Vibrations of Structures

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Lecture No. # 12

Forced Vibration Analysis - I

We have been looking at the response of one-dimensional continuous systems to initial conditions; and we have also looked at how this initial value problem can be recast as a as a forced vibration problem. Now, in this lecture and in the next two lectures we are going to concentrate on the forced vibration analysis of continuous systems. Now, the question naturally arises what are the sources of forcing; how or why we should study forces. So, there are various reasons. For example, you can have a system with an actuation, say for example, for vibration control or for some other control. So these actuators, they will excite the system, the mode of the system. Secondly, you can have fluid forcing, for example, bridge or high-rise building that will be excited by the flowing wind. So, that provides forcing to structures. Then there are earthquakes, and such natural sources of forcing. And finally and very interestingly forcing is also used for evaluation and testing of materials, for example to detect flaws or spaces or falls in in material or in a structure. So, from these considerations, it becomes important to analyze forced vibrations of systems. So, what are, let us briefly look, what are the ways of forcing as structure. So, you can have an actuator; you can just attach an actuator to on the structure, and you can force it. You can force a structure like a string by bowing. So, in a violin for example, you use a bow to excite the string by bowing; or you can a put, you can hit the structure with an impact hammer and that gives as impact or impulse forcing to the structures. So, today we are going to look at forcing; we are going to start our discussions on forced vibrations of continuous systems, one-dimensional continuous systems.

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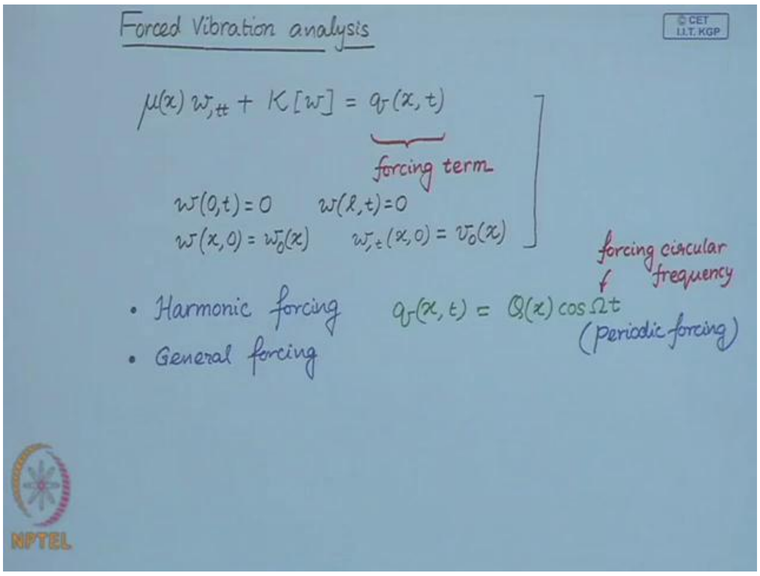

So, let us consider a system; let me first formulate the problem mathematically. So the system which can be put in this form, the kind of system we are discussing, can be put in this form. So, here this represents a general forcing. Along with this, ofcourse you have the boundary conditions, let's say taken as zero, and you have the initial conditions. So, this is the complete formulation of the forced dynamics of the system that can be represented by this differential equation. Now this forcing term, as you can see, makes the equation of motion inhomogeneous. So we no longer have w equals to zero, which is the trivial solution, as the solution of this system. Now, there can be various kinds of forcing. You can have harmonic forcing, which is the most common kind of forcing specially, when we are evaluating or testing a structure, we provide harmonic forcing and try to see the response of the structure, whether it matches with our expected response or not. So, this harmonic forcing is one of the very important types of forcing which we are going to looked at. The second is general forcing, which can be…So in the harmonic forcing, for example, your q(x,t) can be Q(x) times cosine omega t, where omega is the forcing frequency, forcing circular frequency. So here, as you can, this kind of forcing is separable in space and time, possibly separable. For example, one term, one frequency is forcing like this; so you have this as separable forcing in space and time. Now, this is the amplitude function or distribution of the force; and this is the temporal variation of the force. And any periodic forcing, as you know, can be represented as a series of harmonic forcing. So, if you know the solution for the harmonic forcing, then you can also find out the response to any periodic forcing. So, we can also deal with periodic forcing if we know how to find out the response to harmonic forcing. Now, when you have this non-separable, when this space and time part non-separable; we are going to look at some examples of this forcing. We have actually looked at one of the examples of general forcing, when we write the initial value problem as a forced vibration problem; and we are going to discuss this shortly in later lectures. So, today we are going to focus on the harmonic forcing.

