Vibrations of Structures

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Lecture No. # 10

Modal Analysis: Approximate Methods - II

In the previous lecture, we have discussed about the approximate methods of modal analysis; one class of methods which is based on the energy, the kinetic potential energy, the Lagrangian etc; so which are broadly classified as energy based methods for modal analysis. Now in today's lecture, we are going to look at yet another class of methods, which are known as projection methods.

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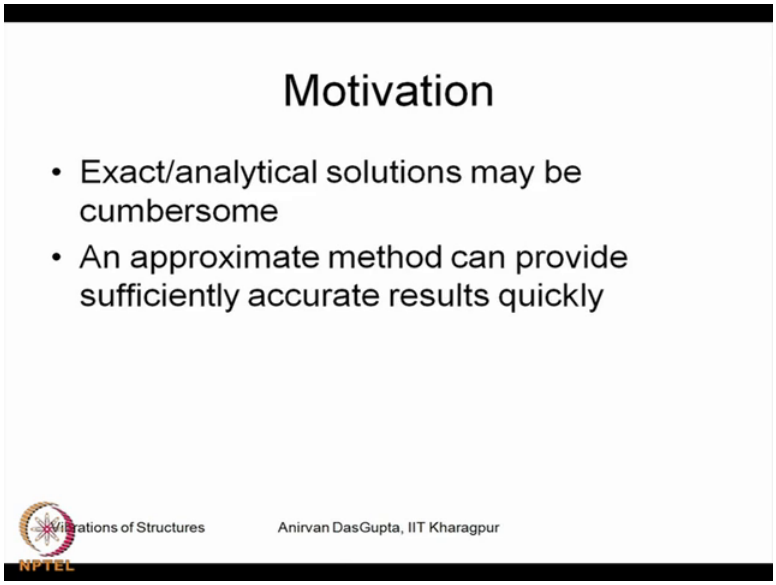

Our motivation for studying approximate methods, we have already discussed that analytical method, though they are more preferable, but are quit cumbersome; and we can have approximate methods, which can give quick and sufficiently accurate results.

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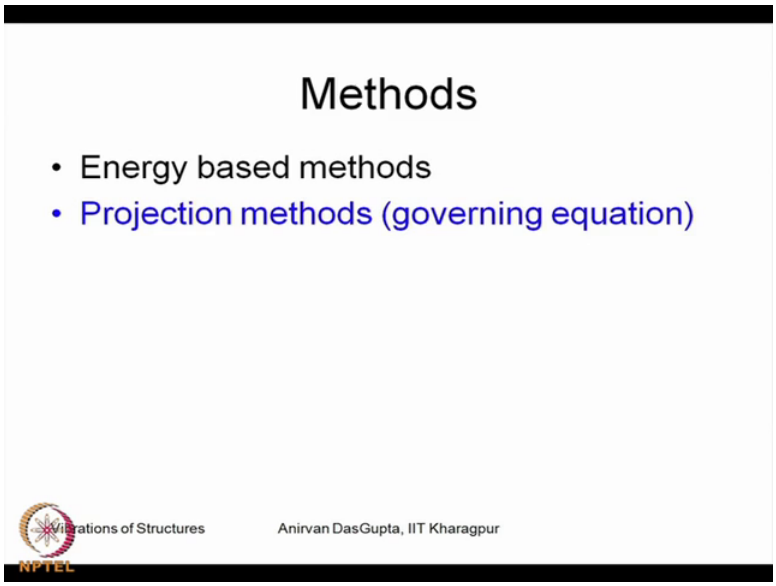

In today's lecture, we are going to look at these projection methods, which work directly with the governing differential equation of the system.

