Computational Fluid Dynamics

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Lecture No. # 08

Approximate Solutions of Differential Equations: Variational Principles and Weighted Residual Approach

We continue with our discussions that we started with in our previous lecture. That, we are now going for a variational formulation for solution of the differential equations.

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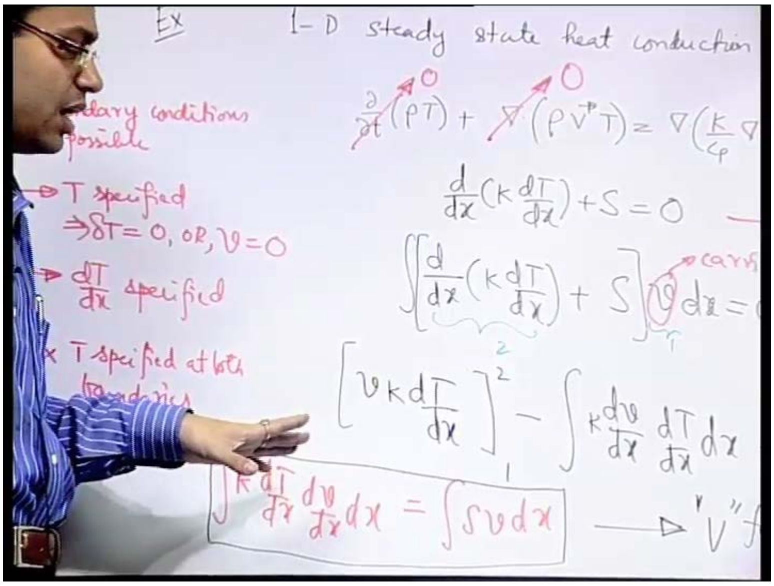

To do that, we have taken up a prototype example, where the example is a one-dimensional steady state heat conduction problem. So, in the one-dimensional steady state heat conduction problem, we have come up with this particular simplified form with the constant source term. Then, we have multiplied the equation with a variational parameter v and integrated it over the domain. And then, we have considered a special type of boundary condition that, the temperature is specified at both the boundaries, let us say at x equal to 0 and x equal to l, and based on that we have come up with this particular form. So, this particular form is also known as the V form - variational form or weak form. The D form is often called as a strong form. If there is something which is weak, there must be something else which is strong. So, the V form is the weak form and D form is the strong form. So, where from the strength of the weakness comes.

If you look into the differential form or the D form, it requires the continuity up to second order derivative of the variable T, whereas in the V form, it requires the continuity up to the first order derivative. So, the weak form or the V form has weakened the requirement of continuity of the highest order derivative, and that is why the weak form. So, it is as if the formulation has weakened the requirement of highest order continuity of the highest order derivative. Hence the name weak found.

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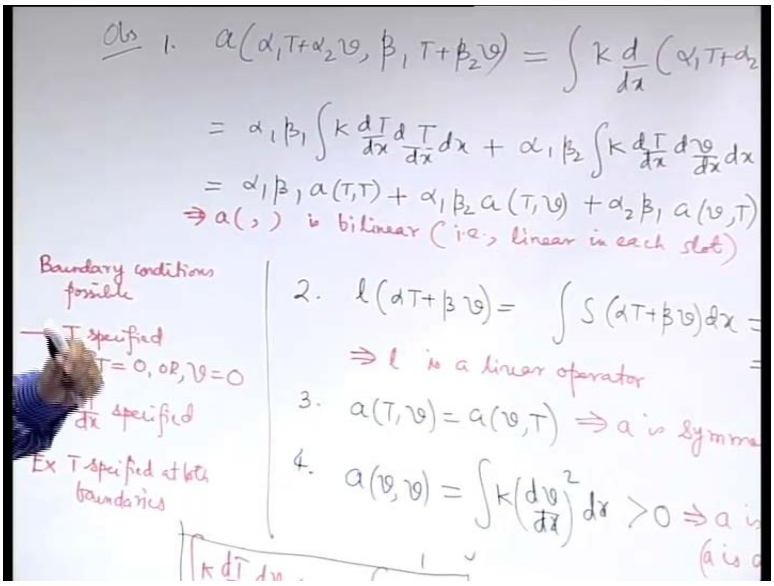

