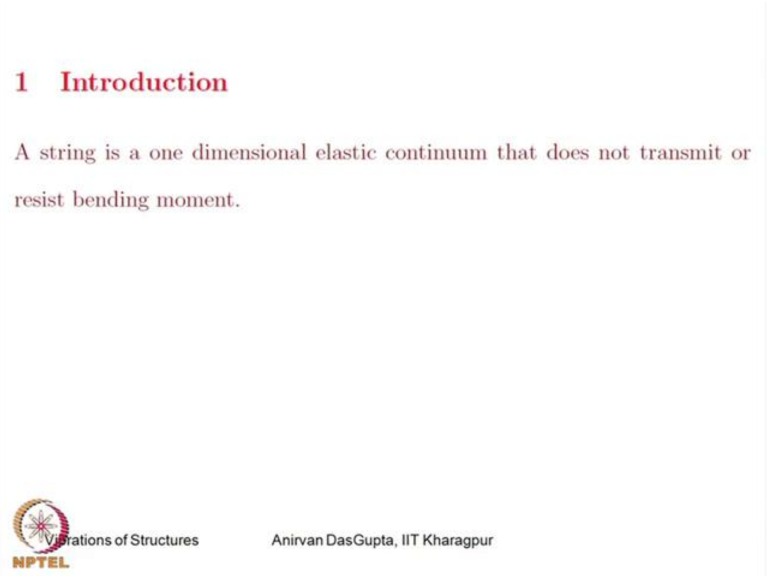
Vibrations of Structures
Prof. Anirvan DasGupta
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur
Lecture No. #01
Transverse Vibrations of Strings-I

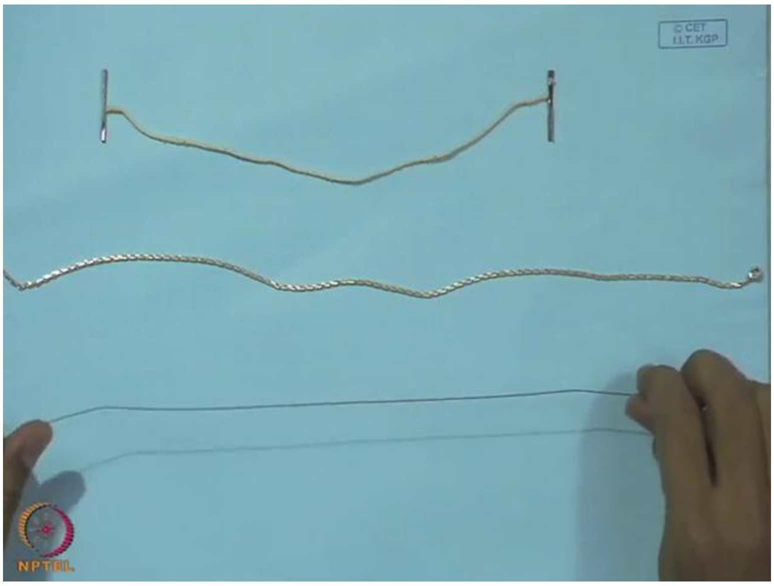
In this lecture, we will discuss transverse vibrations of strings. So, before we start discussing about vibrations of strings, let us look at what a string is. So, here on the view graph, you can see the definition of the string.

(Refer Slide Time: 00:39)



So, string is a one-dimensional elastic continuum that does not transmit or resist bending. So, this is the definition that we will use for a string. So now, I will show you some examples of strings.

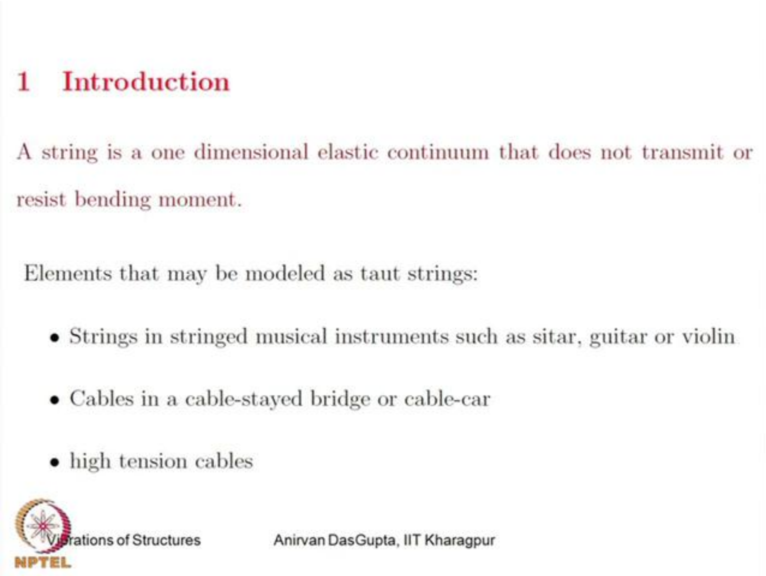
(Refer Slide Time: 00:57)



So, as you can see here, this is an ordinary tag, which is like a string, because it satisfies the definition that I gave you that, it does not transmit; it is the one-dimensional continuum and its does not resist bending in anyway. So, whatever shape I give it, it will retain that. So, the restoring force comes, when I make it taut. So, here it so, this is, this tension that I give to the string acts like the restoring force, the restoring force is produced by the tension in the string; otherwise the string will take any shapes. So, it does not resist bending. This is another example of a one-dimensional continuum that also does not resist bending. This is the chain and this is the hanging chain. So, this also qualifies to be string.