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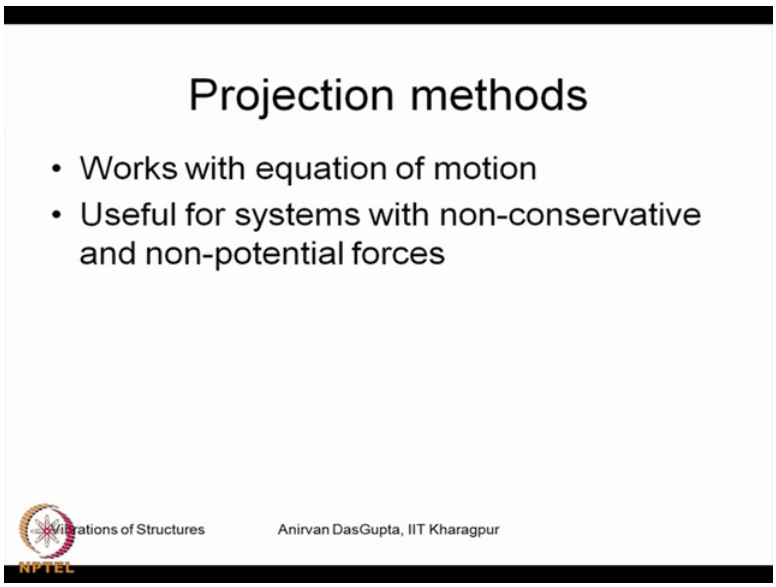

Since these methods work with the differential equations directly, they can be used very easily for dealing with non-conservative forces, non-potential forces etc. Since these terms, they can be directly written in the equation of motion and they are little tricky to introduce in the Lagrangian as such, though they can be done in that way; but if a

method works directly with the equation of motion then these non-conservative terms and non-potential forces they can be handling quite effectively.

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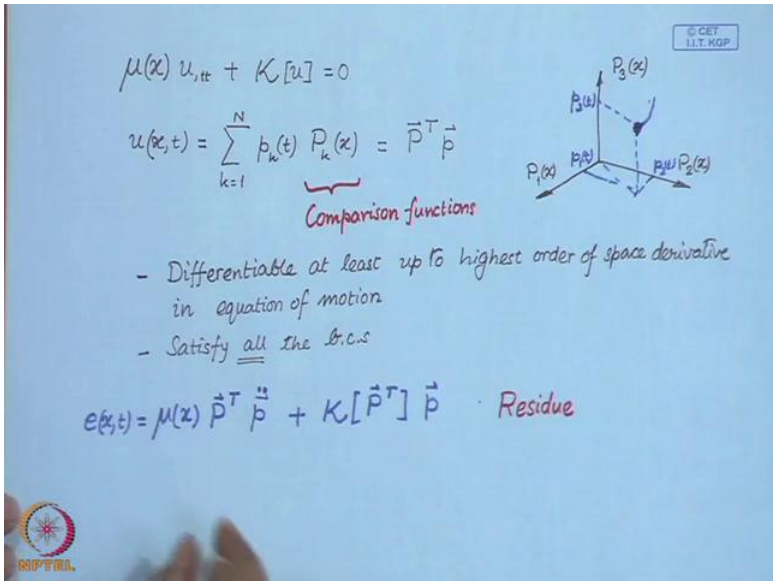

What do we do in these projection methods? Let us discuss this, in the context of this equation of motion… Let us consider a differential equation, which has a structure like this, so mu(x) represents the inertia operator and K is another linear differential operator. Essentially, what we are going to do is discretize; so this is the continuous system described by this partial differential equation. What we aim to do is to discretize this equation of motion.

We use the idea of expansion of the field variable in this form. One thing, that may be mentioned here that even though this looks like a separable solution, but actually it is not because this is an expansion; had it been only one term then it is a solution that is separable in space and time. But once you take this expansion is no longer a separable solution. This may be written in terms of vector multiplication. Now here in these projection methods, there is a restriction on the kinds of function using which we do this expansion. These functions are known as comparison functions.

Now, what are comparison functions? These are function that satisfies two important properties. They must be differentiable at least up to the highest order of space derivative in the equation of motion. The second important property is that they should satisfy all the boundary conditions of the problem. This is very important to note that these

functions must satisfy all the boundary conditions of the problem. This actually makes this method little more difficult to apply compared to the energy based techniques where we were using admissible functions.