Now, this weak form has certain interesting property. So, let us make some observations out of the properties of the V form. So, the V form we generically write in this way a(T,v) is equal to l(v) where a is a bilinear operator, we will see what is the bilinear operator in this case. So, it is a operator involving two functions and l is a linear operator which involves only valve function. Why linear, why bilinear or what are the implications of this, let us make certain observations. Let us find out…

This a is a function of two parameters that is what we first understand. Now, each of these parameters we make some change. We replace T with alpha 1 T plus alpha 2 v and v with beta 1 T plus beta 2 v. Let us see that what we get out of that through this example. So, in here we replace k d dx in place of T alpha 1 T plus alpha 2v in place of dv dx.

So, this you can write alpha 1 beta 1 integral k dT dx dv dx plus sorry dT dx dT dx. So, this term with this term, remember alpha 1, alpha 2, beta 1, beta 2 all these are constants; plus alpha 1 beta 2 integral of k dT dx dv dx dx plus alpha 2 beta 1 integral of k dv dx dT dx dx plus alpha 2 beta 2 integral of k dv dx dv dx dx. Now, you can write this as alpha 1 beta 1 a(T,T) plus alpha 1 beta 2 a(T,v) plus alpha 2 beta 1 a(v,T) plus alpha 2 beta 2 a(v,v). So, you can see that it is at operator where it is linearly operating on each slot. So, it is like alpha 1 beta 1 into the first two that is a(T,T) then plus alpha 1 beta 2 into a(T,v) plus alpha 2 beta 1 into a(v,T) plus alpha 2 beta 2 a(v,v). So, it is as if an operator where it is linearly operating on each of the slots. And that is why it is called as a bilinear operator. So, it is a bilinear operator which means it is linear in each slot. So, symbolically a bilinear operator may be written in this way. There are two arguments and if it is linear in each argument that is linear in each slot we call it a bilinear operator. So, this is the first observation. Let us make the second observation.

What is the difference between a(T,v) and a(v,T)?

a a(T,v) means in you in place of T it is T, in place of v it is v; a(v,T) means in place of T it is v and v it is T, here of course they are the same. So that that is what we call as symmetric that we will come next. But there are many operators which need not be symmetry. So, the second observation, l alpha T plus beta v...

This is nothing but alpha l(T) plus beta l(v). So, if such a property is satisfied then the operator l is called as a linear operator. So, l is a linear operator.

Third you can easily observe that a(T,v) is equal to a(v,T) from this example. If you swap T and v it makes no difference. So that means a is a symmetric operator - a is symmetric or sometimes it is called as self adjoint. The fourth observation is let us find out what is a(v,v). That is integral of k dv dx whole square dx; k is a property which is positive - thermal conductivity of the material is positive, and dv dx whole square no matter whatever is dv dx it is positive. So, its integral over the domain is positive. So, this is greater than 0, this is called as this implies that a is a positive definite operator or in another technical term it is called a is a scalar product on V a is a scalar product on V. V is the space in which you have this variations small v. So, these are some interesting properties and we will make use of these interesting properties. So, we have still now seen two forms; one is the D form or the differential equation form or the strong form, the V form or the variational form or the weak form.

Now, we will subsequently see a third form which we call as M form or a minimization form. And we will see that how we can come up with that form starting from either the M form or starting from the V form or even the D form. So, to do that we will utilize some of these properties and let us has an example, consider a function with which we will starts utilizing this properties that we have just now derived.

