Vibrations of Structures

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Lecture No # 11

Initial Value Problem

Today, we are going to look at what is known as the initial value problem in dynamics or vibrations. So, we are going to look at the initial value problem for continuous systems.

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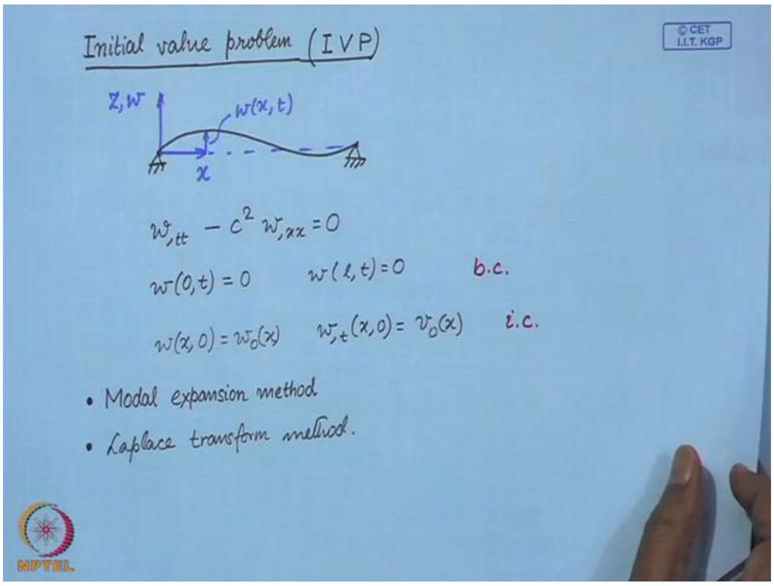

So, what is this initial value problem? So, suppose you have, let us say, a string and it is given some initial shape or some initial velocity distribution over the string. How will to system evolve? So, we want to determine the evolution of this system. So, given this system, described by this equation of motion, these boundary conditions, and these two initial conditions, how will the system evolve as time progresses? So, this is the central problem in the initial value problem. Now to solve this problem, there can be various approaches. Today we are going to look at the modal expansion method. The initial value problem can also be solved by the Laplace transform method. We are going to look at this Laplace transform method slightly later. So, we are going to concentrate today on the mostly modal expansion method.

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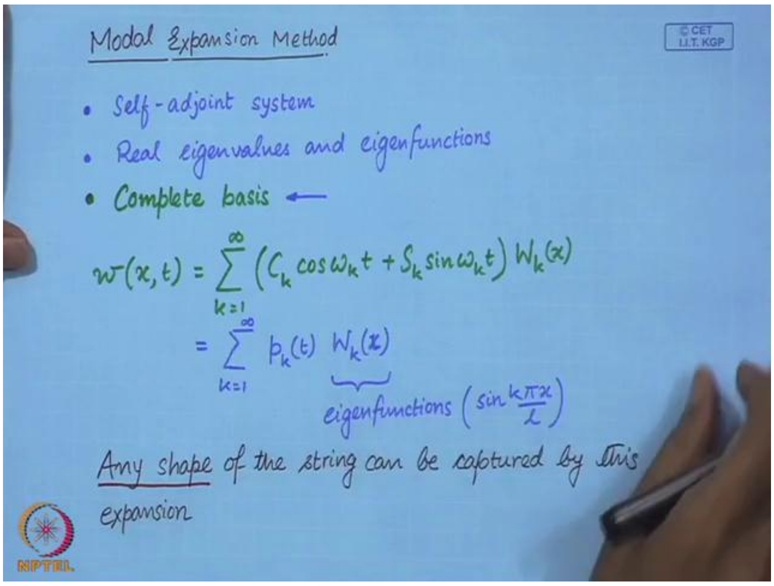

Now, we have performed modal analysis of this kind of system, and we have observed that when we have a self-adjoint system, which is to say that the stiffness operator is self-adjoint, then the Eigen value problem has real Eigen values and real Eigen functions. Now, these Eigen functions have this additional property that they form a complete basis. This is the underlying thing that is used in modal expansion method. So, this property is of fundamental importance in the modal expansion method.