We expand our field variable in terms of these comparison functions and unknown coordinates of time. If I substitute this expansion in the equation of motion, then of course, I do not except that this solution will satisfy this equation of motion because in any case this in an approximate solution. So what we generate is known as residue; this is known as the residue. We do not except that this will be zero throughout the domain. This function is known as the residue. Now we project this residue in a certain space. So, we are looking, I mean, this can be thought of as expanding the solution as a linear combination of certain functions as we had discussed previously as well and so on. This point represents the configurational system and as these temporal functions change, this point moves in this space. Now, we have generated this residue, because this solution is the approximate. What we can do is that we can try to make this residue zero at certain points. Now there are various ways of doing it.

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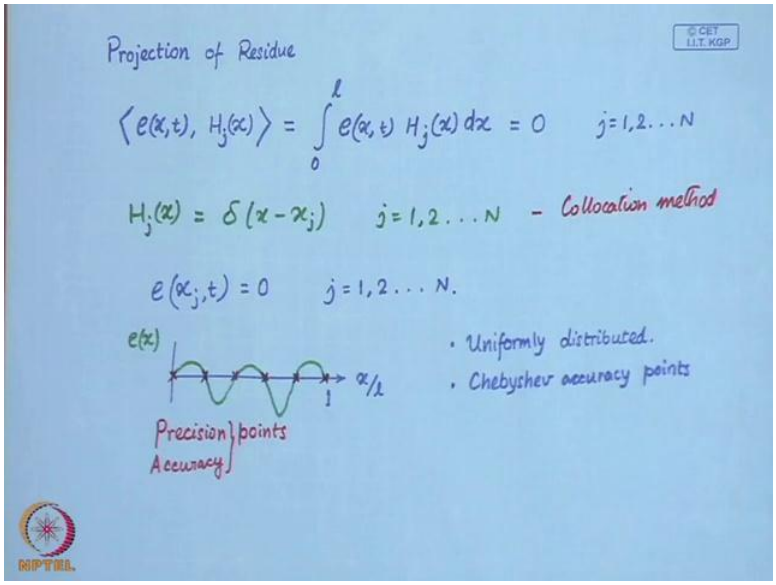

Here, we first introduce the idea of projection. We have a suitably defined inner product of two functions. One is the residue that we have generated and another function H(x), which we can simply define in this case, and say that this projection is zero. Rather than the residue vanishing identically, we say that the projection of this residue on certain

function that we will qualify shortly, so, this projection is zero. Since we are searching for approximate solutions, this solution structure that we have considered, that is approximate, since it is finite expansion. Once we substitute that in the equation of motion, we generate this residue. Since this will not be identically zero, so, what we say is instead of having this identically zero, we have weaker condition which says that the projection of this residue along certain function directions, this is zero. We can take N such functions suitably chosen to generate N equations and thereby we can attempt to solve for this N unknowns p_k that we have in the expansion the temporal functions. So the choice of this functions H that decides the method. What are the different ways of choosing these functions on which we project? The simplest choice is this Dirac delta function. Once we choose these functions as this Dirac delta functions, this method is known as the collocation method.

Now what does it mean that, to choose H_j as Dirac delta functions. So, if you substitute these functions here then what you will obtain is… What this means is that this residue is zero at certain points, not at all points, but at certain points over the domain. Say for examples for the bar, if I choose this x_j points, so, these points… something like this. So, at these points, the residue must vanish. There can be various ways of choosing. These points are known as the precision points, or also sometimes known as accuracy points.

Now, there can be various ways of choosing this precision points or accuracy points. They can be uniformly distributed or there can be other methods of choosing precision points. Good way of choosing precision point is given by Chebyshev method and they are known as Chebyshev accuracy points.

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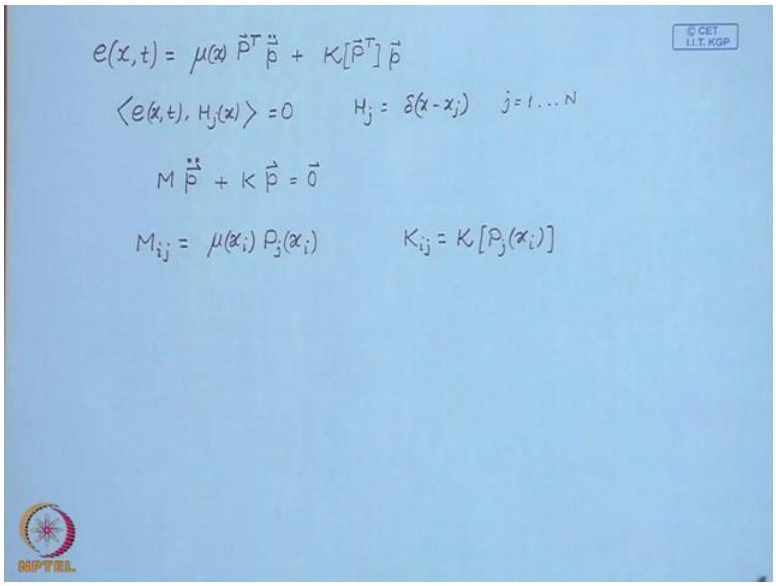

