the slope of the string. Now string does not resist bending moments. So, it can have slope discontinuity. So, therefore, this actually can be written as… and that implies… So, now we have all the conditions required to... So, here we have one condition and here we have three more conditions. So, these 4 conditions can be used to now solve for these unknown coefficients A_L, B_L, A_R, and B_R.

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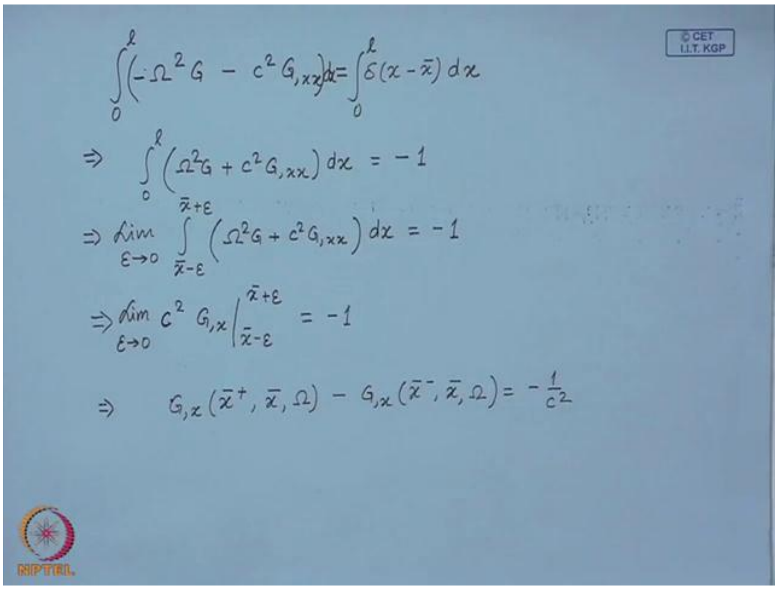

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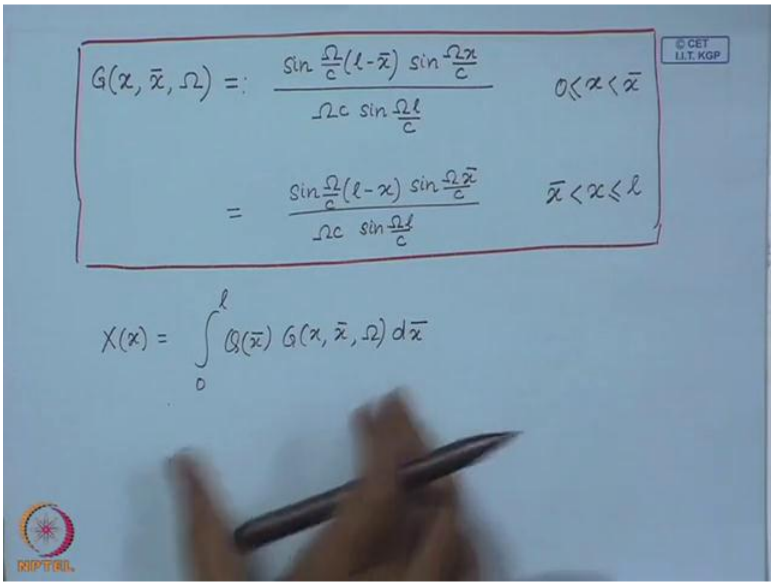

So, if you use these four conditions and solve for these unknowns and finally put in the Green's function then what you obtain is… So this, when x is between 0 and x bar… So, this is our Green's function for the string, when I concentrated harmonic force of circular frequency capital omega at that x equal to x bar. Now once we have this Green's function then corresponding to any force distribution capital Q(x), we have seen that this can be, the amplitude function of the particular solution can be written as… Now this, now we are going to now solve an example and see how this integration can be performed. So, we consider string with uniformly distributed harmonic force.

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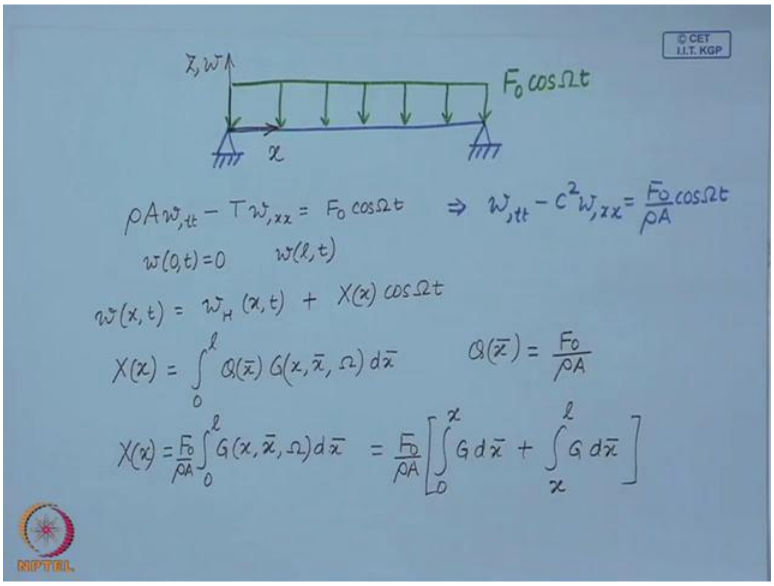

So, let us see this. So, this is a string, taut string on which we have uniformly distributed harmonic force. We have looked at this example in the previous lecture also. So, let us now solve this same problem with this Green's function method, using the Green's function method. So, we have the equation of motion with boundary conditions. So, we are right now more interested in solving the particular solution. So, this is, I mean this is the formulation of the problem for this uniformly harmonically forced taut string. So, the solution is a homogenous solution; plus we have the particular solution in this form and we have already derived the Green's function and we need to solve this amplitude function of the particular solution by performing this integral. In our case Q(x bar), so, I will rewrite this equation so as to match the differential equation for which we solve the Green's function. So, our capital Q(x bar) is nothing but this...