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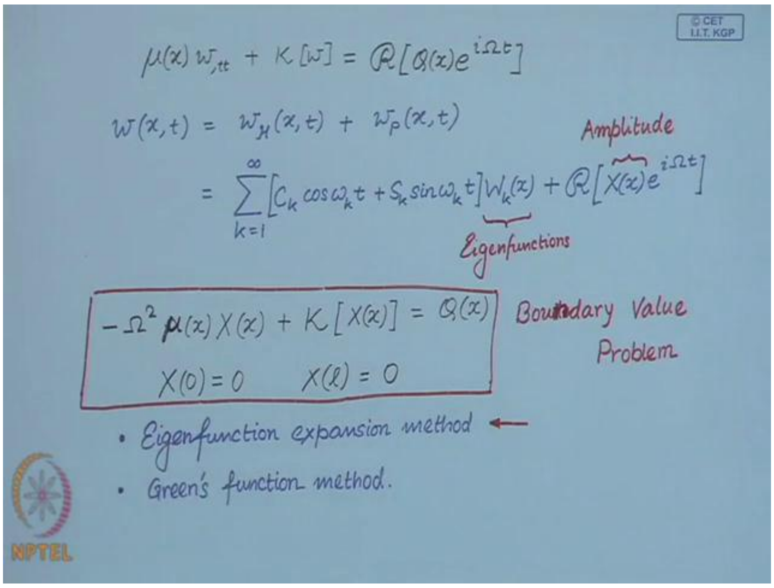

So, let me write the differential equation of motion as… So, the forcing that I discussed just now can be represented as a part of this complex forcing term, where as I mentioned this is the force distribution function, this is the circular frequency of forcing, and this R represents the real part of this argument. Now the solution, this of course, along with the boundary and the initial conditions will completely specified the forced vibration problem. Now, first we must write down the general solution of this differential equation. So as you know, the general solution of such a differential equation can be written as the homogeneous solution, which means the solution with zero forcing, plus the particular solution, which is due to this forcing. So this kind of solution satisfies the differential equation since the homogeneous solution will actually go zero; once you substitute here the particular solution will satisfy or equate the right hand side. So the solution, the homogeneous solution we have been looking at for the last few lectures, it can be represented as… So, this is the homogeneous solution which is expanded in terms of the Eigen functions of the corresponding Eigen value problem. So the Eigen value problem was obtained by considering the homogeneous problem and searching for special solutions which are separated in space and time. So from there, we have obtained these Eigen functions and we have been representing the solution for solving various kinds of problems, for example the initial value problem. So here again we come across the solution; so this is the homogeneous solution and this is the particular solution which satisfies or which meets this non-homogeneous term of the right hand side of the differential equation. Here, this is the amplitude function, the amplitude function of the response. So as you know that in undamped system, the response is proportional to this harmonic time function, so we have written this out as the real part of exponential of i Omega into t. Now, if we substitute this solution in the equation of motion, the differential equation, then what we obtain is that this term is going to go to zero; so what remains is that if you substitute this and make a little bit of simplification of the equation, then what you will obtain is… So, this is the differential equation in X, so this is the amplitude function X in the space co-ordinate x. So, this is the differential equation that you obtain by substituting this solution in this equation of motion; along with this you also have the boundary condition which this amplitude function must satisfy. This comes from the boundary conditions which we wrote out, when we formulated the problem. So what we obtain is a differential equation in this amplitude function along with these boundary conditions. This specifies what is known as the boundary value problem. So, this is the boundary value problem corresponding to the amplitude function of the particular solution. So we must solve this boundary value problem in order to solve this amplitude. So there are various ways of solving this boundary value problem. One is the Eigen function expansion which is what we have been looking at the past few lectures. This works on the premise or the fact that for the self-adjoint problems, you have the Eigen functions which are all real and which form a complete basis for the system. So by complete basis, I mean that if any configuration or shape of the system can be represented in terms of these Eigen functions, and we have been looking at this method in past few lectures. So you can represent any shape using this Eigen functions as we have done even for the homogeneous solution. So this Eigen function method is one of the methods that can be used to solve the boundary value problem. The other one is the Green's function method. So, this is another method which we are going to look in the next lecture.