Now let us first look at what happens when we substitute this residue in the projection equation. So, I rewrite the residue, and if I consider the projection, the functions in this form using direct delta functions then we arrive at this equation of motion. This is the discretized equation of motion where these matrix elements are obtained like this. So, let us once again look at the example of the tapered bar. So, here we have a fixed-free bar once again and we have the geometric boundary condition at the left end and the natural or dynamic boundary condition at the right end. Now, we have to choose functions, we have to choose the comparison functions which satisfy, the comparison functions must satisfy all the boundary conditions of the problem, which means they must satisfy the geometric boundary conditions as well as the natural boundary conditions. So, let us look at the particular choice as shown here. So, we have these functions p_j. We can check very easily that $p_j(x)$, so we can check… These functions satisfy the geometric boundary condition as well as the natural boundary condition at the right hand. Now we can choose this accuracy points as we have discussed. We can choose them as uniformly distributed. So, if you divide the domain of the bar in N parts, then you can take this uniformly distributed; or you can also have the Chebyshev accuracy points which are determined by this expression… Now this has a nice geometric visualization. This is the domain of the bar. If you draw a semi circle with this domain as the diameter, and you put in regular polygon, for example, if you want to take three accuracy points you have to inscribe half hexogen and the projection of this corner points on the domain will give you the accuracy points. So, these are the Chebyshev accuracy points. As, you can see this

Chebyshev accuracy points will never fall on the ends of the bar. So, once you use either this uniform spacing or the Chebyshev spacing and calculate the Eigen frequencies, for uniform spacing what you obtain, so, these are the first two circular natural frequencies when you consider uniform spacing. When we consider Chebyshev spacing, then the first two natural frequencies are obtained like this. However, if you look at the exact circular natural frequencies, they are obtained like this. So, here what we find is that the fundamental circular natural frequency is in some error from the exact circular natural frequency, while the second one is more close. Now this collocation method has disadvantages that these not have the upper bound property as we saw in the energy based methods.

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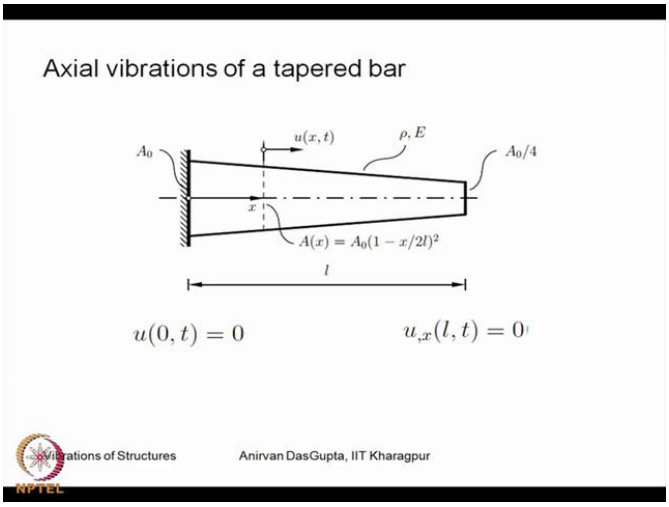