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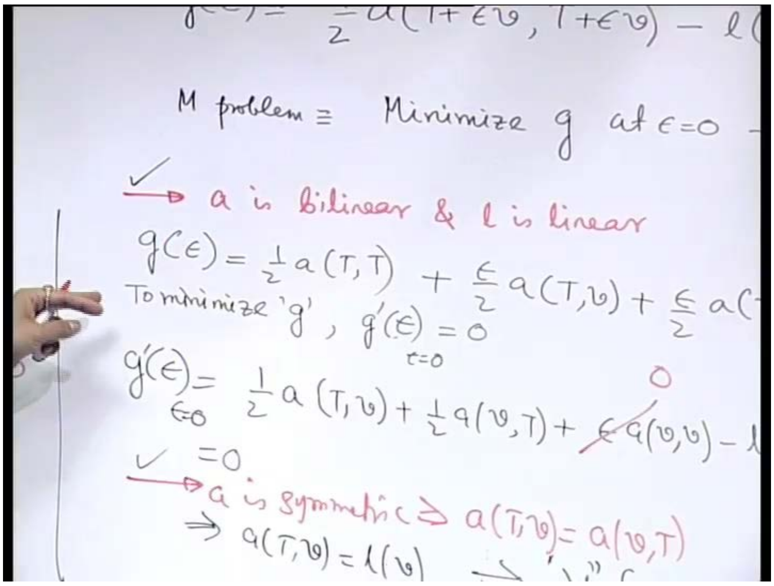

So, let us say that we have a function g of epsilon which is given as follows. And we defined a minimization problem, say that the M problem or the minimization problem is equivalent to minimize g at epsilon equal to 0; that is minimize half a(T,T) minus l(T). Let us see whether we can arrive at this M problem from the V problem or if we assume that the M problem is true then can we arrive at the V problem starting from the M problem. So, what are the assumptions that we will make? The first assumption that we make is that a is a bilinear operator. We have seen by this time through an example that what is a bilinear operator? So, g epsilon is half a(T,T) plus epsilon by 2 a(T,v) plus epsilon by 2 a(v,T) plus epsilon square by 2 a(v,v) minus l(T) minus epsilon l(v). So, we have assumed a is bilinear and l is linear.

To minimize g we must have g dash epsilon is equal to 0, and since we are minimising at epsilon equal to 0 we must substitute g dash epsilon at epsilon equal to 0, equal to 0. Remember epsilon equal to 0 is not first substituted; obviously, you first find out g dash epsilon and then substitute epsilon equal to 0, but because at epsilon equal to 0 your minimization condition is satisfied.

Now, g dash epsilon, what is g dash epsilon? Half a(T,v) plus half a(v,T) plus epsilon a(v,v) minus l(v). So, g dash epsilon at epsilon equal to 0 will have this term go away. At epsilon equal to 0, you can substitute now epsilon equal to 0. You can see that from this we are disparate to get back the V form. How can we get the V form? If a T comma v was same a v comma T then this would have been... So, for minimisation of these this is equal to 0. So that would have given us a(T,v) equal to l(v), provided a(T,v) equal to a(v,T). That means a is a symmetric operator. So, a is symmetric implies a(T,v) equal to a(v,T) which implies a(T,v) equal to l(v) which is the V form.

So, what is our conclusion? Conclusion is M form will imply V form provided, what are the things which are provided? a is bilinear, l is linear and a is symmetric. These are the conditions which are mentioned. So, that means starting from the M form you could arrive at the V form. Question is - could you do it the other way? That is starting from V from could you arrive at the M form. So, let us let us try to do that; let us try to see whether that is possible.

So, to do that we will start with this g epsilon.