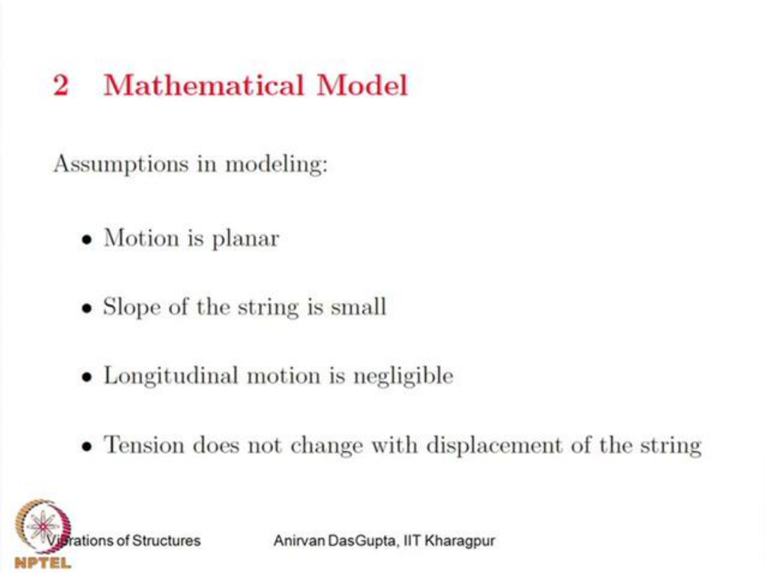
Here is a guitar string, as you can see, this is the guitar string. Now if I give it some bending, if I bent it, it restores back as you can see here, but still this is called as string. To understand the reason for this, let us see what happens in a guitar. In a guitar, the string is under tremendous amount of tension. Because of this tension, the primary restoring force is because of the tension in the string. Of course, there is this bending in the string, in this string which also kind of restores to its original states, shape. But when it is put in a guitar under tremendous amount of tension, the tension becomes the dominant restoring force; and hence any structural element, which is under high tension, qualifies to be analyzed in the first approximation as a string. So where do we find strings?

(Refer Slide Time: 03:28)



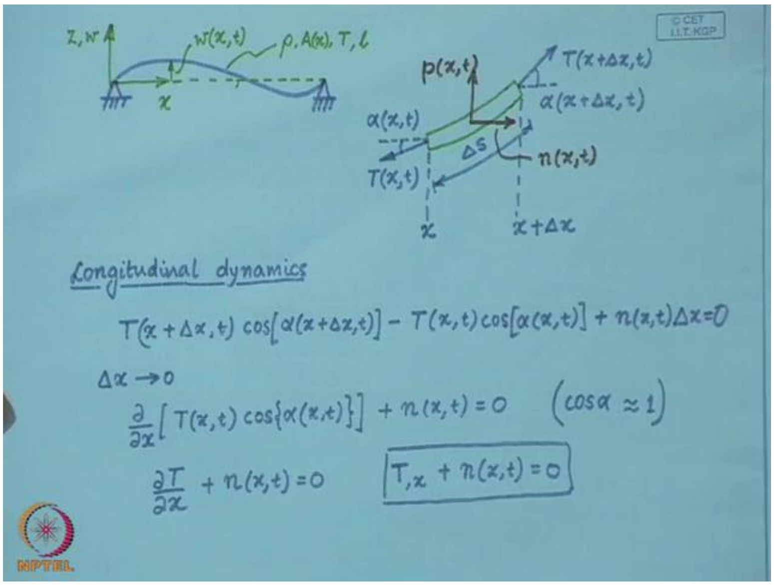
So, elements that may be modeled as taut strings are found in stringed musical instruments such as sitar, guitar, violin, even in the piano. So, we have seen such instruments, in which the sound is basically produced by the string. Then in the cables, in a cable-stayed bridge or a cable-car, so these structures have… Actually cables which are under tremendous amount of tension, and hence they can be analyzed as strings. In high tension cables, which are again under very high tension, they may be treated as taut strings.

(Refer Slide Time: 04:22)



So, we start with a mathematical model. So how do we model strings? Now in order to model strings, we will make some assumptions. So, here I have listed out some of the assumptions that we make in modeling the string. So, the first assumption says that the motion of the string is planar. So for example, here what I have, I will assume that the string vibrates only in this plane. Then the slope of the string is small. So when the string deforms, the slope at any point of time is small. The third assumption says the longitudinal motion is negligible. So, if I make a mark here, if you make a mark here and trace the motion of this mark as the string vibrates, you will find most of the time this mark moves transverse to the string, there is hardly any axial motion, there is hardly any motion in this axial direction or the longitudinal direction. So this is our third assumption. The fourth assumption says that the tension does not change with displacement of the string. So, I have put some tension in this string and as I displace, the tension, the change in tension is negligible. So with this, with these assumptions, we can now start modeling our string.