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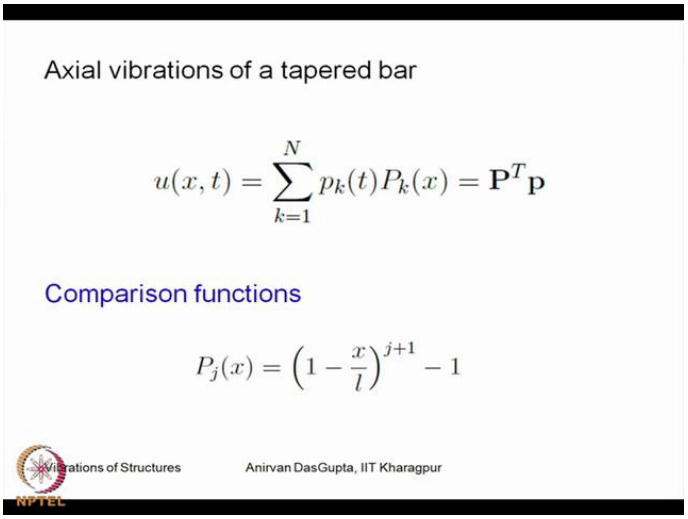

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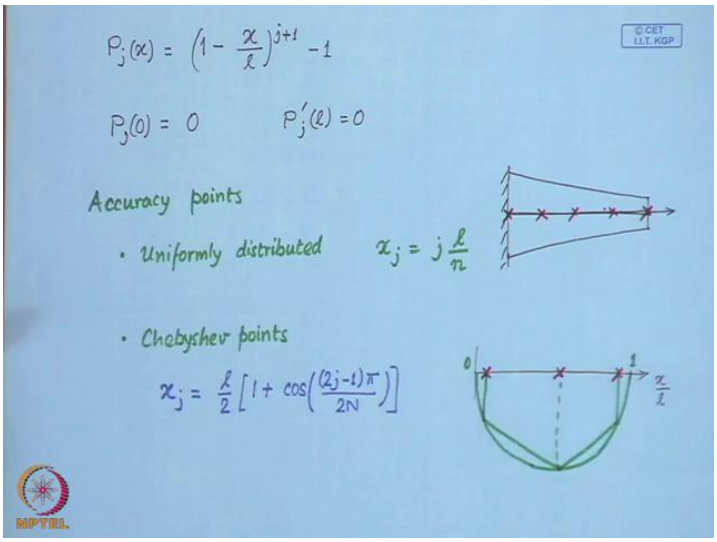

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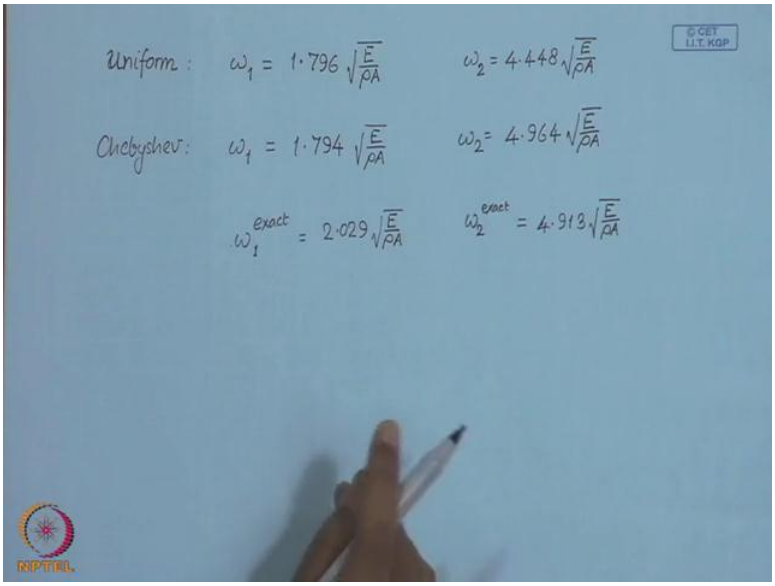

This approaches the exact natural frequency from below, but it can also, for example, if you see this comparison, this is approaching from other. So, this is a disadvantage of the collocation method. Now if you calculate the Eigen functions corresponding to these Eigen circular natural frequencies then, what we obtain is shown in this screen.