So, what is this complete basis? We have discussed that, suppose let us consider the string again. So, the solution was obtained, was first considered. So, the general solution, we have expressed in this form like this, which I can write like this… Now what this says is that, say for example at a certain time instant t, the solution or this field variable is an infinite expansion in terms of this Eigen functions, which for the string, it happens to be… So, for a fixed-fixed string, it happens to be this sine k pi x over l. Now, when we say that this is a general solution, that means, any shape of the string can be represented by this expansion. So, the keyword here is any shape. So, any arbitrary shape of the spring can be represented by an expansion of this type; and we have also discussed that these Eigen functions are orthogonal. So, if I make, if I want to have a visualization of this that these Eigen functions are orthogonal, and these forms a function space with this as we call as the basis. Now, of course there are infinitely many basis functions. Here I am drawing only three; and with a slight stretch of imagination you can very well imagine that there are infinitely many such axes which are all orthogonal to one another to represent the orthogonality property with respect to certain inner product as we have discussed. So, in a space which is known as the configuration space or the modal space, in such a space, the points with coordinates P_1, P_2, P_3, P_4 etc; so, these points representsthe configuration of the string at the time instant, when the coordinates are P_1, P_2, P_3 etc. So, at a particular time instant, this expansion, therefore, represent the shape of this string at the time instance. So, this point represents a shape, the configuration of the system; and any shape of the string is the point in the space. There is no shape that lies outside the space. So, this is the important thing. So, any shape of the string can be captured by this expansion. This is known as the expansion theorem, the modal expansion theorem. So, the modal expansion theorem says that any shape of the spring can be represented in terms of these Eigen functions. So these Eigen functions, they form a complete basis that means any shape can be represented in this basis, what I have shown here in three dimensions. So this is the key to the modal expansion method for solving the initial value problem. So for the problem that we have here or similar problems we will now try to solve using modal expansion method. Let us see this general system with certain boundary conditions, let us say of this form, and some initial conditions. SO, we intend to solve this problem using the expansion this… So, if you substitute this expression here, so, this is what you will have. Now K is a linear differential operator and the kinds of system we are considering, this K has only spatial derivatives and this is linear. So, therefore, I can interchange the summation and the operator. So, finally, I can simplify this and write… So, this is what we obtain. Now, we recall that this Eigen value problem for this system was… So, this was the statement of the Eigen value problem for this system. So, therefore in this summation, I can replace this operator acting on the k^{th} Eigen function with this; and therefore, I can simplify the whole thing and write like this. Now, this is a summation again in terms of these Eigen functions. So, I can use now the orthogonality property. So, I will multiply both sides by j^{th} Eigen function and integrate over the domain of the problem. So, this I will say we take the inner product with, let's say, U_j. If I take inner product with U_j then that filters out the j^{th} term in this expansion. So, I will have... and this I can do for all j's, all values of j; and the solution, the general solution of this system can be easily written as… and therefore, I will substitute this in the original expression, and write… So this is our general solution. Now, once I have this general solution, now I have to use the initial conditions. Let us say, these are the initial conditions that are specified for the system.

Now, using these initial conditions, we have to actually solve for these coefficients C_j and S_j.

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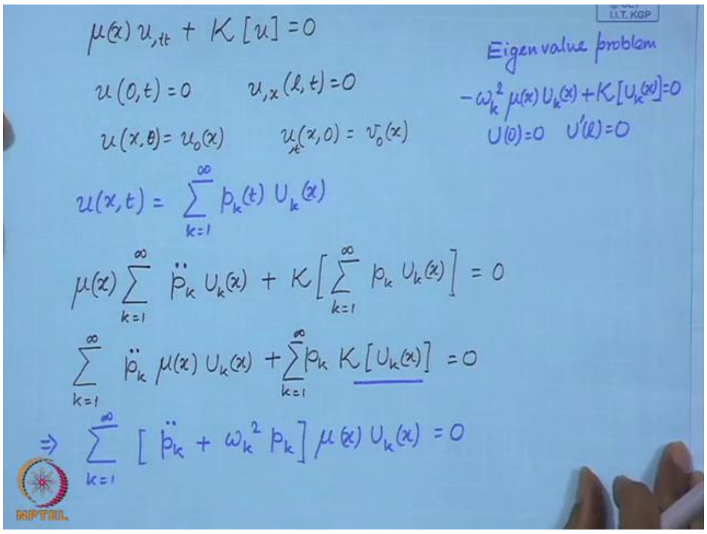