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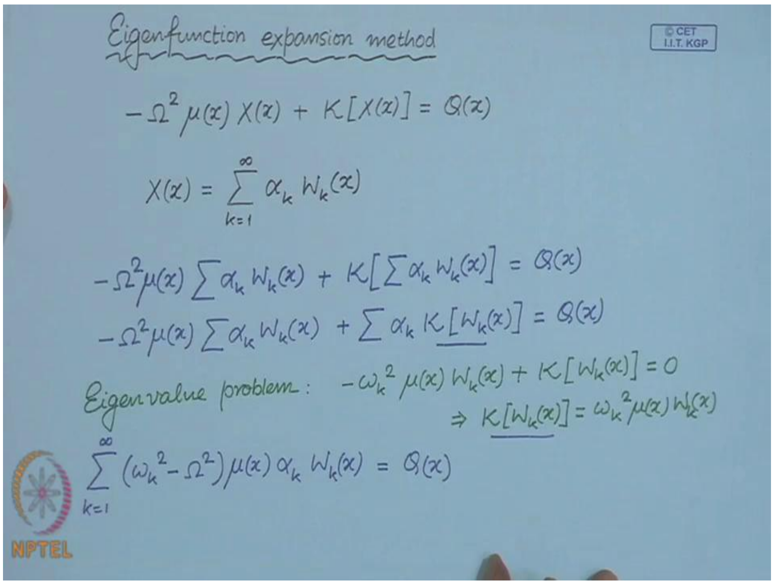

Today we are going to focus on this Eigen function expansion method for solving the boundary value problem. So what we are going to… So in this Eigen function expansion method, we have our differential equation of the boundary value problem in this form. We are going to expand this solution, the general solution of this differential equation in terms of the Eigen functions of the problem. So, these are the Eigen functions of the problem which we have obtained previously by solving the homogeneous problem. So if you substitute this expansion in the differential equation, then… So, here of course these alpha k's are constant which are to be solved. So, this is what we obtain. Now, this K is the linear differential operator. So, I can exchange the summation and the operator and write it like this. Now recall that the Eigen value problem for this operator, for this system read… So therefore I can write the operator acting on the k^{th} Eigen function on this form. So, this is what I am going to replace here. So if I do that and simplify… So, this is the equation that I obtain which has these unknown coefficients alpha k, which I now need to solve. So, for this which I use is the orthogonality conditions for the Eigen functions.

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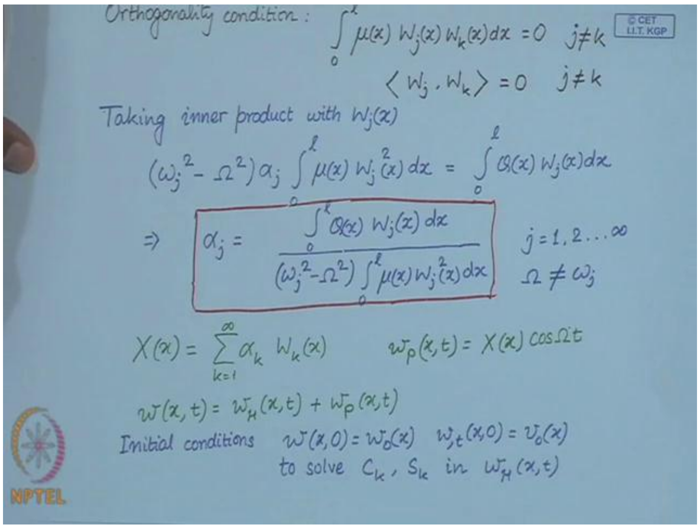