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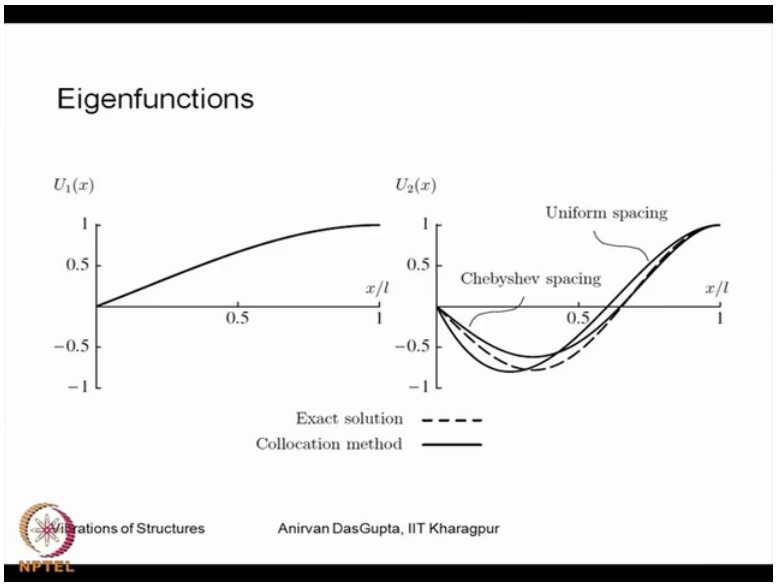

If you look at the fundamental Eigen function, the Eigen function corresponding to the fundamental mode, then within the accuracy of this plot, you see, you cannot distinguish the exact method with the collocation method. Though the natural frequency is in some error, but this can be of course rectified by considering more and more terms in the expansion and making it more accurate. On the other hand, if you look at the Eigen function of the second mode, you see, with the Chebyshev spacing it appears to be more accurate because the location of the node matches quite well with that obtain from the exact solution, while that of the uniform spacing is in some error.

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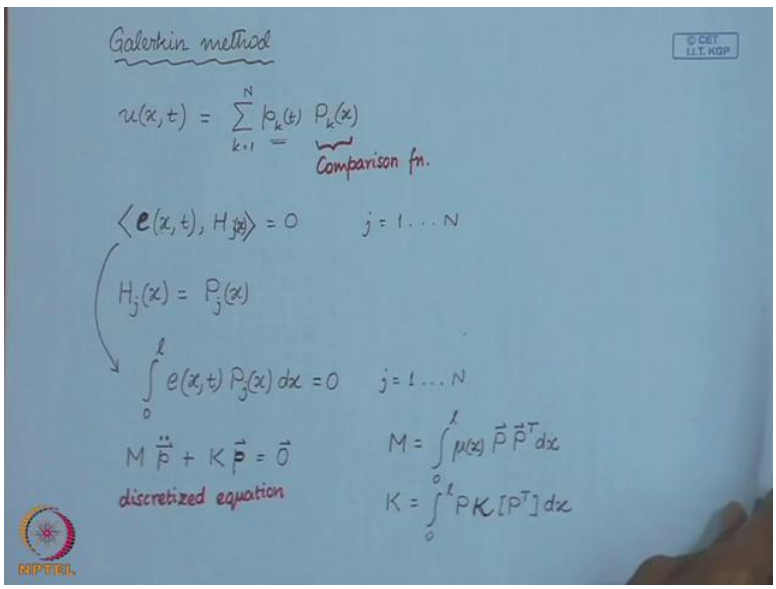

So, we have looked at this choice; we have looked at one choice of the projection function H(x). Now will look at another choice which gives us what is known as the Galerkin method, which is another powerful method; which uses the projection technique. In the Galerkin method, remember that, we have expanded… where these capitals, these are comparison functions, then we generated the residue; we projected the residue on certain functions and put them to zero. Now the Galerkin method, it takes this projection function same as the comparison functions we used for this expansion of the field variable. This projection gives us these N equations from which, we are going to solve for this N temporal functions p(t). This when we choose this projection functions as same as the comparison functions in the expansion, then we have the Galerkin method and when we do this projection, we obtain this discretized equation of motion where this matrices M and K are obtained in this form. So this structure, so this is again… So we obtain the discretized equation of motion of the system.

