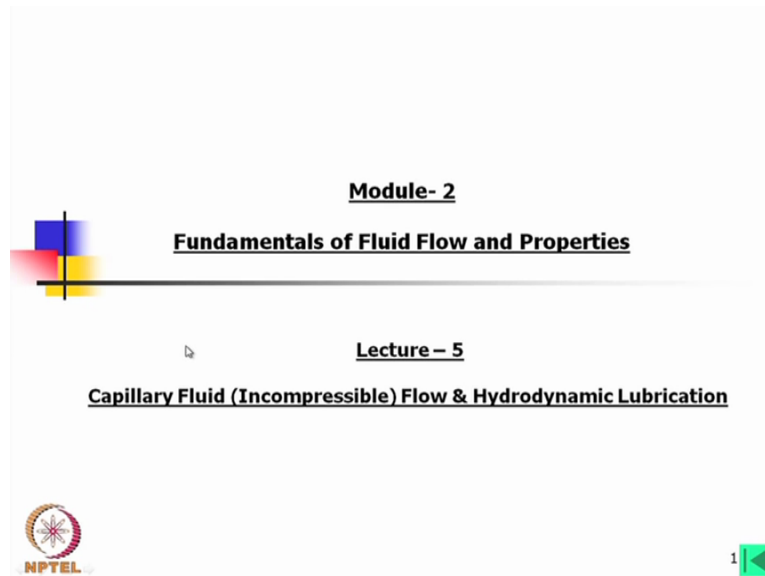


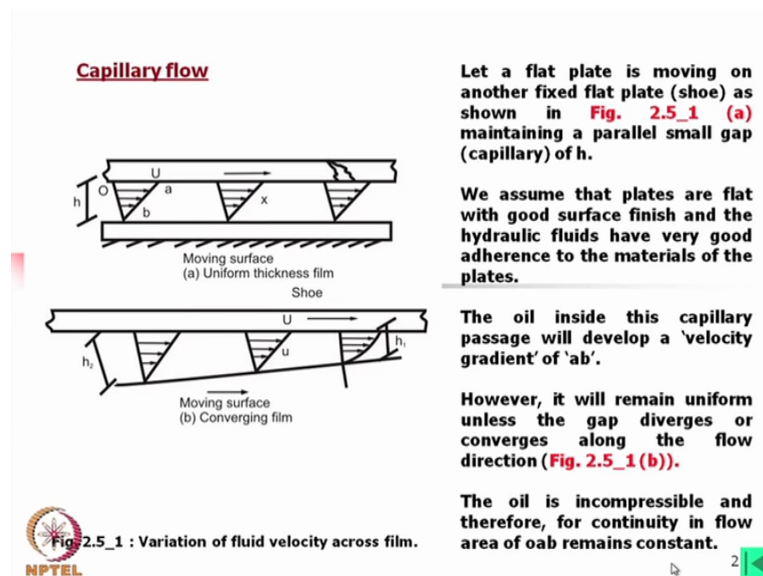
Fundamentals of Industrial Oil Hydraulics and Pneumatics
By Professor R. Maiti
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur
Module02 Lecture05
Capillary Fluid (Incompressible) Flow and Hydrodynamic Lubrication

(Refer Slide Time: 0:20)



Today's lecture is on capillary fluid incompressible flow and hydrodynamic lubrication. This is under module 2 which is fundamentals of fluid flow and properties. In the last classes, we have learned some basic properties of fluid and also, basic mathematics on the fluid flow such as momentum energy equations etcetera. In this lecture, we shall learn about the mathematical development of the incompressible fluid flow in capillary passage.

(Refer Slide Time: 1:17)



Now if we look into this figure what we find at the top on a parallel plate another parallel plate is moving and the gap is h and it is maintained throughout the plate or in other words if bottom one is larger the moving one is smaller, then we can consider the field is the area of the upper one and bottom what we see that the h is gradually decreasing from h_2 to h_1 within this field. Now we assume that let us a flat with good surface finish and the hydraulic fluids have very good adherence to the materials of the plates. Now if we think of the one plate is moving on the other and in between there a fluid is there. Then we may think of that if perhaps there is slight roughness the lubrication will be better, it is not that.

It is always better that surface should have good finish, but the fluid should adhere to the plates. This means that the layer of the fluid is touching the bottom plate that will never separate and which the layer touching the upper plate that will not separate, then what may happen if there is it is viscous fluid, the when one plate is moving to the other then the oil will try oil will be dragged and it will try to move, not oil it might be any fluid as such. Now this gap you may ask what is the dimension of this gap. This gap are in the order of few microns 20-25 microns or slightly more than that.

Now therefore I have writtun that it should have good adherence property to the plate material. The oil inside this capillary passage, which develop a velocity gradient of ab . Now this line is you can say, this is a straight line, this is a straight line. What this point is not moving. So what the oil if I take a point on the oil bottom layer, it is touching here and if this point is not moving, whereas the in upper layer o the point o has moved through a. So there is a velocity gradient okay.

However, it will remain uniform unless the gap diverges or converges along the flow directions. What we have shown here, it is converging at the bottom one we are showing that it is a converging gap. In that case, what is happening if we think of this point is moving over here. So at the top also this point will move the same distance. Here also it is the same distance, but if we look into this the gap has reduced here, but this is an incompressible fluid and there is no slippage. So amount entered here has to go out through this h1 gap. Therefore, definitely this line will take a it will gradually take a concave shape or in other words, what it should actually? Here it will be a concave, then straight line and then it is a convex so that area of these 3 are same that only on a unit length along the transverse directions will give the same volume.

Now if this why considered this is h2, this is h1 and if this velocity is U in both the cases and h2 plus h1 divided by 2 is equal to h say here. Here you will find this triangle and this triangle will the same? Now this is quite interesting what we find that this oil is taking gradually the velocity of the mid points are increasing in this case. The oil is incompressible and therefore fort continuity in flow area of oab remains constant. This area will always remain constant which I have already told you.

(Refer Slide Time: 7:19)

Capillary flow (Cont...)

Consider an element of fluid as shown in Fig. 2.5_2,

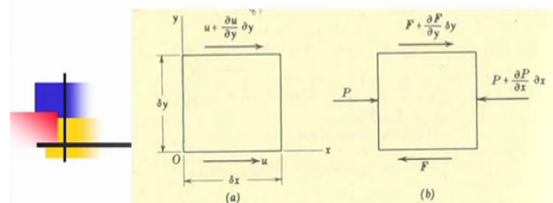




Fig. 2.5_2 : Element of fluid.

For force equilibrium in x direction,

$$\left(\tau + \frac{\partial \tau}{\partial y} \delta y \right) \delta x \delta z - (\tau) \delta x \delta z = \left(p + \frac{\partial p}{\partial x} \delta x \right) \delta y \delta z - p \delta y \delta z \quad \dots 2.5_1$$

Now consider an element of fluid as shown in figure 2.5-2. Now this is the figure. What the element we have considered? This is a like a cube. The axis system is that along the direction of flow or along the direction of motion, the capital U is the plate velocity, small u is the fluid velocity in that direction capital U direction axis is x and along the h direction the axis is y and along that transverse directions there will be z axis okay.