So, let us see how we can solve using the orthogonality conditions of the Eigen functions. So, the orthogonality condition of Eigen functions of the system that we are considering that can be represented as… So for j not equals to k, we have this orthogonality condition. We sometimes denote this as the inner product like this. So to solve these coefficients, what we can do is we can multiply the both sides with j^{th} Eigen function and integrate over the domain of the problem. So, we will say that we take the inner product with the j^{th} Eigen function; and when we do that we this does in effect because of orthogonality is that only the j^{th} term is filtered out; so which means if we do this inner product, if we take the inner product then what we are going to obtain this condition; and therefore… So, that is the solution for alpha j. Now, I can take j from pone to infinity and I can solve for all alpha j's. But this is contingent for the condition that the forcing frequency is not equal to any of the natural frequencies of the system. So, the circular forcing frequency is not equal to the circular natural frequency, any of the circular natural frequencies of the system; otherwise the corresponding alpha j will go to infinity. So, you do not have a finite solution in that case. Now, let look at the situation when… So, this completes our solution for the non-resonant case. So, finally you can, as I have wrote, can write this again. So, we have the solution; and the particular solution will be obtained as in this form. This now has to be substituted in the complete solution and remember that this homogeneous solution has these unknown constants C_k and S_k, which I have written a few moments ago; those constant are to be determined from the initial conditions. So, we have to apply the initial conditions to solve for the C_k and S_k in the homogeneous solution. So, that will complete the solution for the forced vibration problem. So, this part we have already done in our previous lectures, how to solve for these constants using initial conditions. So we will not repeat that here again.

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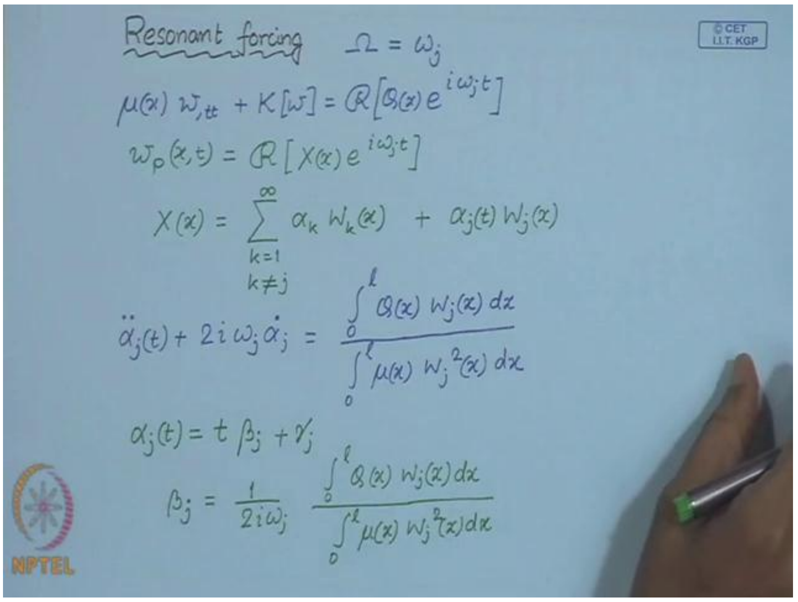

Now, we are going to look into this condition, what happens if we have a resonance forcing. So, let us consider the case of resonant forcing. Let us assume that the forcing frequency is equal to one of the circular natural frequency of the system; let us say the j^{th} circular natural frequency. So, the forcing frequency is equal to the j^{th} circular natural frequency of the system. In that case, our solution; so, let me write this equation once again. So, what we have is, we have… So, this is what we have. So, therefore our particular solution; we must consider this; so let me write down this particular solution as… Now, we have been expanding this X, the amplitude function of the particular solution in terms of the Eigen functions; we will do that. But now, because of this resonant forcing, and if you look at the solution that we just now derived; when this capital omega equals some omega j, so this is going to be undefined. So, to prevent that, we are going to expand this as this for all k except k equal to j. So, the same expansion words for all the non-resonant modes. For the resonant mode, we are going to considered or assume that this coefficient is now a function of time. So, this is going to be our expansion. Now, here this we have considered to be a function of time as we do in, for example in variational methods, so variation of parameter. So, we assume that this is the function of time, and we substitute this expansion in here, and the particular solution into our differential equation. Then for all the non resonant modes, we have a way of solving just as we have discussed just now. For the resonant mode, we are going to obtain differential equation corresponding to alpha j which is obtained as… If you substitute this and take the inner product with W_j, this is what you are going to obtain in account of orthogonality. Now, this differential equation, we know from our previous studies that this differential equation admits the solution of this type, where beta j is now a constant. It could be this; but then when you substitute and evaluate this will be vanished. So, this will be only beta j; and beta j if you substitute this solution form in here, then… So, this is the solution for beta j; and once you have the solution for alpha j, then you can substitute in this expansion and what you will obtain is… So, here as you can see this beta j is, there is one beta j in the denominator, so this is imaginary and when you substitute this, and take the real part as here; so, when we substitute this whole expansion here, and take the real part, what we are going to obtain finally, upon simplification… We obtain this sine omega j t when we take the real part, because of that i sitting in the denominator of beta j; and along with this, we have the other terms. So, this completes the particular solution. Now, once again you have to add it with the homogeneous solution and use the initial conditions of the homogeneous solution.