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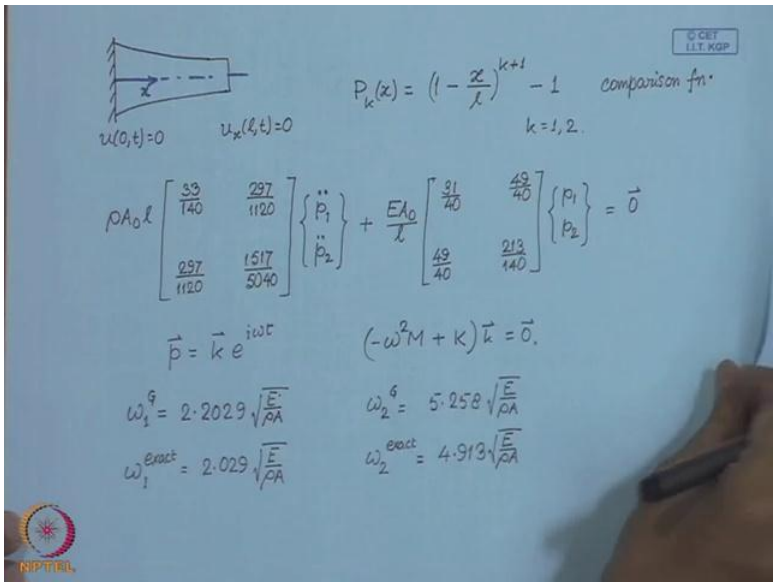

Now, let us look at this example once again. So, these are the boundary conditions at the fixed and free ends. So, once again we choose the comparison functions. We will take two comparison functions and discretize the equation of motion. So if you do that, you will obtain the discretized equation in this form. Then as usual, we do the modal analysis assuming this structure of solution. We obtain the Eigen value problem, discretized Eigen value problem and from where we obtain the first two circular natural frequencies. Now these superscripts G indicate obtained from Galerkin. Now if you compare, so as you

can see that the fundamental frequency compares very well with exact, while the second modal frequency is in some error. Here again, if you want to have accurate modal solutions for the first n modes, you use an expansion with 2n terms. So, here we can see that the error in the in the fundamental frequency is very small as compared to the second frequency. When you solve this Eigen value problem, you also obtain the Eigen vectors k. Now these are obtained, so one can easily determine these Eigen vectors corresponding to the Eigen frequencies and the corresponding Eigen functions. So, the Eigen functions are obtained by using these Eigen vectors. So, for the first Eigen functions, you use the first Eigen vector in this vector product. Similarly, for the second Eigen function, use the second Eigen vector in this vector product. So, these are the two Eigen functions that I have obtain from the Galerkin method.

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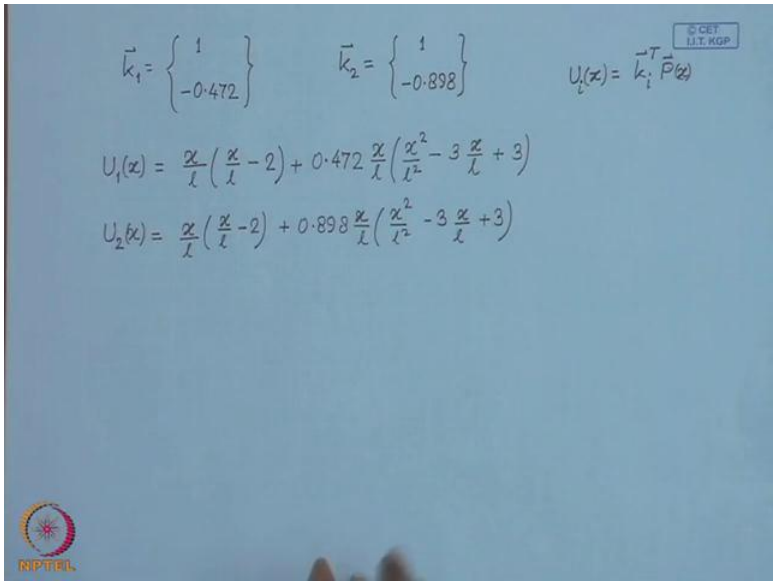