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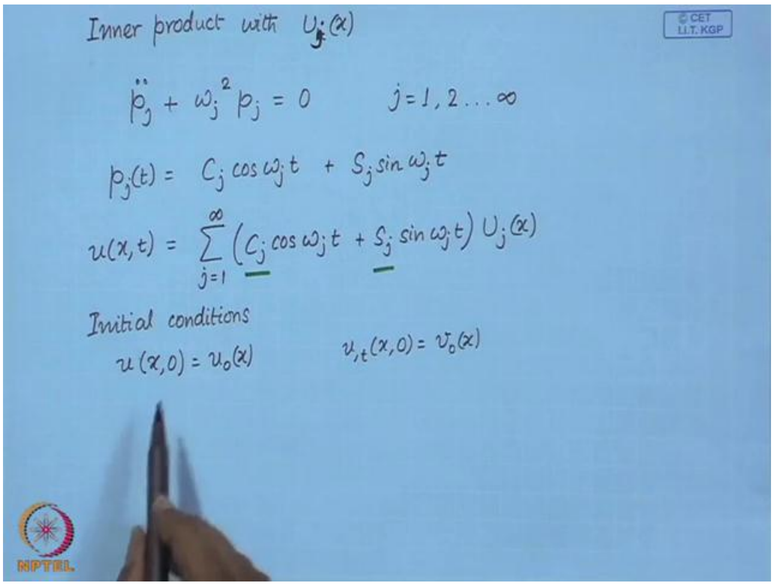

Let us see, how we can do that. So, substituting this expression in the initial conditions… So this is from the first initial condition, and from the second initial condition, this… So, we have to solve for the coefficients. So there are infinitely many coefficients, so, for C_j and similarly, infinite coefficients for S_j, and we have these two equations. But then remember that these Eigen functions, they are orthogonal. So, we can use once again the inner products. So, suppose I take the inner product with let's say U with U_k, then I can write… So, once I take inner product with U_k, this filters the k^{th} term and that gives us… So, if I write in the integral form, so, that solves for all the C_k. Similarly, so that solves for all the values of S_k. So once that is done, I have all these values of C_k and S_k, which I can substitute here and I have the final solution. So, we have solved the initial value problem using the modal expansion technique.

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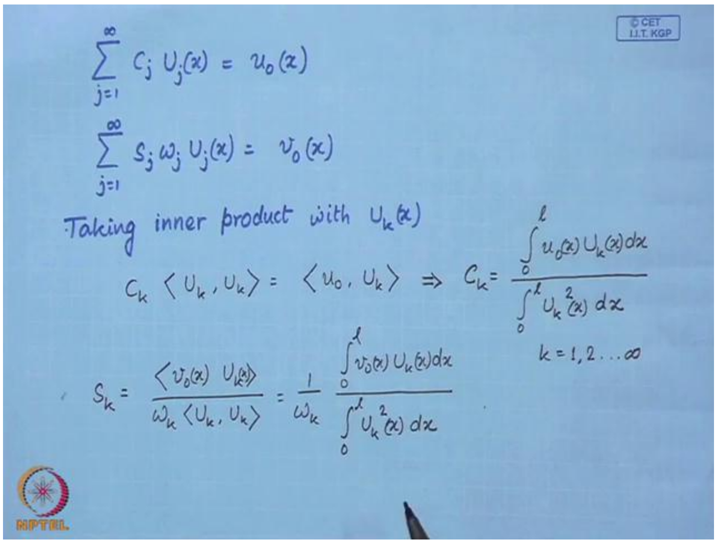