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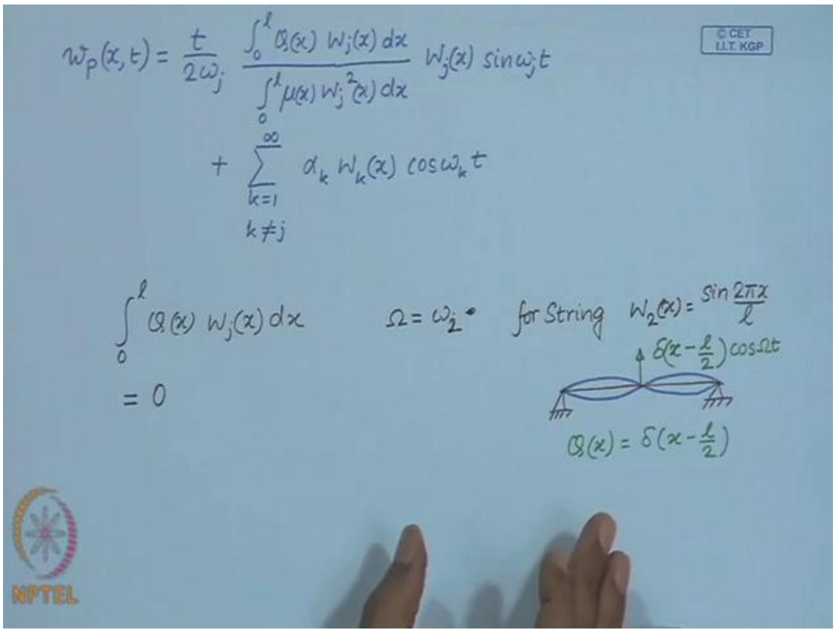

Now here, we see something interesting in this solution form. In the numerator of this term; so as usual the resonant mode, the amplitude of the resonant mode has an envelope which is linearly increasing with time. This is what we all know; this is what happens for resonance; in a resonance mode this is what happens. Now, in a continuous system like this, we have this integral sitting in the numerator of this resonant solution. Now, this integral in general will be non zero. But there are special instances when this integral will actually vanish. So, let us look at some situations. So, if you consider the omega j to be the second natural frequency; so omega, the circular forcing frequency is equal to the second circular natural frequency of the system. In that case as you know for a string, let us say; so for a taut string, we have the Eigen function sine of 2 pi x over l; so this is the Eigen function for a string in second mode. So, if we have a string which is forced at the middle. So, this is let us the string and suppose the forcing is of this form is being applied at the middle. If you substitute this, so Q(x) in this case; and if you substitute this here, you can easily see that this integral vanishes. So, if this integral vanishes even though you are exciting at the second natural frequency of the system; so this is omega 2; capital omega is omega 2; the second mode will not show the resonant behavior, which means, because of this integral vanishing, this term will drop out from the solution; so the response of the system will still be finite. So force like this cannot excite this mode; this cannot excite this mode, because the forcing is at this node. So, this is one situation where there would not be forcing. There can be other situations, for example, one such example we are going to look into very shortly. So, what we have seen here is that in a continuous system, just forcing the system in a resonant frequency does not mean that you will observe the resonance solution. So, the location of the force is also important in these situations.