(Refer Slide Time: 06:15)



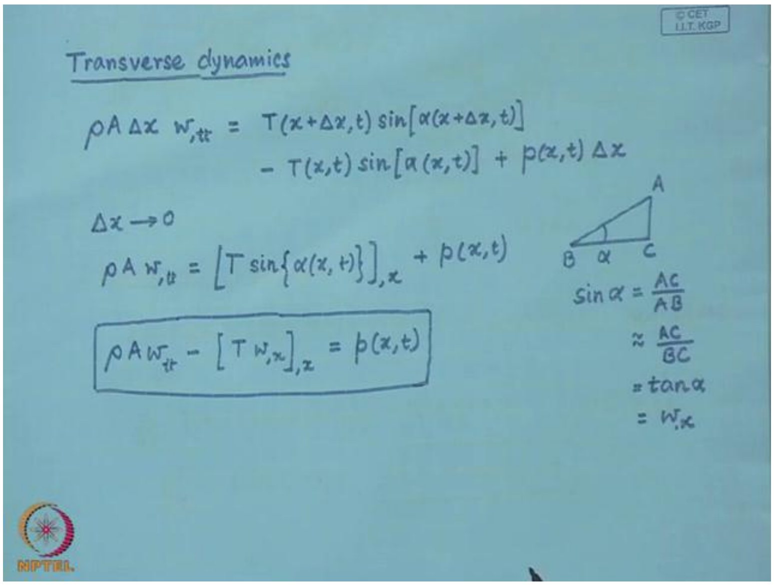
So, consider a string made of a material of density rho, has an area of cross section, which may be a function of the positional coordinate x; it is under a tension T and has the length l. So, this transverse motion of the string is measured by this variable w at a location x at a time instant t. So, so this shows a string, a stretched string or taut string, which has been displaced from its equilibrium position, which is the x axis. Now to write out the equations of motion, we will consider an infinitesimal element as we do in Newtonian mechanics.

So, we will draw the free body diagram of this infinitesimal element. So, this element lies between x, the coordinate x, and x plus delta x. On the left this is under a tension T (x, t), and it makes an angle alpha. On the right end, the tension is T (x plus delta x, t), and the angle it makes is similarly alpha (x plus delta x, t). The stretched length of this element is delta S. Now to begin with we are going to write the longitudinal, the equations of longitudinal dynamics of this infinitesimal element. Now as we have assumed that the longitudinal motion of this element is negligible, so we will neglect the inertia force that is the acceleration in the longitudinal direction; so, we will neglect that.

So, then the longitudinal dynamics reduces to just a force balance equation in the longitudinal direction; so, let me write out this force balance equation. So, this is the tension at this right end cosine of the angle minus the tension at the left end times the cosine of the angle, and along with this, you may have some external distributed forces, external forces. So you may have some external force distributions as I have indicated here, so these are force per unit length of the string. So, in the longitudinal direction I have for example, this n (x, t). So, I will introduce that also in this equation, and that must be equal to 0. Now if I divide the sole equation by delta x and take the limit delta x tends to 0. So, that will imply… Now we have assume that alpha thus the angle made by the string is very, very small. So, I can safely assume that cos alpha is almost 1.

So, if I make this simplification or this assumption, then this equation simplifies to… Now this I will write in a shorter form, so this comma x in the subscript would indicate the partial derivative with respect to x, and we are going to follow this notation throughout this course. So, this then is our equation for the longitudinal dynamics, which is essentially a force balance equation. Now we are going to look at the transverse dynamics. So, I am going to use this free body diagram for the Transverse dynamics.

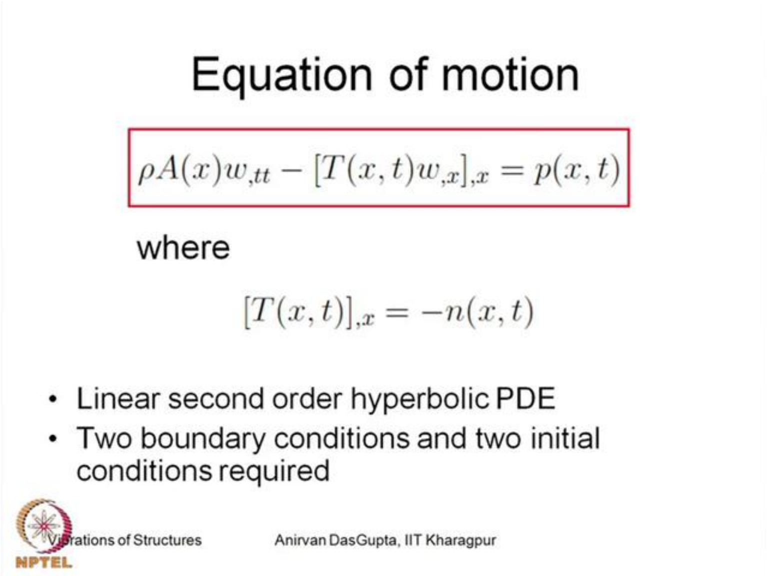
(Refer Slide Time: 14:42)



So, if I write out the equation of motion, so mass of this little element can be written as rho times A is the mass per unit length and the length of this element is almost delta x, up to a linear approximation. So, rho A delta x would be the mass of this little element, times its acceleration; so which I will now write, since w is the motion in the transverse direction, so, w,tt indicates the acceleration, it represent the acceleration of the element in the transverse direction; so, mass times acceleration that must be equal to all the forces in the transverse direction. So, on the right end, we have… and the left end we have minus T (x, t) sin of the angle the left end and plus the distributed force in the transverse direction. Now if I once again divide this whole thing by delta x and take the limit, delta x tends to 0; so, we have partial derivative of this term with respect to x.