Now, let us look at some examples of actual systems. So, the first example that we are going to see is shown in this slide. So, this is the collapse of a stretched bar. So what we have is… So, we have a fixed-free bar which is under tension because of this string. So, here there is a string which is attached to this free end of the bar and this is under a tension T. So, naturally this bar is under tension. Now imagine that you cut this string, so this string snaps. So, when this string snaps, this bar is going to collapse back. So, we are going to study the collapse of this bar. Let us mathematically formulate the problem. We have this uniform bar. So, we are going to write, formulate the problem at the moment the string snaps. So, in that case, the right end of the bar is force free. So, this boundary condition is… zero. Now the initial conditions we find out… you can find out the initial conditions. So, here what was the condition that this was under the tension T? So, you can write the string in the bar. So, that is del u/del x. The stress in the bar is the young’s modules times the strain and then this was under the tension T. So, the force was T which is sigma times the area of cross section of the bar. So, when the bar was under this tension, so, this was the condition. This is the equation of statics of the bar, which can very easily integrated out to determine… Since at x is equal to zero, u is zero. So, the constant of integration is zero. So, this is the initial conditions of the bar.; and the initial velocity, as soon as the string snaps, its initial velocity is zero. So, this is our problem. This is our initial value problem for this collapsing bar. So, as usual we are going to, as we have just now discussed, we are going to express the solution of this collapsing bar in terms of its Eigen functions. So the Eigen functions of the fixed free bar, they are given by… So these are the Eigen functions of the fixed-free bar, that we already known. Here omega k, these are given by these values. So here, this is the general solutions and these are coefficients that we must find out from these initial conditions. So, let us see.

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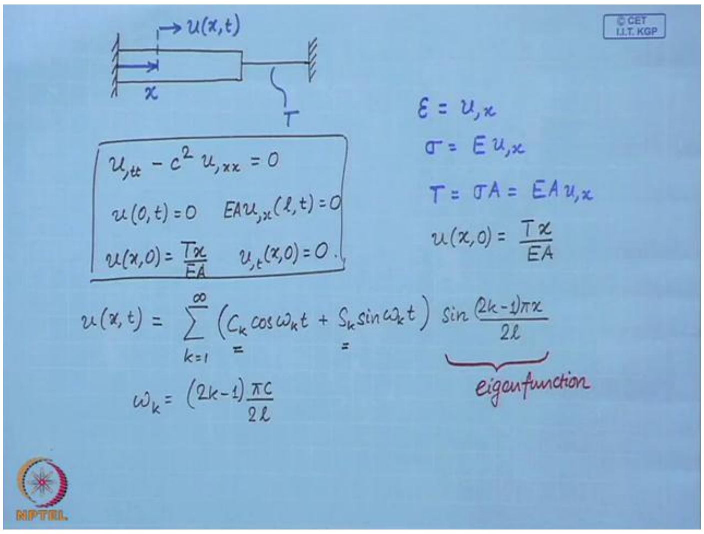

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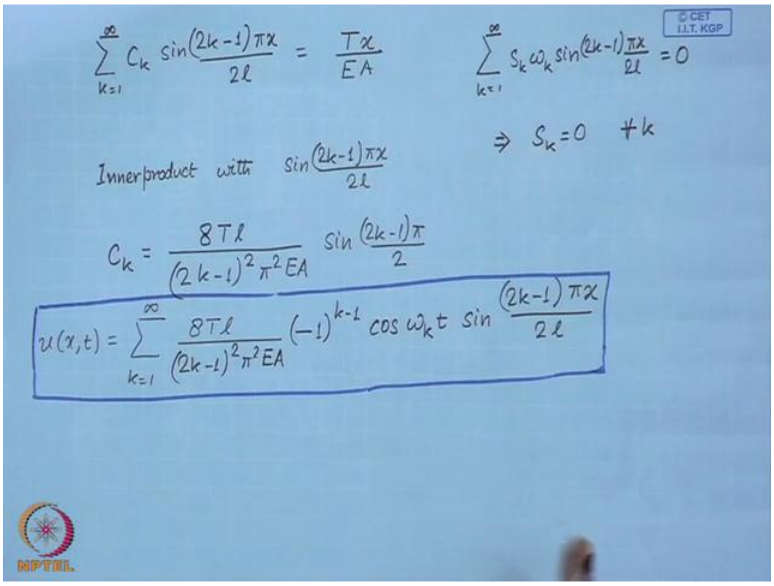

So, when you substitute this in the first initial conditions… and similarly for the velocity condition… these are all zeros. So, this immediately tells us, for all k, this coefficient S_k must vanish. So, we are left with this. So, we take inner product with this k^{th} Eigen function and if you perform this integral, so you multiply this and integrate over the length of the bar; so, you can check… So, these are the coefficient and finally, therefore, the complete solution… So, the complete solution is… Now this solution I have plotted at certain time instants in these sets of figures.