Now inside that fluid film, we have considered a cube of fluid. In other direction, it might be Δz other direction is Δz that means volume of this fluid is Δx into Δy into Δz , okay. Now look at this, if at that point velocity is u then at the top level of this element the velocity will be u plus Δu Δy into Δy . This should be Δy like this anyway. Now if we think of the force balance. What the force is acting mind it we have a taken a fluid at the middle not at the bottom or not at the top for our clarity, then what we find this side. There is a pressure field p and here definitely pressure will be p plus Δp by Δx into Δx .

Similarly what force is acting over there? This is moving like this, so what force F is acting that will be here F plus dF Δy by Δy . Now this is we can see this one component may be 0, anyway if we write down the equilibrium equation for this element then clearly τ plus $\Delta \tau$ Δy by Δy the τ is the shear stress between the layers into Δx into Δz that means this x into Δz , we have consider this area minus τ plus Δx Δz is the force in this directions and here, I think here this will be τ not F . Here, this will be p plus Δp by Δz , consider this is small p not capital p , Δy into Δz minus p into Δy Δz , okay. So this will be the equilibrium we are considering this force, this minus this force is equal to this minus this force p minus this force, this will be τ . So this is for the equilibrium of this element.

(Refer Slide Time: 11:03)

Capillary flow (Cont...)

The above expression reduces to:

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} \quad \dots 2.5_2$$



This means that change in shear stress in y direction is equal to the change of pressure in the x direction.

Now, $\tau = \mu \frac{\partial u}{\partial y} \quad \dots 2.5_3$

[Dimension check $N / m^2 = Pa \text{ Sec } \frac{m / Sec}{m}$]

Differentiating w.r.t. y ,

$$\frac{\partial \tau}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} \quad \dots 2.5_4$$

Now we can see that all terms will be cancel out except the terms these terms will remain there. So what we find that the change in shear stress. This is the shear stress change in shear stress in y direction is equal to the change of pressure in x direction. Now this is again the theory from the theory of shear stress and fluid. The shear stress is given by μ Δu Δy

where, μ is known to us which is dynamic viscosity. So if we can check the dimension this is newton per meter square Pascal's. So μ is given in pascal seconds and this is meter per second by meter. So this is Pascal's means newton per meter square. Now differentiating with respect to y what we get, $\frac{\partial T}{\partial y}$ is equal to $\frac{\partial \tau}{\partial y}$ is equal to $\mu \frac{\partial^2 u}{\partial y^2}$, we have differentiated this one and we get this. Now $\frac{\partial T}{\partial y}$ is equal to $\frac{\partial p}{\partial x}$.

(Refer Slide Time: 12:44)



Capillary flow (Cont...)
 Substituting eqn, ... 2.5_2 we get,

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \quad \dots 2.5_5$$

 It is the reduced form of Navier-Stokes equation
 Clearly if, $\frac{\partial u}{\partial y}$ is constant in case of flow between two parallel plates (Fig. 2.5-1a), then,

$$\frac{\partial p}{\partial x} = 0$$

 Hence, no pressure will be built up. Therefore, wedge (Fig. 2.5-1b) is essential for varying $\frac{\partial u}{\partial y}$
 i.e., to generate pressure gradient and thus the lubrication fluid film.

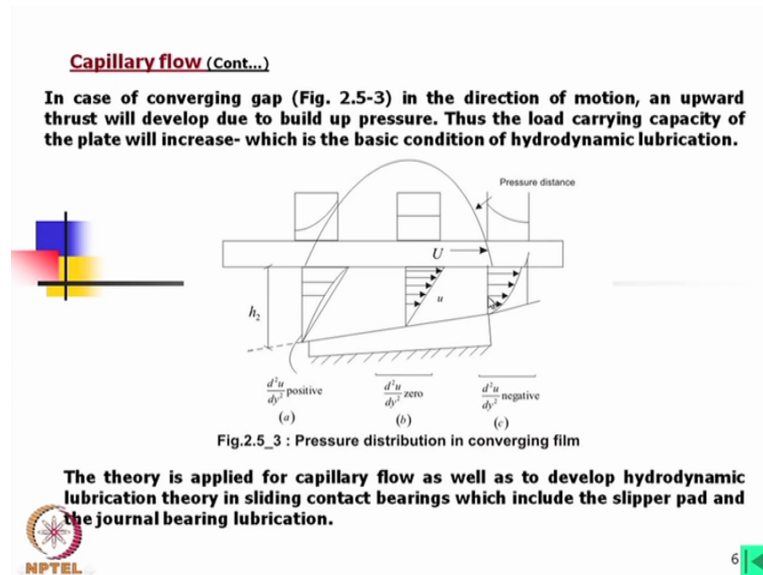



So we can next phase we can write $\frac{\partial p}{\partial x}$ is equal to $\mu \frac{\partial^2 u}{\partial y^2}$. So from the equilibrium of the fluid in a capillary passage, we have arrived into an equation that change in the pressure in x direction is defined by viscosity into $\frac{\partial^2 u}{\partial y^2}$. This means that a rate of change of u in y . Clearly if $\frac{\partial u}{\partial y}$ is constant in case of flow between two parallel plates, then $\frac{\partial p}{\partial x}$ will be 0, because second derivatives of u by y will become 0. So $\frac{\partial p}{\partial x}$ is 0. This means that no pressure will be built up. Therefore, wedge is essential for varying $\frac{\partial u}{\partial y}$ that means plate one plate is to be inclined to the others gap should gradually decrease to have positive pressure that means to vary this because there will be change in u then only there will be $\frac{\partial u}{\partial y}$ that is to generate pressure gradient and thus the lubrication fluid film.

Now this lubrication fluid film needs, there should have pressure gradient in the field or in the flow in the capillary passage only then this film will be generated. Now I was talking about that if there is a rough surface then what might be the problem? Apparently at the top surface some fluid will be more more adhere to the plate, but due to this roughness there will

be breakage in the fluid frame it will be disturbed, but if the plate surface are good finish and the fluid film or the fluid is adhere to that then this fluid film will exist there.