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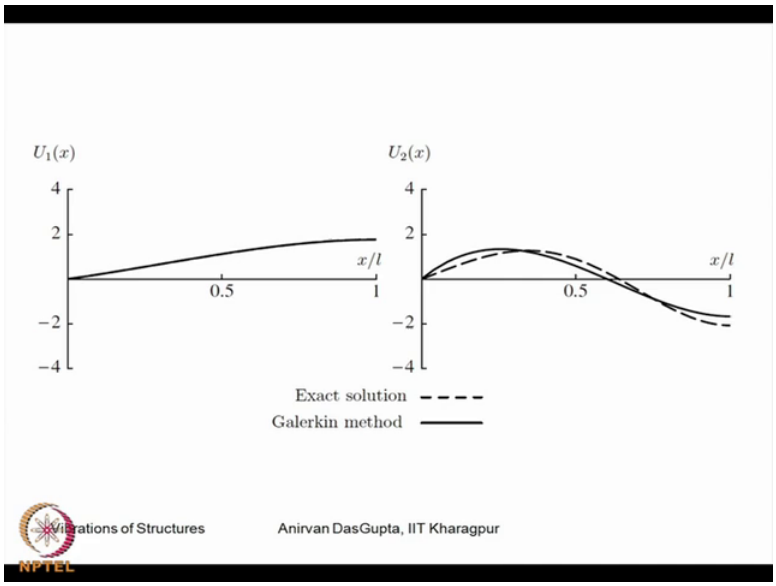

Now, here we can see the comparison of these Eigen functions with the exact solution that we have obtain previously. You can see the within the accuracy of the plot, the fundamental Eigen function is indistinguishable from the exact, for the fundamental Eigen function is solved from the Galerkin method is the indistinguishable from the exact solution. For the Eigen functions of the second mode, this is fairly close and you can see now that the boundary condition at the right end, the free end, is also matching though the location of the node is in slight error. So, in these plots, we have compared these two Eigen functions obtain from the Galerkin method with the exact solution.

Now, this Galerkin method, since as we discussed that this is the method for discretizing the equation of motion of a continuous system, so we can use this, also when we have external forcing and which we will discuss later on in this course.

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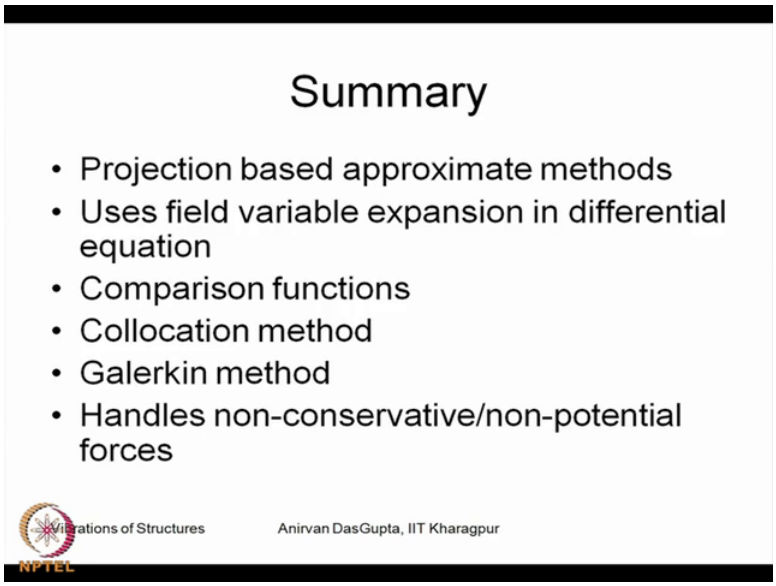

To summarize, we have studied in this lecture, projection based approximate methods, in which the field variable is expanded in terms of comparison functions and substituted in the differential, in the partial differential equation of motion of the system. Then what we generate is the residue, and this residue will not be zero uniformly with the domain. We use the weaker condition; we project this residue on to certain functions, and the choice of these functions decides the method. When we use the Dirac delta function, we have the collocation method; while if you choose the comparison function used in the expansion of the field variables themselves as the projection functions, then you have the Galerkin method; And these methods, so the projection methods, they can handle non-conservative and non-potential forces. So, with this we conclude this lecture.

Keyword: comparison functions, collocation method, Chebyshev points, Galerkin method.