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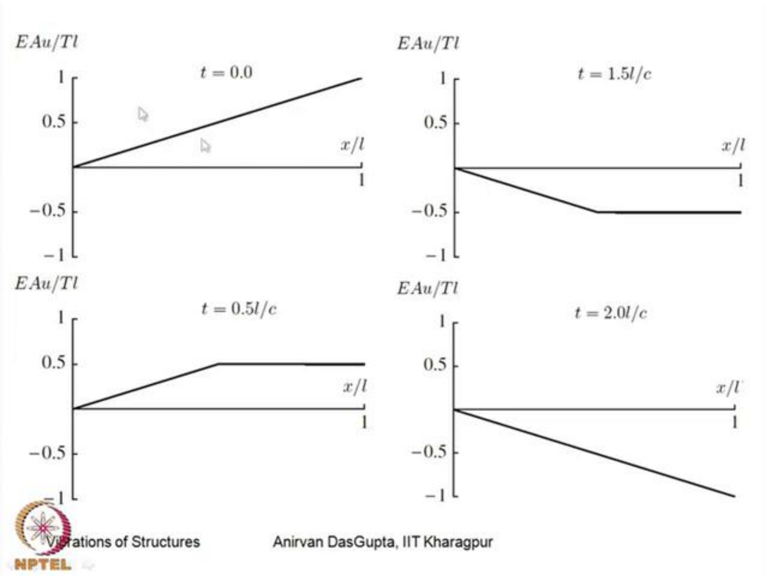

So, you see at time t is equal to zero, this is the axial displacement, you see this is the axial l displacement of the bar. So, this line is nothing but, t x over EA. this is nothing but t x over EA. So, this is the straight line with x. Here of course, I am plotting with this non-dimensional free variable x over l. So, initially the actual displacement is like this. So, as time progress at 0.5 l over c, this is the axial displacement profile of the bar. So, here it is still linear, the same as it was here. So, you can imagine this portion of the bar does not know as yet that this has been released. So, this information has progressed in 0.5 l over c time to approximately half. So, it is half; half of the bar knows that this has been released. This half still does not know. So there is the propagation of information from the end, which was released at time t is equal to zero in to the bar. So, at l over c, the bar comes to complete the full, whole of the bar is that equilibrium consideration, which is not shown in this figures; and then at 0.5 l over c, there is a compression taking over a bar, here this are all the bar is under tension; when it comes to the equilibrium configuration at l over c, the bar is in the equilibrium, but then it has an acquired velocity, it still goes and hits against the or compresses against the wall; and then there is a compression generating the bar and that compression is complete at 2 l over c. So, there is the as you can imagine, there is a propagation of information of this compression wave, as it is usually known as compression or tension wave, that progresses in the bar, and it reflects back and forth and that with the bar with vibrating. Now this propagation of this waves, we are going to discuss in this course at … So, we are going to discuss that in detail later in this course.