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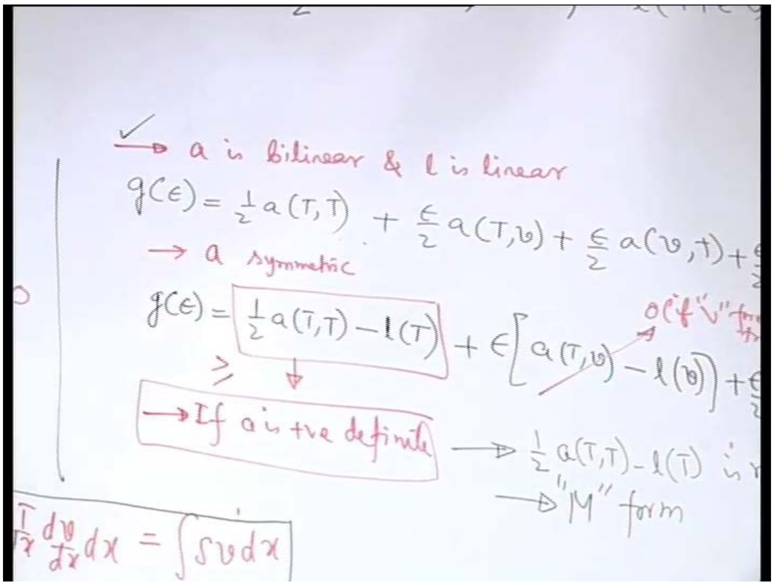

So, now we have utilised what conditions that a is bilinear, l is linear and a is symmetric. So, we considered a as symmetric that means you can write g epsilon is equal to half a(T,T) minus l(T) plus epsilon a(T,v) minus l(v) plus epsilon square by 2 a(v,v). Now, if we assume that the V form is true then this is equal to 0 - if V form is true. So, if V form is true this term is 0, then we can say that g epsilon is always greater than or equal to this term in the box provided a v comma v is positive. When that is true P? When a is a positive definite operator. So, we have just now seen that what is the meaning of that one. So, if a is positive definite, then only this is true. So, g epsilon is greater than equal to this one provided a is positive definite. That means this term in the box is the minimum of g right. So, this implies that under this condition you have half a(T,T) minus l(T) is minimum of g epsilon at epsilon equal to 0, which is nothing but the M form.

So, you can see that we can do it in the other way; we can start with the V form and come to the M form, but it requires an additional constrain that it requires a to be positive definite. You could start with the M form and come to the V form without requiring a to be positive definite. So, this additional constraint is necessary to convert the V form into that equivalent M form. So, with this understanding, let us see that how these forms are interchangeable? Let us let us take this example, whatever example that we were considering that in this example let us see how we can swap different forms.

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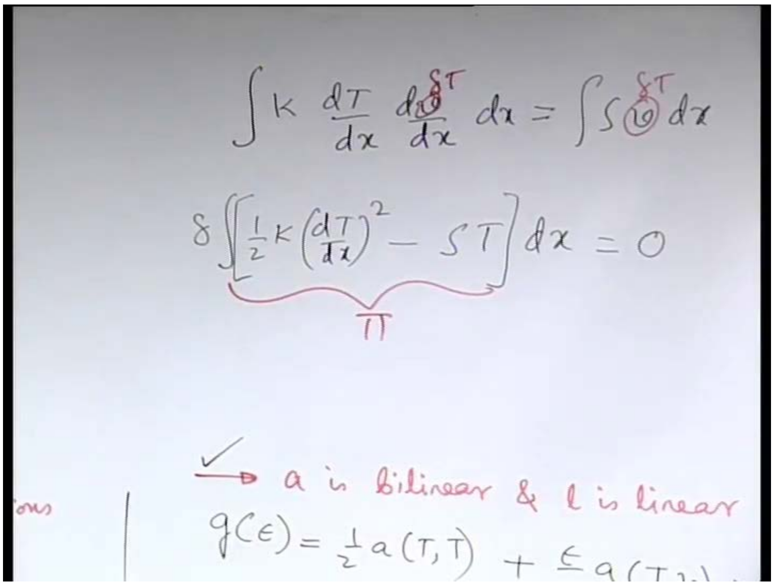

So, you have k dT dx dv dx dx is equal to integral of S v dx. Here in place of v if you substitute delta T. So, you can write. So, this is like the function which we we can give a name. This is like, we just give it a name capital pi, this is the capital pi which you are minimizing. So, del of that is 0. So, this is like half K(d,T) minus ST. So, we can see that we can swap one form from one form to the other form, and in this example this interchangeable is possible in both interchangeablity is possible in both directions, because it is symmetric as well as positive definite - a is both symmetric and positive definite.