(Refer Slide Time: 15:31)



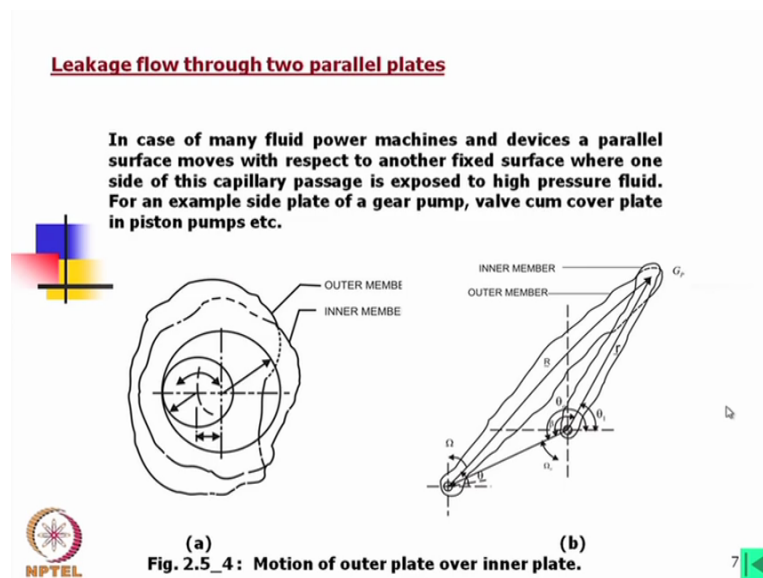
Now in this figure we have shown this converging field and the plate is moving in these directions then what we see that to keep this area constant, initially there was a concave curve. From this side it was a concave curve and then the straight lines and then it is a convex curve that means the fluid velocity is gradually increasing when the plate is moving in these directions. Now this mean that here sorry $\frac{d^2u}{dy^2}$ is positive. In this case, it is 0 at the middle, where the gap is h , you can say and here, it becomes negative which is also shown in this form.

Now as this at this point and this point, the plate there is no existence of plate and definitely pressure will be the atmospheric pressure and 0. So this must be that pressure map the pressure gradient will be such that we can see this maximum pressure at the middle. So now we can consider on this plate this is the pressure distribution. So this pressure in this plate will float and we can estimate if this is there then, obviously if there is no load the gap will try to be increase. So if we can put a load here and the magnitude of this load will depend on how much pressure is being developed here. Definitely this gap has a role this magnitude of this gap will have a role depending of this gap, the either I mean pressure will be more. If the gap is less, but sorry, if the gap is less then the pressure will be more. However, we cannot make this gap very small at least there should have some magnitude that depends on fluid property and other parameters.

So in case of converging gap in the direction of motion and upward thrust will develop due to build up pressure. Thus the load carrying capacity of the plate will increase, which is the basic condition of hydrodynamic lubrication we call this hydrodynamic lubrications. Now contrary to that there is also hydrostatic lubrication and in many fluid power components, there are hydrostatic lubrications. These are done by creating a hydrostatic pressure from external source, for example in that case, one plate is moving on the on another plate, which is having a gap uniform gap. In that case, from the stationary plate we can make several holes and we can allow the high pressure oil to go in. Anyway, we will look into the hydrodynamic lubrications and the pressure field in case of capillary passage.

The theory is applied for capillary flow as well as to develop hydrodynamic lubrications theory in sliding contact bearing which include the slipper pad and the journal bearing lubrication. Now even if we would like to find out the pressure distribution in a capillary passage between two stationary plate or one is moving and at the entry side and the exist side, there is a pressure difference, the same equation can be used only in that case U will become 0. Let us see how it can be derived.

(Refer Slide Time: 20:12)



Now in this figure, what I have shown, there is one outer member and this is the inner member. Now one is moving over the other not only that, it has some not a simple motion not just rotary motion or not sliding motions, rather we can consider that one plate is moving with respect to other. One is the plate is having rotational motion as well as it is revolving on other plate with on a fixed centre, say this plate is moving with respect to this point, one is the rotation about its own axis, other is that it is revolving about this axis. Now here we can

say that say this is outer member and this is the inner member. If I consider this is fixed then this one is moving like this. It is rotating about its own axis as well as it is revolving around this axis, okay. Say one can develop the equation of motion of one plate with respect to the other.

Now why we have taken such a case, because in case of many fluid power machines and devices, a parallel surface moves with respect to another fixed surface where one side of this capillary passage is exposed to high pressure fluid. Say for example, side plate of a gear pump, uh I think you may not have the idea of gear pump, but in case of gear pump, just imagine two external gears are rotating within a enclosed space and there are two side plates and it is pumping oil from one side to other side. So definitely there is a pressure filled inside the cavity inside the space and then there will be leakage through the side plates that can be analyzed with the theory what we are going to study now.

(Refer Slide Time: 22:52)

Leakage flow through two parallel plates (Contd....)

Flow in such capillary passages occur both due to the hydrostatic and the hydrodynamic pressure gradients, although in many cases hydrostatic pressure dominates, in case of Fluid Power System.



Referring to Fig. 2.5_4 the equation for pressure distribution can be derived as follows.

The equation 2.5_5 can be rearranged and written for two dimension flow for r, θ (Cylindrical Coordinate system) directions as,

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{\mu} \frac{\partial p}{\partial r} \quad \dots 2.5_6$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{\mu r} \frac{\partial p}{\partial \theta} \quad \dots 2.5_7$$

Where, u = fluid velocity in r direction, &
 v = fluid velocity in θ direction.

Flow in such capillary passages occur both due to hydrostatic and the hydrodynamic pressure gradients, although in many cases hydrostatic pressure dominates, in case of fluid power system. This I have explained a little, but I would always say usually in case of hydrostatic machines, in many occasion you will find that between in the capillary passage one side there is a high pressure, in other side there is a low pressure, for example the case we have considered in that case also it is like that. So referring to this earlier figure which I have shown.

The equation of pressure distribution can be derived as follows. Let us consider the earlier equation which we have derived can be rearranged and written in two dimensional form as follows. So $\frac{\partial^2 u}{\partial z^2}$ is equal to $\frac{1}{\mu} \frac{\partial p}{\partial r}$, okay. Now here we have consider the cylindrical co-ordinate system that means there is a from a centre, there is a r direction and theta directions and z direction is the opposite (24:28) towards the thickness and u is the velocity along r directions and v is the velocity along theta directions. So this can be derived as $\frac{\partial^2 v}{\partial z^2} = \frac{1}{\mu r} \frac{\partial p}{\partial \theta}$. For u we can derive this and for v we derive this. This the same equation which we develop for x directions. Now here applying the same theory for r and theta directions okay. Where, u the fluid velocity in r direction, v is the fluid velocity in theta direction.

(Refer Slide Time: 25:15)

Leakage flow through two parallel plates (Contd....)

And from the bulk continuity,

$$\int_0^h \frac{\partial}{\partial r} (ru) \delta z + \int_0^h \frac{\partial v}{\partial \theta} \delta z = 0 \quad \dots 2.5_8$$



To solve these three equations simultaneously, integrating twice eqn. 2.5_6,

$$u = \frac{1}{2\mu} \left(\frac{\partial p}{\partial r} \right) z^2 + k_1 z + k_2 \quad \dots 2.5_9$$

Similarly, integrating twice eqn. 2.5_7,

$$v = \frac{1}{2\mu r} \left(\frac{\partial p}{\partial \theta} \right) z^2 + k_3 z + k_4 \quad \dots 2.5_10$$

Where, $k_1, k_2, k_3, \& k_4$ are integration constants.