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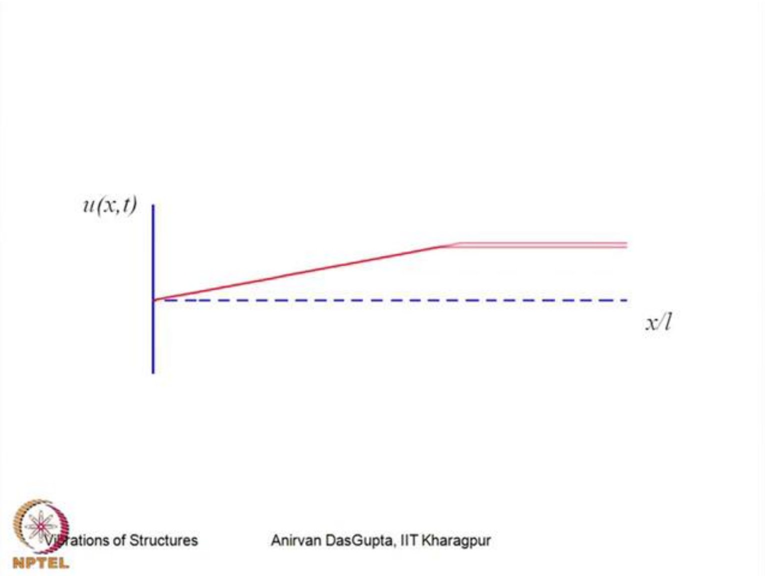

Now, here I have an animation, which shows the same thing. So, you can see this, so at this instant, now the propagation, now the compression wave is the propagating. So, this now in tension, on this side it is in compression, and this wave propagates back and forth in the bar, which is shown in the animation. Next, let us look at the another initial value problem, where we are going to have velocity initial condition.

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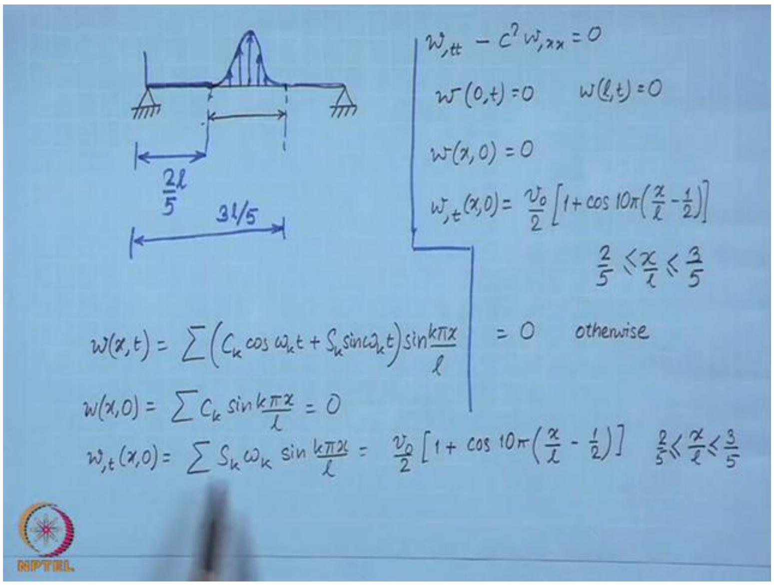

So, let us consider a string, on which I specify a velocity profile. So, the velocity profile that I considered, so, we will consider that the initial displacement of a string is zero, but the velocity has this profile. So, let me mathematically formulate the problem. So, the initial displacement of the string is zero, while the velocity is given. So, we have velocity profile like this over this region. So, the velocity initial condition is provided only over this small region, which is of length one-fifth, one-fifth the length of the string. So, central portion, we have this kind of the profile. So for this problem, once again… So, we consider a solution of this form. Now when you substitute this solution in the initial conditions, you will obtain these two equations. So, we have to solve these two equations in order to solve the coefficients. So, immediately you can, from here it is immediate that all the C_k's will be zero; while if you once again take the inner product and solve this problem, solve for the coefficients S_k, then S_k turns out to be… for k 1, 3, 5, so for all the odd values of k, we have this and zero for all the even values. So, for all the even values, S_k is zero; for all the odd values, we have this expression of S_k. So, now with these expressions in here, you can write down the solution of the string, the motion of the string.