Now we have again assume this alpha to be very small, so let us see what sin alpha turns out to be when alpha is small. So, sin of alpha is equal to AC over AB, and if alpha is small, you can very easily see that this, up to the linear approximation, is almost equal to AC over BC; since BC is almost equal to AB, when alpha is small; and this is the tangent of the alpha, of the angle alpha; and this tangent of alpha can be very easily seen to be del w/del x. So, by using this approximation up to the linear order, I can safely write sin alpha almost equal to del w/del x. So, if I make this substitution here and make some rearrangements, I obtain the equation of motion of transverse dynamics of the string.

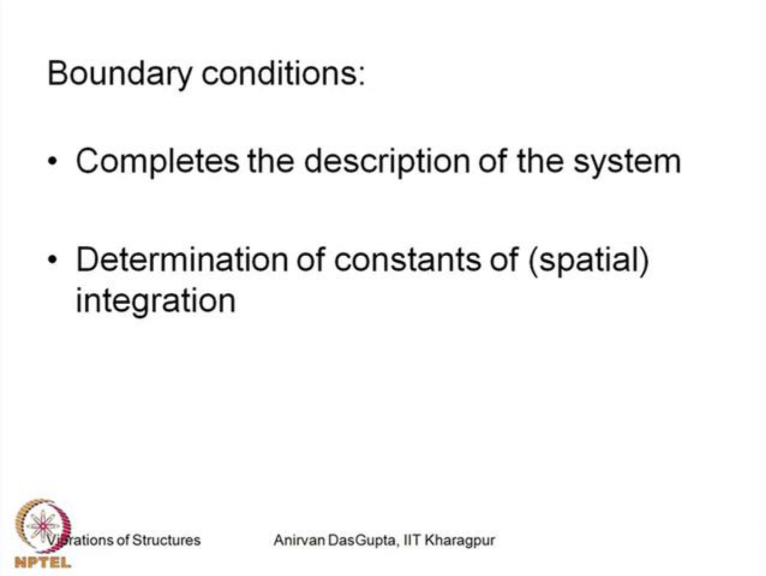
(Refer Slide Time: 19:55)



So, finally, if I look at this slide; here I have put together these two equations that we are just derived. So, the boxed equation is the equation of the transverse dynamics, where this tension, which may be a function of space and time; if there is an external distributed force n. Now this equation of motion, this is a partial differential equation, it is a linear second order hyperbolic partial differential equation. Now to solve this, we need boundary conditions and initial conditions. So, as you know that, since we have second order in space and second order in time, we will need two boundary conditions and two initial conditions. So, let us see, I mean first why do we need these conditions? So, so briefly, so this equation of motion as we have seen is derived by considering and infinitesimal element of the string, this in no way tells us, how the string is connected to the ground or if it all it is connected to the ground. So, we will need the description to, I mean to complete the physical description of the whole systems.

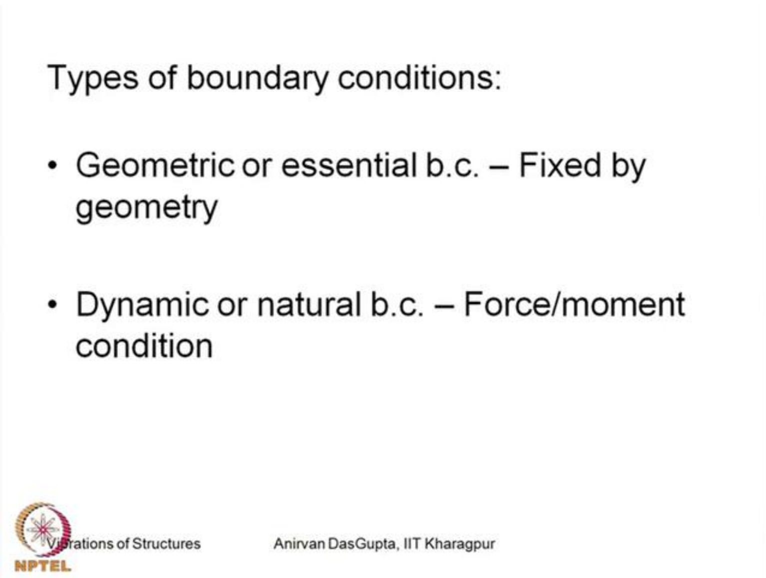
So, we will need these boundary conditions at the two ends of the string; now and mathematically, if you want to understand this, you see there are, I mean, this equation has is second order it in space. So, it is the space derivative is of second order. So, upon integration, we will generate two constants of integration, and hence to determine these constants of integration we need the two boundary conditions. In a similar manner when we integrate the time part, we will again generate two constants of integration, which will be solved from the two initial conditions that are provided.