And from the bulk continuity, in cylindrical co-ordinate system the valve continuity is writturn in this form, r is the general radius. So and h is the gap the limits from 0 to h is the limit. So $\frac{\partial}{\partial r} (ru) \delta z$ plus integration over 0 to h $\frac{\partial v}{\partial \theta} \delta z$ is equal to 0. These are the standard equation which we have adopted here. What is this continuity bulk continuity? This means that the in case of incompressible fluid the amount of fluid entering to the capillary passage, it will go out at the same amount by the same amount; there will be no conservations of mass inside the capillary passage. So that equation can be written in this form. That means accumulation of fluid inside the capillary passage is equal to 0. So we developed the pressure verses velocity equations and this bulk continuity equations.

Now to solve these 3 equations simultaneously we integrate twice the equation 2.5.76 at fast. So we get we have integrated that $\frac{\partial^2 u}{\partial z^2}$ and we have got these equations. This

integration is not difficult you can carry out and you will arrive into this equations and similarly integrating 2.57 we get $V I$ equal to 1 by $2\mu r \frac{\partial p}{\partial \theta} z^2$ plus $K_3 z$ plus K_4 where, k_1, k_2, k_3, k_4 are integration constants. So again repeat, so we have got earlier equations in terms of $\frac{\partial^2 u}{\partial z^2}$ and $\frac{\partial^2 v}{\partial z^2}$ and from that equation we have carried out this integration twice and we have got these two equations.

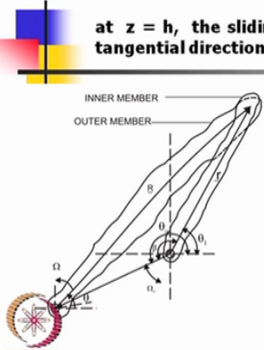
(Refer Slide Time: 27:51)

Leakage flow through two parallel plates (Contd....)

Referring to **Fig.2.5-4**
Now the boundary conditions are,

at $z = 0$, both $u = 0, v = 0$, (i)

at $z = h$, the sliding velocities of an element in radial and tangential directions are derived as,



$$\begin{aligned} u &= (\Omega - \Omega_o) C_o \sin \beta \\ v &= (\Omega - \Omega_o) C_o \cos \beta - \Omega r \end{aligned} \quad (ii)$$

Substituting these boundary conditions in eqn. ... 2.5_9,

$$k_2 = 0$$

$$k_1 = \frac{1}{h} (\Omega - \Omega_o) C_o \sin \beta - \frac{h}{2\mu} \left(\frac{\partial p}{\partial r} \right)$$

10

Now again coming into this motion of this plate one over the other. What will be the boundary condition now? At z is equal to 0 in that case we are considering the capillary passage the capillary gap is in the z directions. So at z is equal to 0 both u is equal to 0 and v is equal to 0 no sorry, z is in the direction of the velocity when z is equal to 0 , u is equal to 0 and v is equal to 0 . One moment ya z actually we here instead of y we have consider the z . So at the bottom that means the on the stationary plate z is equal to 0 , u is equal to 0 , v is equal to 0 . Now at z is equal to h you see this is in the east direction, this sliding velocities of an element in radial and tangential directions are derived as, in this case if I consider this point velocity of this point. This we can have sorry, this is I think this figure has come over here this is you, this one is u and this is v , okay. So u is coming ω minus ω_0 into $C_0 \sin \beta$. Here v is equal to ω minus ω_0 to $C_0 \cos \beta$ minus ωr .

Now what is ω ? This is rotating at the speed of ω and this one is rotating at the ω_0 . Now this ω and ω_0 that has a definite relations mind it. This plate we have not describe about the motion of this plate, but this equation you have to accept and what is C_0 ? C_0 is this distance between these two centres, okay. So these equations probably say for

example if you are asked to find out the equation distribution of these two plates, these equations will be given, because you are not deriving these two equations. So this is the boundary condition 2 at z is equal to h , u is equal to this and v is equal to this one. So this is the boundary conditions. Now what we do substituting these boundary conditions in equation 2.5 minus 9 we get directly k_2 is equal to 0. If you look into that equations and if you substitute this boundary conditions, the first boundary condition you will get k_2 is equal to 0 and then from the other one you will get k_1 is equal to 1 by $h \omega_0 C_0 \sin \beta$ minus h by $2 \mu \frac{dp}{dr}$.

(Refer Slide Time: 31:08)

Leakage flow through two parallel plates (Contd....)

Therefore,



$$u = \frac{1}{2\mu} \left(\frac{\partial p}{\partial r} \right) (z^2 - hz) + \frac{z}{h} (\Omega - \Omega_0) C_0 \sin \beta \quad \dots 2.5_{-11}$$

Similarly, from the boundary conditions and eqn. 2.5_10,

$$v = \frac{1}{2\mu r} \left(\frac{\partial p}{\partial \theta} \right) (z^2 - hz) + \frac{z}{h} \{ (\Omega - \Omega_0) C_0 \cos \beta - \Omega r \} \quad \dots 2.5_{-12}$$

Substituting in the continuity equation 2.5_8, we get-

$$\begin{aligned} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) \frac{1}{2\mu} \int_0^h (z^2 - hz) dz + \frac{\partial}{\partial r} \left(r \right) \frac{C_0}{h} (\Omega - \Omega_0) \sin \beta \int_0^h z dz \\ + \frac{\partial}{\partial \theta} \left(\frac{\partial p}{\partial \theta} \right) \frac{1}{2\mu r} \int_0^h (z^2 - hz) dz \\ + \frac{\partial}{\partial \theta} \frac{1}{h} \{ (\Omega - \Omega_0) C_0 \cos \beta - \Omega r \} \int_0^h z dz = 0 \quad \dots 2.5_{-13} \end{aligned}$$

 11 

Then u will become $\frac{1}{2\mu} \frac{dp}{dr} (z^2 - hz) + \frac{z}{h} \omega_0 C_0 \sin \beta$, okay. Similarly from the boundary conditions and equation 2.5-10 the same way we will get this equation for the velocity v in theta direction is equal to $\frac{1}{2\mu r} \frac{dp}{d\theta} (z^2 - hz) + \frac{z}{h} (\omega - \omega_0) C_0 \cos \beta - \omega r$. This derivation will not be difficult, but the $\omega - \omega_0 \sin \beta$ components that we have accepted for the velocity. Substituting in the continuity equation 2.5 minus 8 we get. Now what we are doing now? We are substituting u and v in the continuity equations, then we get this big equation, $\frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) \frac{1}{2\mu} \int_0^h (z^2 - hz) dz + \frac{\partial}{\partial r} (r) \frac{C_0}{h} (\omega - \omega_0) \sin \beta \int_0^h z dz + \frac{\partial}{\partial \theta} \left(\frac{\partial p}{\partial \theta} \right) \frac{1}{2\mu r} \int_0^h (z^2 - hz) dz + \frac{\partial}{\partial \theta} \frac{1}{h} \{ (\omega - \omega_0) C_0 \cos \beta - \omega r \} \int_0^h z dz = 0$. This means that for incompressible fluid we find this whole is equal to 0, okay.