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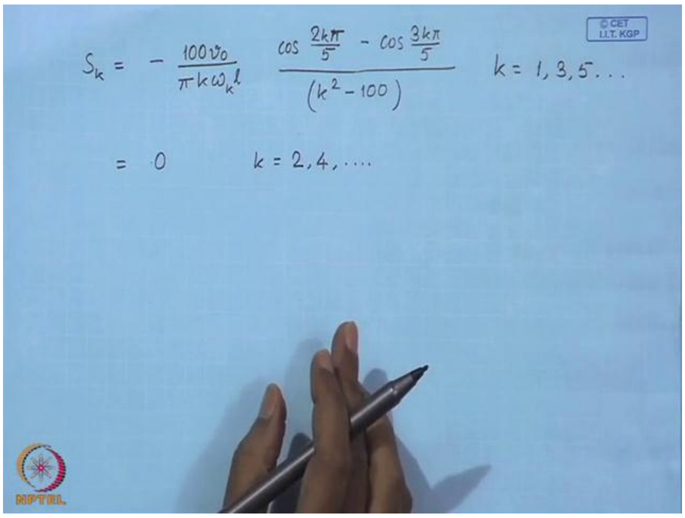

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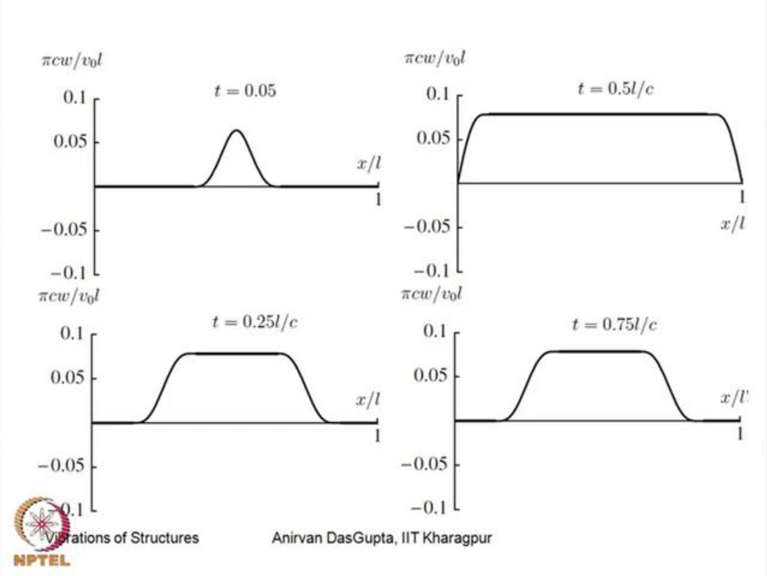

So here, I have the snapshots of the string at certain time instants. So, at time t is equal to zero, as you can realize the string is in equilibrium position, so, I have not shown that. At 0.05, t equal to 0.5, there is a hump that is generating, that is developed in this string; and as the time progress 0.25 value of this hump, its spreads in this string; but see this portion of the string, yet does not know that the disturbance has been created in this string. At this time instants, the full string is displaced; beyond this the disturbance reflects back from these fixed ends and the hump shrinks, and then it comes to the other side of the of the equilibrium position of the string.

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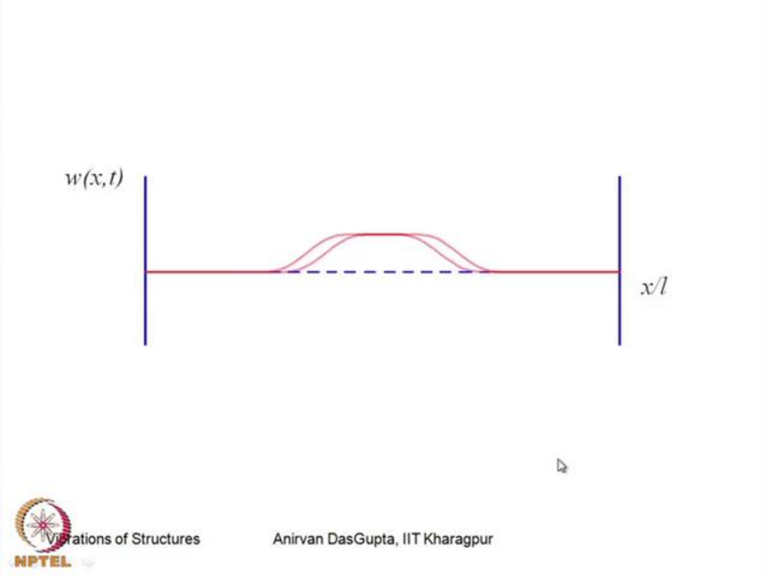

So, this shows this animation. So, as you can see that the hump develops, spreads reflects, collapses and comes on the other side, and does the same thing. Remember that this is a slow motion of what actually happens, I mean, this propagation of this possibly cannot be observed by the human eye as such, you have to see it in slow motion to observe this kind of the propagation of this disturbance. So, we have looked at the initial value problem for a continuous system, and we considered two examples and using the modal expansion technique, we have solved this problem.