(Refer Slide Time: 22:17)



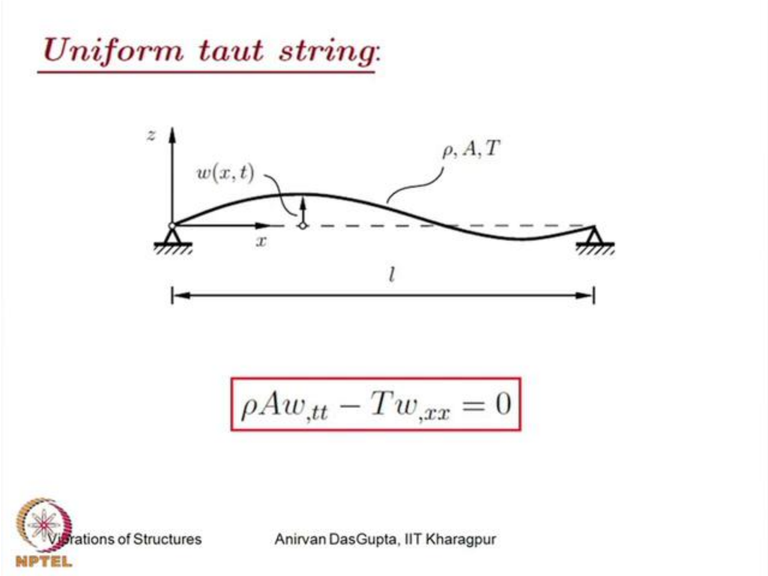
So, the boundary conditions therefore, complete the description of the system, and they are used for determination of the constants of the spatial integration; now these boundary conditions are of two types.

(Refer Slide Time: 22:33)



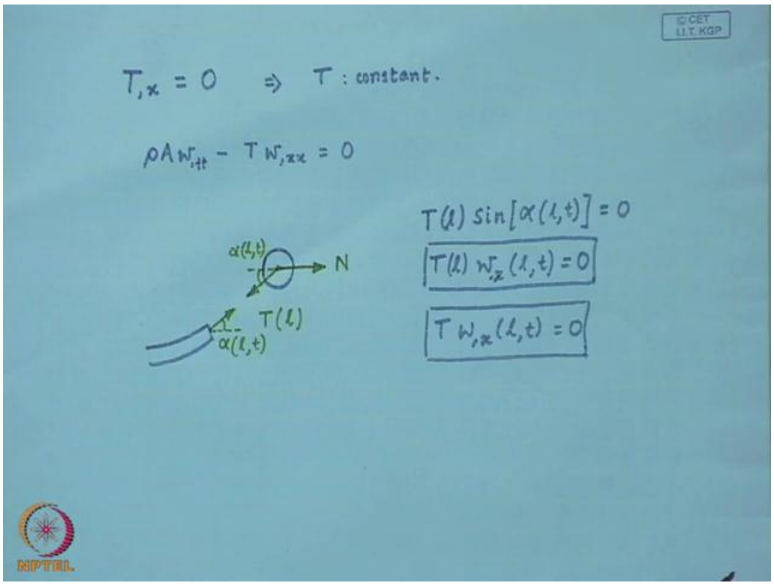
The first type is known as the geometric or essential boundary condition, such boundary conditions are fixed by geometry of the problem, and the second is the dynamic on natural boundary condition, which comes because of some condition on the force or moment, mostly in a in a string it will be force. So, in the dynamic boundary condition on natural boundary condition is a result of some force condition on the string.

(Refer Slide Time: 23:13)



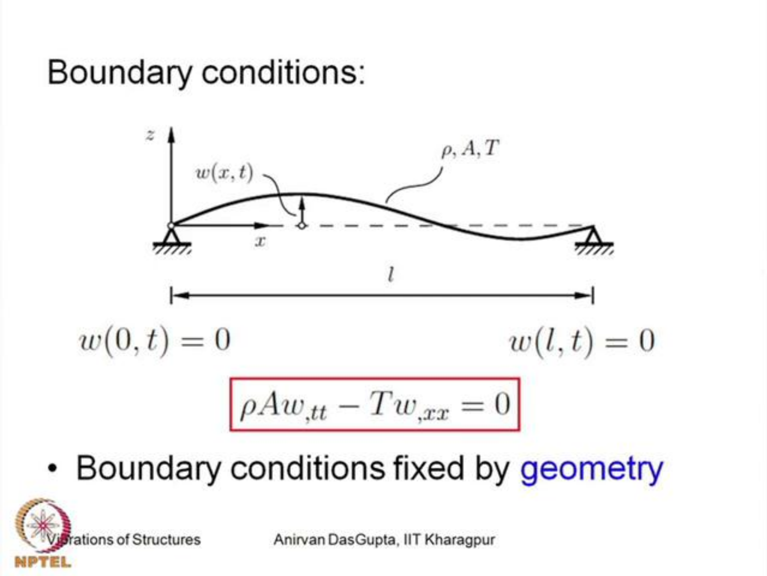
So, let us look at some examples. So, this shows a taut string which I drew before; now immediately… So this is the uniform taut string.