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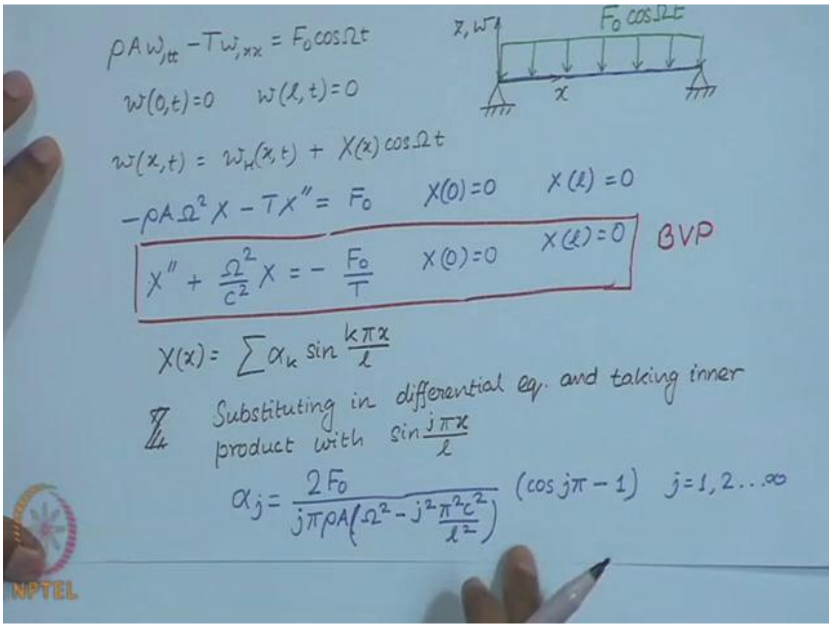

Now to look at what we have been doing, let us look at an example. So, this is an example of taut string with uniform harmonic forcing. So, let me represent the situation that we have. So, this is the string; and the forcing is a uniform; so the distribution is uniform. So, the mathematical problem, the differential equation, the boundary conditions on the two ends; so it is a fixed-fixed string. Now, we are going to look at solutions in this form. If you substitute this in the equation of motion and removing the cosine omega t term throughout, we get this along with the boundary conditions. Now, we will simplify this by dividing the whole thing by the tension and T over rho A is C square. I will write it like this. Now, this is the boundary value problem of our system which we are now going to solve. So, as we have done, we know that the Eigen functions of the taut string of this form. So, we are expanding in terms of these Eigen functions; and when we substitute in here and let me; so these steps are quite simple; let me write out the solutions. So, I substitute this in the differential equation and take inner product with the j^{th} Eigen function. So, when I do that, I can obtain the solution of alpha j. These steps are straight forward. So, you obtain the solution for alpha j; and you can put in any value of j, and you can get this alpha j; and now, you can substitute in the expansion to obtain the amplitude function. So, let me write out the particular solution then… so, cos of k pi minus 1; so that completes the particular solution.

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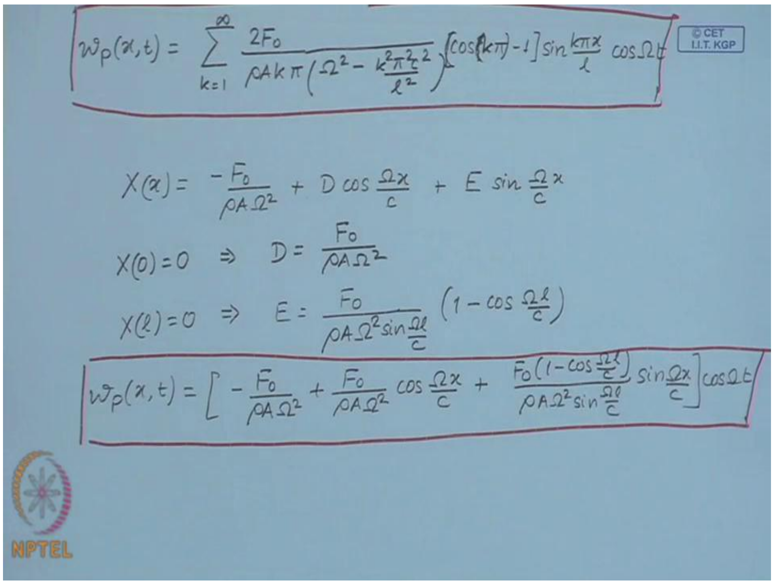