Now, let as briefly finally look at this initial value problem. How an initial value problem can be actually converted to a forced vibration problem or a forced dynamic of a continuous system? Now this is very interesting because then, once we discussed of a forced vibration analysis, the method, that we discussed here, will be applicable for solving the initial value problem as well. So, which means that we can solve the forced dynamics, for the forced initial value problem as a forced vibration problem; so, we then have a unified way of it treating initial value problem, forced dynamic etc.

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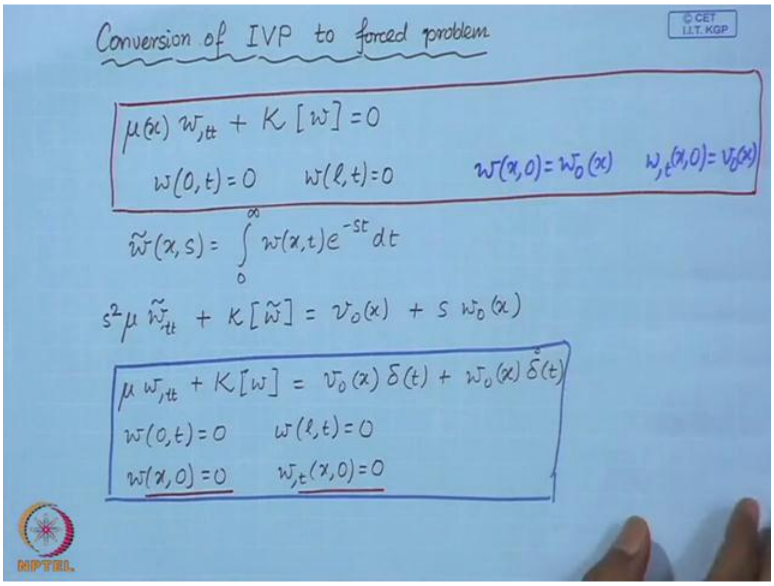

So, let us look at this a conversion of an initial value problem to a forced problem. Now, you see this, we have been considering, let say, this kind of a system and with certain boundary conditions and initial conditions. Now, if you take the Laplace transform of this equation. So, we define this Laplace transform. So if you define the Laplace transform in this manner, then in the Laplace domain here S is the Laplace variable. So, in the Laplace domain, you can write this… this you can very easily check. So, this is what you are going to get. So, this is the equation in the Laplace domain. Now, this is for this problem with these initial conditions. So, now the same equation can be obtained, so, this same equation of Laplace domain can be obtained for this system. So, in the Laplace domain, the equation of motion for this system is same as the Laplace transform of this equation. But now, you see, this system has zero initial conditions. So, here instead of these initial conditions, non-zero initial conditions, we have this forcing, this in-homogeneity of the equation of the motion. So, we have been able to convert a system with initial conditions to a system with zero initial conditions, but with forcing. So, this is the system, which is now a forced system with zero initial conditions; and therefore, the solution of this will be the solution of the solutions of the original system with non-zero conditions. So, this way we can convert or bring an initial condition problem, or convert the initial value problem to a forced vibration or a forced dynamic problem.

So, in this lecture we have looked at the initial value problem; we have solved using a modal expansion technique; we have looked at some examples of how the solution behaves; and we have found some interesting, made some interesting observation in the solution; how the motion actually take places as propagation of disturbances in the continuous system. And finally, we have looked at converting an initial value problem to a forced vibration problem, and so with this we can solve the initial value problem and the forced dynamics problem in a unified manner, in a later lecture. So, with that we conclude this lecture.

Keyword: initial value problem, modal expansion theorem, collapse of a bar, striking of a string, Laplace transform.