(Refer Slide Time: 23:32)



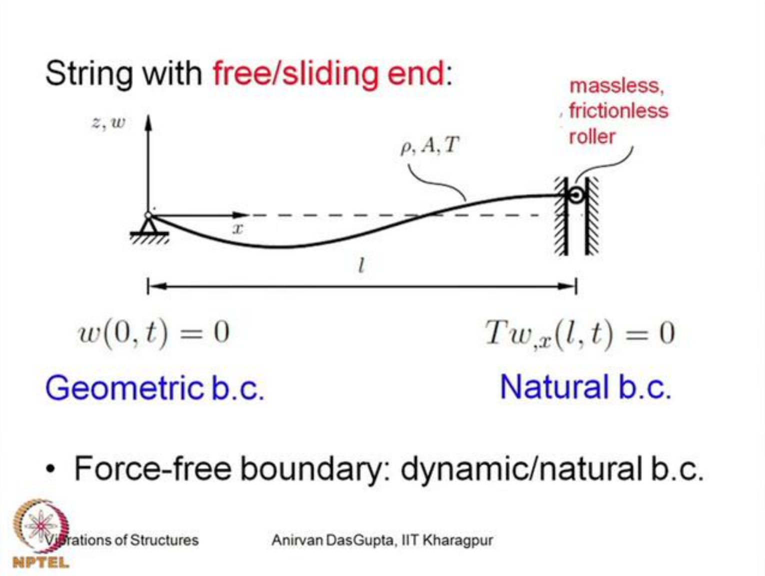
So, you can immediately see that the partial derivative of the tension in the string is 0; there is no external force in the in the longitudinal direction of the string. So, therefore, this is our equation for the tension, which implies the tension is a constant. So, if the tension is a constant, then our equation of motion reduces to… Here we consider that the there is no force even in the transverse direction. So, so we see in this slide that the equation of motion is given by these box equations, so this is an equation for a uniform taut string with no external forces.

(Refer Slide Time: 24:50)



Now the boundary conditions now this is very simple. From geometry you can easily identify that at the left boundary and the right boundary, at both the boundaries, the displacement of the string must be 0. So, these boundary conditions are set by the geometry of the problem. So, the geometry of the problem says that the displacement, the transverse displacement of the string at the two boundaries must be 0. So these are the geometric boundary conditions for the string.

(Refer Slide Time: 25:28)

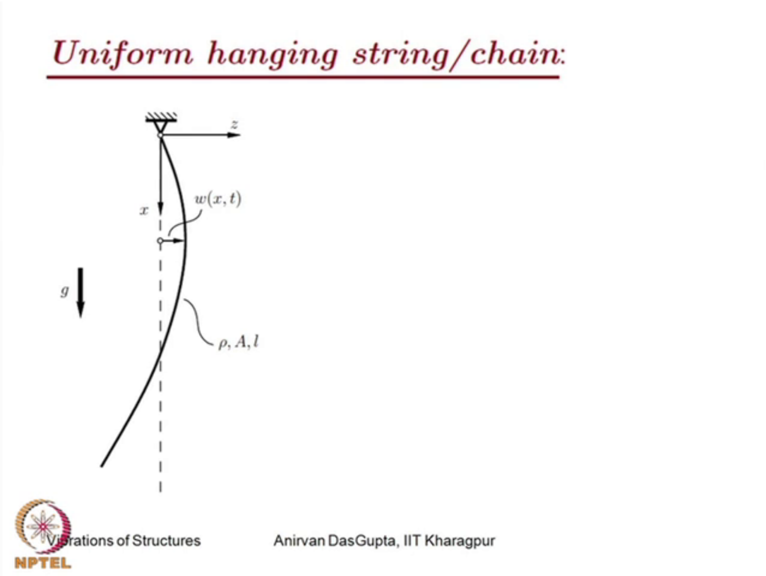


Next we are going to look at a string with free or sliding end. So here as you can see, the equation of motion. So the string is uniform, there are no external forces; so, the equation of motion will remain the same as we have discussed before; for the boundary conditions at the right boundary is what we want to now see. So, that at the left boundary we again have a geometric boundary condition which is the displacement is 0 at x equals to 0; at the right boundary, so to understand the boundary conditions at the right boundary, let us look at the right boundary in a little detail.

So, this is the mass-less frictionless pulley to which this string is attached. So, now, I will draw the free body diagrams of these of this connection. So, at the pulley, you have one normal force from the gait, a frictionless gait, so there is only one normal force, and from the string, we have this tension at x equal to l. So, if alpha is the angle at any instant at this end of the string, then we can see that from the free body diagram of the pulley, we can write down the equations of equilibrium for this pulley. So, if I write out the equilibrium in the transverse direction, so I will write out the equation of equilibrium in the transverse direction for this pulley; this pulley is mass less and frictionless; so we have… this turns out to be 0.