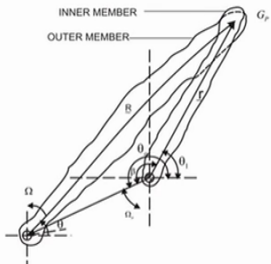
(Refer Slide Time: 33:35)

Leakage flow through two parallel plates (Contd....)

$$\text{or, } \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) \frac{1}{2\mu} \left| \frac{z^3}{3} - \frac{hz^2}{2} \right|_0^h + \frac{C_o}{h} (\Omega - \Omega_o) \sin \beta \left| \frac{z^2}{2} \right|_0^h +$$

$$\left(\frac{\partial^2 p}{\partial \theta^2} \right) \frac{1}{2\mu r} \left| \frac{z^3}{3} - \frac{hz^2}{2} \right|_0^h - \frac{1}{h} \frac{\partial}{\partial \theta} \{ (\Omega - \Omega_o) \cos \beta - \Omega r \} \left| \frac{z^2}{2} \right|_0^h = 0$$

Therefore, the above expression is reduced to,

$$-\frac{h^3}{12\mu} r \frac{\partial^2 p}{\partial r^2} - \frac{h^3}{12\mu} \frac{\partial p}{\partial r} - \frac{h^3}{12\mu r} \frac{\partial^2 p}{\partial \theta^2} + \frac{C_o h}{2} (\Omega - \Omega_o) \sin \beta - \frac{C_o h}{2} (\Omega - \Omega_o) \sin \beta = 0$$


2

Then if we put our limits we integrate and then put our limits we get then this equation, $\frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) \frac{1}{2\mu} \left| \frac{z^3}{3} - \frac{hz^2}{2} \right|_0^h + \frac{C_o h}{2} (\Omega - \Omega_o) \sin \beta \left| \frac{z^2}{2} \right|_0^h + \left(\frac{\partial^2 p}{\partial \theta^2} \right) \frac{1}{2\mu r} \left| \frac{z^3}{3} - \frac{hz^2}{2} \right|_0^h - \frac{1}{h} \frac{\partial}{\partial \theta} \{ (\Omega - \Omega_o) \cos \beta - \Omega r \} \left| \frac{z^2}{2} \right|_0^h = 0$ okay. Now sorry, here one conditions this was what coming over here. If we look into this, this angle is this as a direct relations θ with this angle this angle has direct relations that means with respect to this, this will be constant. The rate of varying this angle with this is same. So therefore substituting this condition here, we find that ultimately this equation will be reduced to this and this part will be $\frac{C_o h}{2} (\Omega - \Omega_o) \sin \beta$ and this will also become a $\sin \beta$ this part, if you put that conditions somehow this is missing here, anyway we will arrived into this equation what we find directly this and this will be cancel out. So this part will become 0.

(Refer Slide Time: 36:03)

Leakage flow through two parallel plates (Contd....)


Simplifying,
$$r \frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial^2 p}{\partial \theta^2} = 0 \quad \dots 2.5_{14}$$

Also written as,
$$\left[\frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 p}{\partial \theta^2} = 0 \right]$$

The final equation of pressure distribution is expressed as,

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} = 0 \quad \dots 2.5_{15}$$

It is independent of the velocity of moving plate. It is solved (mostly numerically) knowing the pressure distribution at boundaries.

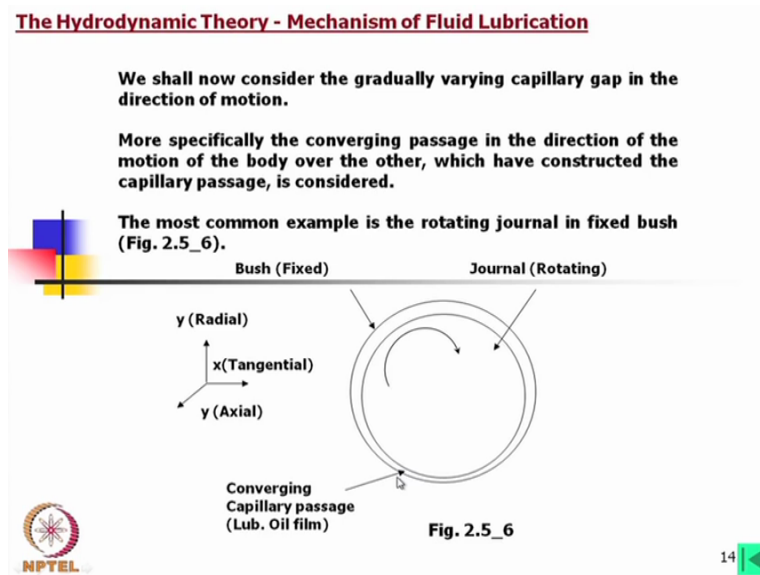


13

So this means that $r \frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial^2 p}{\partial \theta^2}$ is equal to 0, which also can be written as $\frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 p}{\partial \theta^2}$ is equal to 0. In many cases, you will find the equation is written in this form. However, another form we can directly write from here r we cancel out dividing the whole left hand side by r we get this equations. So we will find the equation either in this form or in this form. What is there that it is independent of the velocity of the moving plate? We have consider a fixed plate another plate is moving, we have consider the fluid velocity of the due to this drag we have considered, but while we have arrived into the pressure distribution equation what we find that it is independent of u . This means that if you take a simple case on a plate another plate is rotating simply; the pressure distribution will become the same.

Now the question is that in that case then what is the benefit of developing these equations. These equations we have developed considering the gap is constant, there is no converging field. This means that if the gap is capillary gap is maintained constant then irrespective of the motion of the plates velocity distribution will be in this form, sorry this pressure distribution will be in this form and then whether there will be change in pressure or not that will completely depend on the boundary conditions. If there is no pressure difference so pressure will be equal at each and every point within the plate.

(Refer Slide Time: 38:31)



We shall now consider the gradually varying capillary gap in the direction of motion. So earlier what we have found that is there is no variation in the gap uniform capillary passage. Now we shall consider the capillary gap is varying. More specifically the converging passage in the direction of the motion of the body over the other which have constructed the capillary passage is considered. The most common example is the rotating journal in fixed bush which I have shown here. Now here say this outer one is the bush inner one is the journal and the journal is rotating and outer one is fixed, it may be otherwise also inner one is fixed, outer one is moving. In some cases say for an example planetary gears you will find both are moving but we need only one should have relative rotation with respect to the other.