So, we have to perform this integral of the Green's function from 0 to l. Now remember that this Green's function is the response of a system of the string when concentrated harmonic force is applied at x equal to x bar. So, this is the response of the amplitude at x, is given by this green's function. Now, we are looking at this string from 0 to l. We want to find out the amplitude function X at any location x, an arbitrary location x. Now, this integration has to be performed over x bar from 0 to l. So, if you are interested at a particular location x, the solution at particular location x, then when we perform this integral there will be a region from of this integral from 0 to x and from x to l. So, x bar which must go from 0 to l can be broken up from 0 to x, x bar going from 0 to x and from x to l. So, we are going to actually perform these two integrals. So, let us see what these two integrals. Now from 0 to x so, x bar lying between 0 to x, let us once again look at the Green's function. x bar lying between 0 to x, so, which means x bar is less than x. So, this is the function corresponding to this integral, the integrant of this integral, whereas from x to l, when x bar is greater than x, so, this must be the integrant. So, let us now carry this out.

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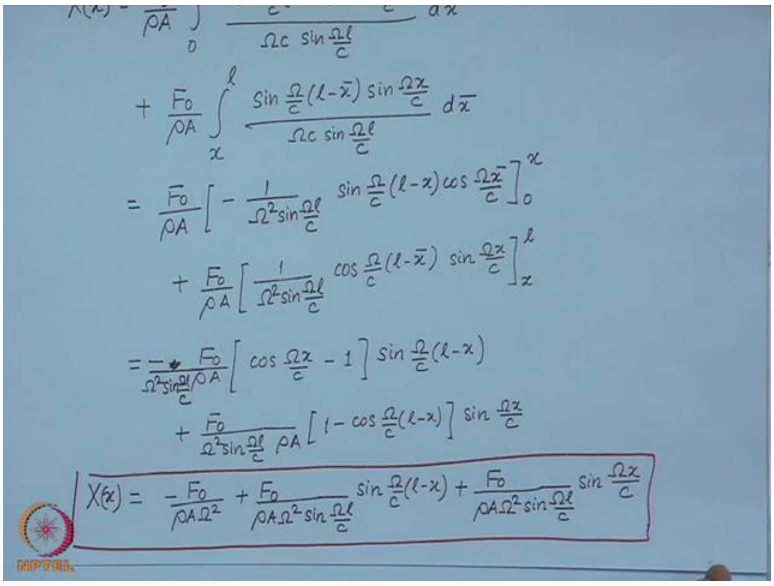

So, therefore… So, this is the first integral and that is the second integral. Now performing these integrals is straight forward. So this… and if you simplify these terms… So, if you open this… So, this is what you obtained and upon further simplification… So, this is obtained as… So, this is the amplitude function of the particular solution. Now you can check this expression with what we obtained in the previous lecture when we solve this problem exactly. So, this was exactly the expression of the amplitude function. So, when we solve problems of forcing with Green's function, what we need to look at is this integral over the domain, and this has to be performed little carefully taking into account the regions of the problem, of the string for example what we have looked at today. Now this Green's function can also be determined using the modal expansion technique that we discussed in the last lecture. So, let us briefly look at this method of solving the Green's function using modal expansion or the Eigen function expansion method. So, our Eigen value problem for the Green's function, the differential equation which is the boundary value problem for the Green's function is given like this; and what we discussed in the previous lecture is that the solution of this differential equation can be represented as an expansion in terms of the Eigen functions of the unforced problem, so the Eigen functions of the unforced problem that we have already solved.

So, if you substitute this expansion in the differential equation of the boundary value problem, then, and remember that this Green's function, this Eigen functions already satisfy the boundary conditions. So therefore, the Green's function is also guaranteed to satisfy the boundary conditions of the problem. Now this is a linear differential operator. So, therefore, we can exchange this summation with the operator and rewrite this… and from the Eigen value problem of this operator, we also know that… So, this is what we have. So, this is the statement of the Eigen value problem, the differential equation of the Eigen value problem. So, if you substitute in here and simplify… Now to solve this equation for this unknown alpha k we use the orthogonality property of Eigen functions, which means I take inner product with W_j on both sides and using the orthogonality then it filters out the j^{th} term of this expansion. So, what we obtain is upon simplification… So, we have obtain the coefficient alpha j and this can be done for all j, 1 to infinity and then we can solve for all this infinitely many coefficients alpha j; and once I have this, I can substitute back in here and I have the series expansion of the Green's function in terms of the Eigen functions of the problem. So, this is how we can also, solve the Green's function in terms of, using the Eigen function expansion, a method which we have also discussed in detail in the previous lecture.

So, what we have looked at in this lecture today we have solve the boundary value problem which arises in the forced vibration analysis. So, this boundary value problem actually gives us the solution of the amplitude function of the particular solution. In this lecture today, we have solved that boundary value problem using the Green's function method; and we have looked at one example that we also, and we took this same example in the previous lecture and we have got the same, we have compared the solution with that obtained in the previous lecture. So, with that we conclude this lecture.

Keywords: forced vibrations, boundary value problem, Green's function.