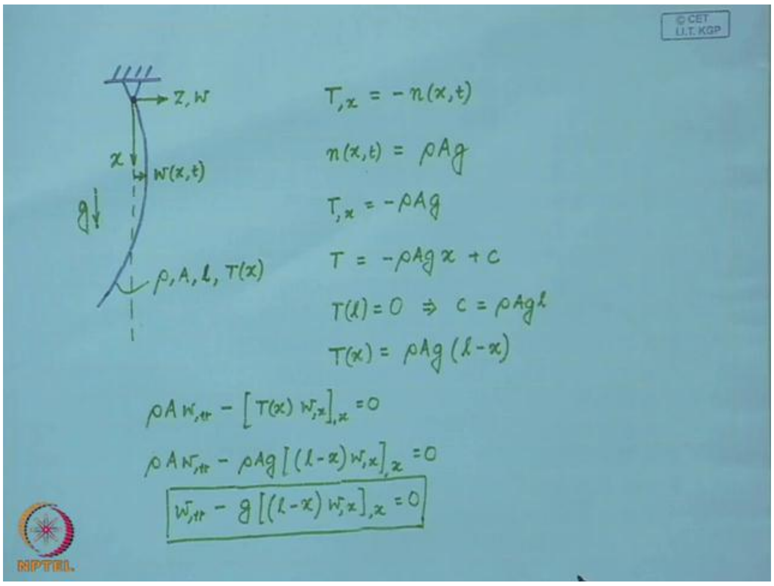
So, what this essentially tells us is that, the force in the transverse direction on the pulley has to be 0; now if I use the approximation that we are discussed, I obtain this condition; now it is so happens that the tension in this string is uniform, so this can be written further in this way. So, this becomes our boundary condition at the right end, so this condition comes from a force condition, that the force in the transverse condition on the pulley must vanish; and if you take the force balance in the longitudinal direction, then you will be able to solve for this normal reaction on the wall. So, the boundary condition as we can now see on this slide, is given by tension times del w/del x, at x equal to l, and at all time t must vanish. Such a condition such a boundary condition is known as the natural boundary condition. So, this comes from a force condition. So, the Force-free condition, Force-free boundary gives us a dynamic or a natural boundary condition.

(Refer Slide Time: 30:27)



Now let us look at a uniform hanging string or a chain as we have discussed. A chain qualifies to be analyzed like a string; so the equation of motion for this chain can be derived like this.

(Refer Slide Time: 30:57)



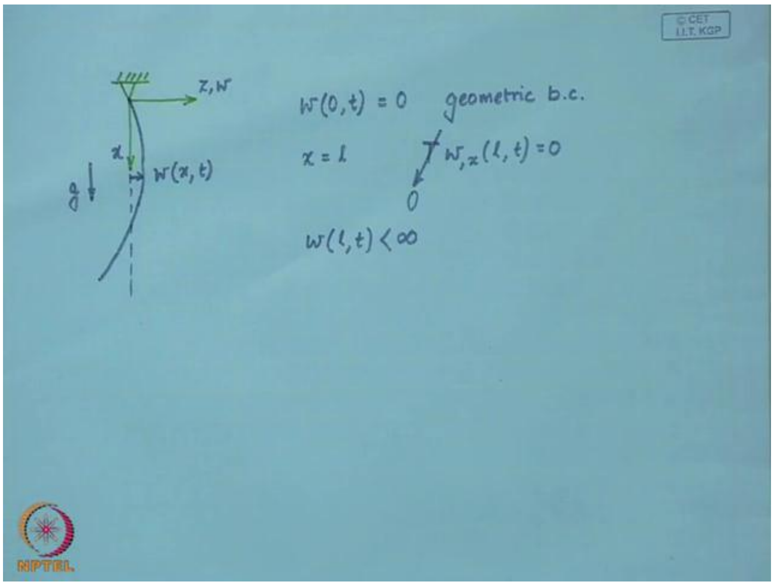
So, here I show a hanging chain, which is made of material of density rho, its area of cross section may be assumed to be uniform, for simplicity and its length is l; now as you can realize this chain will be under varying tension, so the tension in the chain will be a function of the position coordinate x. So, this is what we have to now determine. So, we already know that this is our equation for the in the longitudinal direction, the force balance in a longitudinal direction. So, this is force per unit length, n is the force per unit length in the longitudinal direction.

So, if I express this, so if the density since the density of the chain is rho and area of cross section assume to be uniform is A; so, this is mass per unit length; so rho A is mass per unit length; and if I multiply by the acceleration due to gravity, will assume that this is an uniform gravitational field, then the transverse force distribution or the force per unit length of the chain is rho A g; this is weight per unit length; rho A is a mass per unit length, g is the acceleration, so it is weight per unit length, so which means is the force per unit length in the longitudinal direction in the direction, vertical direction in this case.

So, if I substitute this and integrate out, so that is what I have, the tension is the function of the coordinate; now I will use a boundary condition a condition at one of the ends of this chain; So, it is convenient to see that, to use this condition that the tension at x equal to l is 0. So, that gives us c, which is… And if I substitute back and which can be simplified further…

So, this is the equation of motion of transverse dynamics of a hanging chain in a uniform gravitational field; now again we will need boundary conditions to complete the description of the problem. So, let us look at the boundary conditions for this chain.