Now look at this in usually this journal bearing there will be gap that means if we measure the inside diameter of the bush and outside the diameter of the journal you will find that there is a difference. This difference are in the order of micron. So may be 20, 30, 40 microns (()) (40:18) per there for the journal lubrication. Now if you would like to apply the rectangular co-ordinates system what we? We fix that co-ordinate system at the point of analysis then x is the tangential directions, suppose we have consider this point, so tangential direction will be x. It is rotating like this. So may be the the oil is moving like this, so we will consider the in the direction of motion x is the positive and y is along the radial directions that means in the direction of h the film thickness and sorry, this will be again z there is a mistake. The z is axial direction along the shaft.

(Refer Slide Time: 41:25)

The Hydrodynamic Theory - Mechanism of Fluid Lubrication (contd...)

In a capillary passage the one dimension flow equation can be written as (ref eqn. 2.5_5),


$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}$$

Integrating twice w.r.t. y ,

1st. Integration, $\frac{\partial u}{\partial y} = \frac{1}{\mu} \left(\frac{\partial p}{\partial x} \right) y + k_1$

Finally, $u = \frac{1}{2\mu} \left(\frac{\partial p}{\partial x} \right) y^2 + k_1 y + k_2$... 2.5_16

where, k_1 and k_2 are integration constant.



15

So now we are studying hydrodynamic theory, mechanism of fluid lubrication in a capillary passage the one dimension flow equation can be written as $\frac{\partial^2 u}{\partial y^2}$ is equal to $\frac{1}{\mu} \frac{\partial p}{\partial x}$, which we have derived earlier 2.5-5. Now integrating twice with respect to y we get after the first integration $\frac{\partial u}{\partial y}$ is equal to $\frac{1}{\mu} \frac{\partial p}{\partial x} y + k_1$ and after the second integration we get, u is equal to $\frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + k_1 y + k_2$ where, k_1 and k_2 are integration constant. This is same as we have done earlier in case of two flat parallel plates.

(Refer Slide Time: 42:40)

The Hydrodynamic Theory - Mechanism of Fluid Lubrication (contd...)

Applying boundary conditions,

At $y = 0$, $u = 0$... (i)
 & at $y = h$, $u = U$... (ii)


Also, considering only one dimensional flow and using 1st. boundary condition in eqn. 2.5_16, we get, $k_2 = 0$,

Substituting again in eqn. ... 2.5_16

$$U = \frac{1}{2\mu} \left(\frac{dp}{dx} \right) h^2 + k_1 h$$

[Note: We have now onwards used ordinary differential form rather than partial differential form.]

Now using 2nd. boundary condition

$$k_1 = \frac{U}{h} - \frac{h}{2\mu} \left(\frac{dp}{dx} \right)$$


16

Now we apply the boundary conditions. Here the boundary condition is that when y is equal to 0, u is equal to 0 and at y is equal h , u is equal U , U means the velocity of the moving plate

capital U. Also considering only one dimensional flow now for deriving these equations we are considering the flow is only in the direction of the capillary passage in the x directions. So we apply the first boundary conditions we get k2 is equal to 0 and again substituting in equation 2.5-16 we get U is equal to $\frac{1}{2\mu} \frac{dp}{dx} h^2 + K_1 h$.

Now here if you look into this, we have now onward used ordinary differential form rather than the partial differential form, the earlier while we are deriving the equation we use partial differentiation form to get that variation in pressure is might be all directions. So we must use the partial differentiation equation, but here as we are considering only in one directions we will write the equation in this form. Now using second boundary condition we get k1 is equal to $\frac{U}{h} - \frac{1}{2\mu} \frac{dp}{dx} h$.

(Refer Slide Time: 44:30)

The Hydrodynamic Theory - Mechanism of Fluid Lubrication (contd...)

Finally,


$$u = \frac{1}{2\mu} \left(\frac{dp}{dx} \right) y^2 + \left\{ \frac{U}{h} - \frac{1}{2\mu} \left(\frac{dp}{dx} \right) h \right\} y \quad \dots 2.5_{17}$$

The eqn. 2.5_17 can be rewritten as:

$$u = \frac{1}{2\mu} \frac{dp}{dx} (y^2 - hy) + \frac{Uy}{h} \quad \dots 2.5_{18}$$

The first part of the above equation presents pressure induced flow i.e., flow due to pressure difference

and the second part is drag flow.



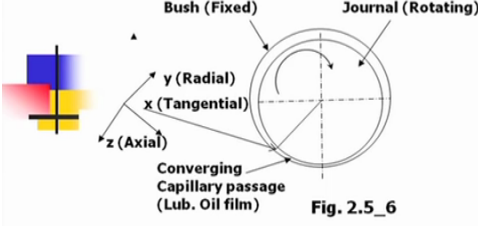
17

So finally we get u is equal to $\frac{1}{2\mu} \frac{dp}{dx} y^2 + \frac{U}{h} y - \frac{1}{2\mu} \frac{dp}{dx} h y$. So we can rearrange this and then we can write this equation in the form $\frac{1}{2\mu} \frac{dp}{dx} y^2 - \frac{1}{2\mu} \frac{dp}{dx} h y + \frac{Uy}{h}$. The first part now look into this equation, this first part of the above equation presents pressure induced flow clearly; this is a pressure induced flow. So this only will exist if there is a pressure difference and the second part is the drag flow.

(Refer Slide Time: 45:31)

The Hydrodynamic Theory - Mechanism of Fluid Lubrication (contd...)


Now applying the continuity in flow (assuming no breakage in film),
for an unit length in z direction, at any section,-



$$Q = \int_0^h u \, dy \quad \dots 2.5_{19}$$

Substituting the expression of u from eqn. ... 2.5_18,

$$Q = \int_0^h \left[\frac{1}{2\mu} \frac{dp}{dx} (y^2 - hy) + \frac{Uy}{h} \right] dy \quad \dots 2.5_{20}$$



18

Drag means one plate the moving plate is dragging the fluid over the other. Now applying the continuity in flow assuming no breakage in film. Now here it is important. This theory cannot be applicable where the oil film is breaking. Now how we can understand this oil film is breaking or not, you will find that for continuity in the fluid flow for continuity in the film or the film will not break, we have to satisfy some conditions so that automatically will come and we will see that whether it is being satisfied, but there are some external region also, say a film is breaking satisfying all hydrodynamic conditions, film may break suppose there is a particle inside the fluid, somehow it has entered or a roughness has developed in a surface that also may cause the breakage, but let us consider there is no breakage, there is no particle inside the film.

So for an unit length in z directions, because if you want to match the dimension you will find that in this equations one dimension is missing. So we have consider that we have taken unit length in z direction. In this case z means the along the axis of the shaft, you can see these directions. So this is along the axis of the shaft and this is the x directions, this is the y direction we have consider and converging capillary passage, it is moving like this, so from here to here where the minimum gap there will be the converging fluid and here the pressure distribution will be there, we will see later, but let us see that what we find now in this case Q is equal to u dy. Now why this equation we have written in this form not in the cylindrical form, because here although tis capillary passage is a circular, but we have considered as if it is a flat one we have consider the developed one capillary passage the same equation we (())

(47:59). Substituting the expression of u from equation 2.5-18, we get Q is equal to integration from 0 to h $\frac{1}{2\mu} \frac{dp}{dx} y^3 - \frac{hy^2}{2} + \frac{Uy^2}{2h}$ by dy .