So, this is the particular solution of the problem. Now, this boundary value problem can also be solved exactly as this is differential equation is straight forward and the solution of this differential equation can be easily written as… So, this is the general solution. Now see, here in the previous solution, we did not have to worry about the boundary conditions, because we have expanded in terms of the Eigen functions, which already satisfied the boundary conditions. But now, we have solved this exactly; now, we have to satisfy these boundary conditions. So if you solve for these constant D and E, you can easily obtain these constants which can be substituted here, and you can once again get the particular solution. But, now the particular solution is closed form expression. So, this is the solution which is now in the closed form. So, we have kind of summed over all these terms to obtain this. Now here, one thing to note is you have this cos k pi minus 1. So, for even values of k this is going to go to zero; this bracketed term is going to go to zero. So, therefore, you will have only odd. So, only for k odd, you will have non-zero coefficients.

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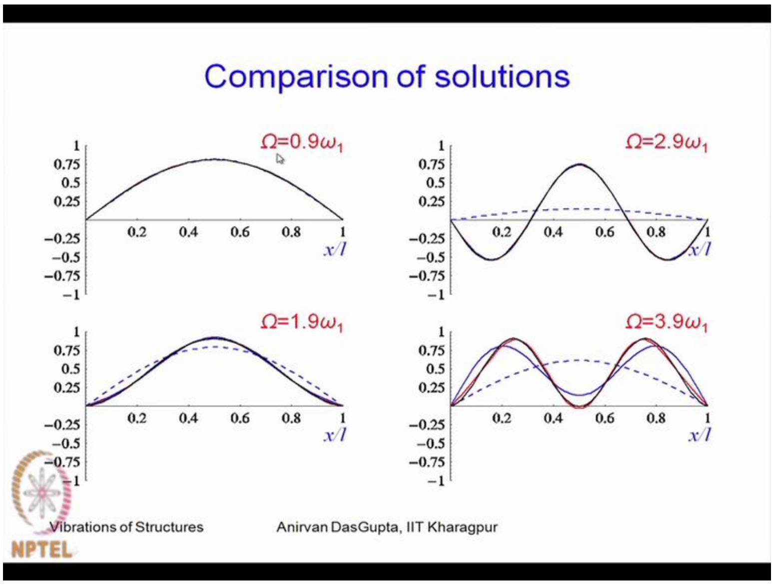

Now, in this figure, in this slide, I have compared these solutions, the exact solution, the closed form solution with the series solution taking up to three terms. So, if you look at… So, I have taken the forcing frequency very close to the first natural frequency of the string; it is 0.9 times the first natural frequency; and you can see that the exact solution and the series solution they match. Actually, I have plotted this series solution taking one term, another solution with two terms, and another solution with three terms. Now, in the first plot they are indistinguishable with the exact solution. This is when you force it close to the second natural frequency. So, two times of omega 1 is actually omega 2 for a string, as you know that they are integral multi force of the fundamental frequency. So, this is close to the second natural frequency; the forcing is close to the second natural frequency. This dashed curve is the series solution with only one term. So, you can see that this deviates considerably from the actual solution; while when you take two terms or three terms in the series, then they are matching quite nicely. This is when the forcing is close to the third natural frequency. You see the one term series solution is quite off; while when you consider two terms or three terms, then they are matching quite nicely with the exact solution. This is when your forcing at the, close to the fourth natural frequency. Again the one term solution is off, the two term solution is this blue solid line, the red line is the three term expansion series solution, and the exact solution is given by this black solid line. So, you can see that slowly as you increase the forcing frequency, and consider higher modes or the higher natural frequencies, then the two term solution is now in error; the three term is still quite close; but as you will go to higher and higher, you will have to take more and more terms in the series to get close to the exact solution. So, we see that the series solution actually converges on to the exact solution.

So, let us look at what we have studied today. We have discussed the forced vibration analysis of one-dimensional continuous systems; we have looked at harmonic forcing; and we have solved this problem using the Eigen function expansion method. We are going to continue this discussion, in the next lecture. We conclude this lecture.

Keywords: forced vibration, harmonic forcing, boundary value problem, Eigen function expansion method.