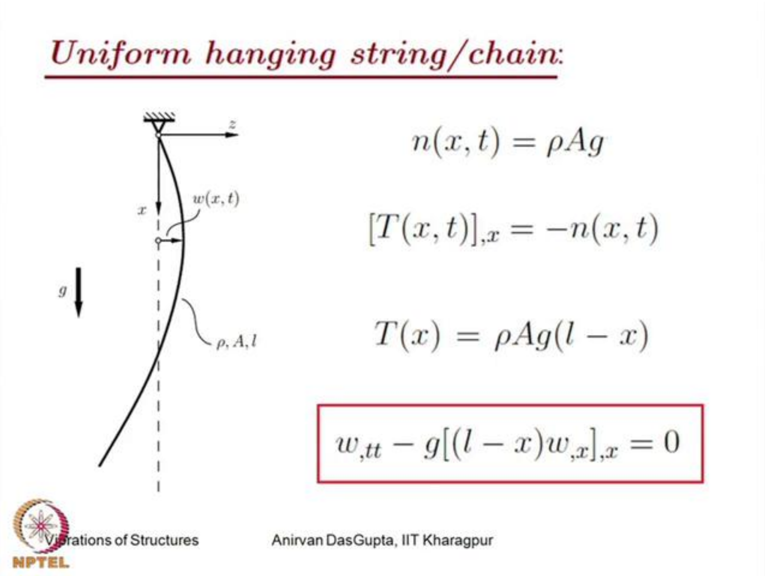
(Refer Slide Time: 36:32)



So, at x equal to 0, as you can see, so we have a geometric boundary condition… Now what happens at x equal to l? So, what is the boundary condition at x equal to l? Now, this is a free end of the string and this is free to string. So, one may write in a manner similar to what we have discussed before. So, this was the Force-free condition, so no force in the transverse direction, but then remember this tension at this free end is also 0; so, the tension is also 0. So, if you imagine the last particle of this chain, it is it has no, almost no restoring force, because remember that in the string the restoring force comes because of this tension in the string; now this end, the free end of the string, the tension goes to 0. So, the last particle of the string is hardly has restoring force in the transverse direction. So, there is the possibility at least theoretically that this displacement might become infinity; because it does not have a restoring force when displacement can become very large; but that definitely we know we have seen a hanging chain or a vibrating chain, and it does not go to infinity, its remains finite. So, from the physical consideration, we must have a finite solution at this end. So, we will we write this in this form, so this is an inequality. So, what this says is that the displacement of the string at the free and must be finite.

Now, this when we discuss the solution of the vibration problem of a hanging chain, the solution of the equation of motion of a hanging chain, we will see that there is a solution, which has an infinity; so what this condition will tell us, is that solutions should not be present, because from the physical consideration, we must have a finite displacement of this free end. So, this we will elaborate or discuss in detail, when we discuss the solution of vibration of a hanging chain.

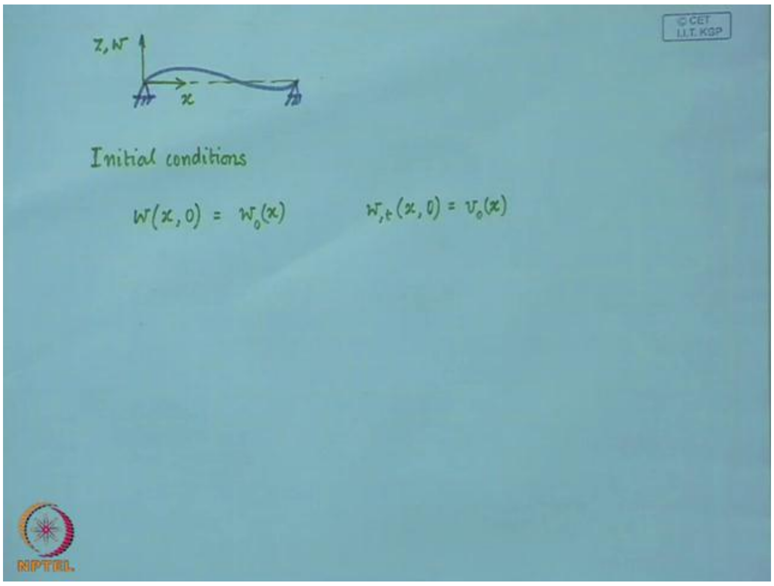
(Refer Slide Time: 41:07)



So, let us see what we have discussed in this lecture. So, we have started with the motion, modeling the equation of motion of a string, the transverse as well as longitudinal. So, in the longitudinal direction, it is a force balance, because we have assume that the motion in the in the longitudinal direction is negligible; then we have derived the equation of motion in the transverse direction of the string, assuming it to be planar; then we have looked at the boundary conditions that come up in vibrations of… transverse vibrations of strings.

There are two kinds of boundary condition as we have seen; one the first one is the geometry boundary condition, the second is known as the natural boundary condition or the dynamic boundary condition; and then we have seen a few examples of strings, of taut strings, and we have looked at the dynamics of the hanging chain, and in each of these cases we have looked at the boundary conditions that governed the equations, the total description of the system; now along with the boundary conditions, we also need the initial conditions, which we have not discussed as yet.

(Refer Slide Time: 42:52)



So, so as I mentioned that we need two initial conditions. So, these initial conditions are usually specified as the initial deformation of the string and the initial velocity distribution over the string. So, with these two initial conditions, one on the displacement, other on the velocity; we can now completely solve or uniquely solve the equation of motion of vibrating string. So, with this, we complete this part on the transverse vibrations of strings.

Keyword: dynamics uniform taut string, boundary conditions and initial conditions, string with free/sliding end, uniform hanging string/chain.