(Refer Slide Time: 48:22)

The Hydrodynamic Theory - Mechanism of Fluid Lubrication (contd...)

$$= \frac{1}{2\mu} \frac{dp}{dx} \left[\frac{y^3}{3} - \frac{hy^2}{2} + \frac{Uy^2}{2h} \right]_0^h$$

$$= -\frac{1}{12\mu} \frac{dp}{dx} h^3 + \frac{Uh}{2}$$

Finally,

$$Q = \frac{Uh}{2} - \frac{h^3}{12\mu} \frac{dp}{dx} \quad \dots 2.5_{-21}$$


19

Now we integrate and put the limits then we get this equation in this form $\frac{1}{2\mu} \frac{dp}{dx} \left[\frac{y^3}{3} - \frac{hy^2}{2} + \frac{Uy^2}{2h} \right]$. This gives the form like this, $\frac{1}{12\mu} \frac{dp}{dx} h^3 + \frac{Uh}{2}$. So we can rewrite the equation again that Q is equal to $\frac{Uh}{2} - \frac{h^3}{12\mu} \frac{dp}{dx}$. Now here if the U is equal to 0 then this part will become 0 and then this will be only the pressure gradient due to the inlet and outlet pressure, but in case of journal bearing you will be always there that is the peripheral velocity and so Q will take this form.

(Refer Slide Time: 49:41)


The Hydrodynamic Theory - Mechanism of Fluid Lubrication (contd...)

Now for an incompressible fluid (say mineral oil) Q is constant.

Therefore,

$$\frac{dQ}{dx} = 0 \quad \dots 2.5_{22}$$

Substituting eqn. 2.5_22 in eqn. 2.5_21 and then differentiating w.r.t x we get,

$$\frac{U}{2} \frac{dh}{dx} = \frac{d}{dx} \left(\frac{h^3}{12\mu} \frac{dp}{dx} \right) \quad \dots 2.5_{23}$$


20

Now for an incompressible fluid say mineral oil Q is constant. Therefore $\frac{dQ}{dx}$ will be 0 and the substituting equation 2.5-22 in equation 21 and then differentiating with respect to x we get $\frac{U}{2} \frac{dh}{dx} = \frac{d}{dx} \left(\frac{h^3}{12\mu} \frac{dp}{dx} \right)$.

(Refer Slide Time: 50:25)


The Hydrodynamic Theory - Mechanism of Fluid Lubrication (contd...)

Now, considering flow in z direction (leakage flow) also (i.e., two dimensional flow in a capillary passage).
It is to be noted that flow in y direction is negligibly small in such cases and ignored in this derivation.

$$\frac{d}{dx} \left(\frac{h^3}{12\mu} \frac{dp}{dx} \right) + \frac{d}{dz} \left(\frac{h^3}{12\mu} \frac{dp}{dz} \right) = \frac{U}{2} \frac{dh}{dx} \quad \dots 2.5_{24}$$

It is Reynolds' differential equation for two dimensional flow with pressure gradient in a converging film (in the direction of motion) and considering end leakages.

No general solution for pressure p exists.
Case to case it is solved applying boundary conditions.



21

Now, consider the flow in z direction, we have consider along the peripheral direction of the hydrodynamic bearing. Now we are also considering the along the axis. Why it is important, because bearing will have certain length. So we must consider the equation also in the z direction. This means that if there is a pressure gradient in z directions there will be also leakage flow. Now the equation we will take the same form, so this is the full equation form

journal bearing lubrications $\frac{d}{dx} h^3 \frac{dp}{dx} + \frac{d}{dz} h^3 \frac{dp}{dz} = 6\mu U \frac{dh}{dx}$. It is Reynolds differential equation for two dimensional flow with pressure gradient in a converging film. In the direction of motion and considering end leakages or considering end leakages, no general solution for p exist. Case to case it is solved applying boundary conditions.

(Refer Slide Time: 51:49)

Application in journal bearing design

Journal bearing design using 'Sommerfeld number'



A solution of Reynolds equation (No. 2.5_24) with side leakage assumed to be zero, was made by *Sommerfeld*. According to the solution:

$$\frac{R}{C} f = \phi \left[\left(\frac{R}{C} \right)^2 \frac{\mu n}{p} \right] = \phi S \quad \dots 2.5_{25}$$

In which, $S = \left(\frac{R}{C} \right)^2 \frac{\mu n}{p} \quad \dots 2.5_{26}$

The 'Sommerfeld Number', a dimensionless quantity.

Check, $\left(\frac{m}{m} \right)^2 \frac{(N / m^2 \text{ sec}) \times (\text{rev} / \text{sec})}{N / m^2}$


 22 

So now we if we consider the application in journal bearing design. Now Somerfield, he proposed that $\frac{R}{C} f$ is equal to function of $\left(\frac{R}{C} \right)^2 \frac{\mu n}{p}$ that is equal to ϕ into S . this means that this he this quantity he proposed which is defined by S and called as Sommerfield number and it is a dimensionless quantity which can be verified from this. It is a dimensional quantity.

(Refer Slide Time: 52:42)

Journal bearing design using 'Sommerfeld number' (Contd....)

Where, S = Sommerfeld Number or bearing characteristics number,
 R = Journal radius (m),
 C = Radial clearance
[= (Exact diameter of bush - exact diameter of journal) / 2] (m),
 μ = Viscosity (Pa-s),
 n = Rotating speed of journal (rps),
 p = Average pressure on projected bearing area
[Load on a bearing / (length of bush x diameter of bush)
= $F / (L \times D)$ Re. Fig. 2.5_5],
 ϕ = A function,
 f = Coefficient of Friction.



23

Now what are these parameters? Where, S is Sommerfeld number or bearing characteristics number, R is journal radius, C is radial clearance. This is exact diameter of bush minus exact diameter of journal divided by 2 in meter. Then μ is equal to viscosity in pascal second, n is rotational rotating speed of journal in revolution per second, p is the average pressure on projected bearing area which is clearly that load on a bearing divided by length of bush into diameter of bush that is f divided by L into D and ϕ is a function, okay, f is coefficient of friction.

(Refer Slide Time: 53:38)

Journal bearing design using 'Sommerfeld number' (Contd....)

Raimondi and Boyd method of bearing design.

For solutions *Raimondi and Boyd* considered practical and particular cases.

They considered both full (360°) and partial (180°) bearings, and infinite (L/D equal to 4 and above) to short (L/D below $1/4$) bearings.


Also, μ considered to be constant.

They used computer techniques to translate the results of the *hydrodynamic* equation to various operating characteristics for various L/D ratios against a range of Sommerfeld numbers.

Interpolation techniques are to be used for intermediate L/D ratios.

They prepared charts (graphs) for various bearing parameters and characteristics.

Those are adequate to select / design a journal bearing for general purpose applications.

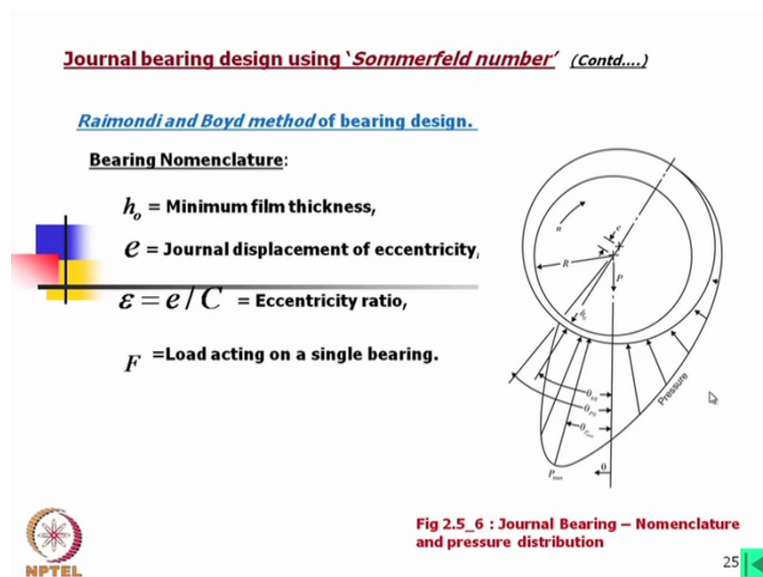


24

Now Sommerfeld, he developed, he proposed that it can be written in this form, but again this is not a closed form solution. So Raimondi and Boyd, the two engineers they developed some

charts out of those using those equations. For solutions Raimondi and Boyd consider practical and particular cases. They consider both full that is 360 degree bearing and partial 180 degree bearings and infinite that is L by D is equal to 4 and above, L is the length of the bush, D is the diameter of the bush, if it is more than 4, it is called infinite L by D below 1/4 is called short bearing and μ he considered is the constant. They considered as a constant. They used computer techniques to translate the results of hydrodynamic equation to various operating characteristics for various L by D ratios against the range of Sommerfeld numbers. Interpolations techniques are used for the intermediate L by D ratios. They prepared charts which is graphs for various bearing parameters and characteristics. Those are adequate to select design a journal bearing for general purpose applications.

(Refer Slide Time: 55:30)




Now in a bearing actually what happens if you look into this, h_o when this is operating, you will find there will be a minimum film thickness, h_o and e is the journal displacement of eccentricity and there will be eccentricity ratio. Now everything will be defined in this form. This means that if we solve a practical case, we have to find that what is the minimum h_o ? How much the eccentricity ratio is coming which will give us a clear idea where this pressure generation will be there or not? So it might be say all gaps everything has same, but due to the speed this is not being generated. Now if it is generated again the question is that how much is the h_o ? Where is the maximum pressure and what is the angle upto which this pressure is there? We are applying the load in this direction. So we have to find out all such angles okay. F is the load acting on single bearing. This is actually F not P , P we have consider for pressure. So this is F .

(Refer Slide Time: 57:11)

Journal bearing design using 'Sommerfeld number' (Contd....)
Raimondi and Boyd method of bearing design.

Along with the described bearing nomenclature, let:

J = Mechanical equivalent of heat (1 Nm/Joule)
 γ = Density of oil (may be taken as 863 kg/m^3)
 C_H = Specific heat of oil = $1755 \text{ Joule/kg}^\circ\text{C}$
 Δt = Temperature rise in $^\circ\text{C}$
 Q = Total flow (drawn) in oil film zone, x direction, m^3/sec
 Q_s = Side leakage flow from oil film, z direction, m^3/sec .

 26

Along with the described bearing nomenclature, there are other factors also. J is the mechanical equivalent of heat and γ is the density of oil, C_H is the specific heat of oil, Δt is the temperature rise in degree centigrade, Q is the total flow in oil film zone, in x directions, Q_s is the side leakage from flow oil film from oil film in z directions that is in the axial directions.

(Refer Slide Time: 57:53)

Journal bearing design using 'Sommerfeld number' (Contd....)
Raimondi and Boyd method of bearing design.


Referring to the bearing nomenclatures as in Fig. 2.5_6, Charts, prepared by Raimondi and Boyd, are listed below.

Graph/Chart:

[i] S vs $\frac{R}{C}f$ (Dimensionless)	[v] S vs $\frac{Q_s}{Q}$ (Dimensionless)
[ii] S vs $\frac{h_o}{C}$ (Dimensionless)	[vi] S vs $\frac{J\gamma C_H \Delta t}{P}$ (Dimensionless)
[iii] S vs θ_{h_o} (deg)	[vii] S vs $\frac{P}{P_{\max}}$ (Dimensionless)
[iv] S vs $\frac{Q}{RCnL}$ (Dimensionless)	[viii] S vs $\theta_{p_{\max}}$ (deg)
	[ix] S vs θ_{p_o} (deg)

Using above charts, general purpose bush bearings and lubrication systems can easily be designed / selected. Technique will be discussed in another lecture through a practical problem.

Ref: P. H. Black & O. E. Adams, Jr. Machine Design (3rd edn.), McGraw-Hill Book & Kogakusha Co. Ltd., 1968.

 27

And you will find that there are chart available for S is a Sommerfeld number. This is dimensionless, R by C into f , R is the radius, C is the clearance and f . So you will get this chart from here we can calculate, f because R and C are known factor. S verses h_0 by C that h_0 is the gap at 0 position and that means directly towards the in the direction of load, then

we will also find $S \theta h_0$ that means what is the angle of h_0 no sorry, h_0 is the minimum thickness minimum film thickness, S verses Q by $RCnL$. So we can have this if we have this L then or directly from there we can find out what should be the L or Q .

Now also we find Q_s by Q , so Q is the flow in the direction of the periphery whereas, Q_s is the side leakage. So knowing this amount we can calculate this amount also. Then we can find out the what will be the temperature rise inside the fluid okay with from this channel and S verses P by P_{max} . What is the pressure, pressure P is the pressure in the projected area, load divided by the projected area of the bearing that is L into D and what is the maximum pressure we can find out from that ratio and then we find out at what angle this P_{max} occur.

Now also we can find the θP_0 where is the P_0 is the pressure at h_0 . Now once we find this then we can verify whether our design is satisfactory or not. Also, we can think of the how much oil is being dragging in and what may be the temperature rise is there and whether we should use any cooling arrangement or we should use more oil there. If the length is satisfactory or not. So basically we can select a bearing and bush bearing and then we can verify all this. If we find these are satisfactory then we can use that bearing for our design. This is general purpose, but for very precision bearing we have to calculate many other parameters also. Now this charts are available in this books however, in the next lecture I will show you some charts and as well we will calculate some parameters. Thank you.