Now, let us pay some attention on the D form or the differential form. So, could you could you get back the differential form from here, how could you get back the differential form from here? So, we have seen that from V form you are getting the M form. How could you get back the D form? So, if you see that the the V form is given to you, so remember you could you derive that V form from the D form itself by integrating by parts. So, here you have to revert it back. So, what you can do? You can again integrate it by parts, now you want to reduce the order of the derivate of the v term.

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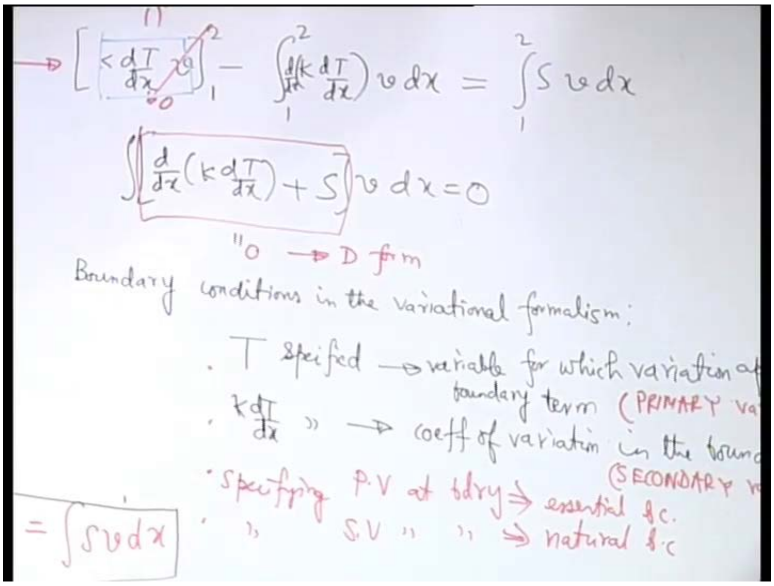

So, let us take this as an example. So, which one will you take as first function and which one you take as a second function. So, this is the first function and this is the second function. So, with that if you integrate, then, first function into integral of the second minus integral of the derivative of first into integral of the second...

With the boundary conditions that we have considered that v is 0 at both the boundaries that is T is specified then this term will become 0. So, you will be left with integral of d dx of k dT dx plus S v dx equal to 0. Since v is arbitrary that means that you have this equal to 0, which is your D form. So, the from the V form you could get back the D form or you could get the M form also, provided all the underlying conditions are fulfilled.

Now, let us pay some attention to the boundary conditions. We have actually taken a very simplified form of the boundary condition. What simplified form we have taken? We have taken that T is specified, but T need not be specified, the dT dx itself could be specified. So, in general, in a variational formulation you could treat these two different cases in a different way; so, one of the possibilities, so boundary conditions in the variational formalism.

You could have T specified, T specified means what? The variable the variation of which appears in the boundary term is specified. The variation of T appears in the boundary term when that variable is specified for which the variation appears in the boundary term. So, T is specified more formalities the variable for which variation appears in the boundary term, because v is variation of T. And k dT dx specified, it means it is the coefficient of variation term, coefficient of variation in the boundary term.

So, if you look at the boundary term carefully, v is the variation so its coefficient is k dT dx. So, specifying k dT dx is actually specifying the coefficient of variation in the boundary term. So, these are the two types of variables; the first type of variable this is called as the primary variable and the second type of variable is called as a secondary variable. Specifying the primary variable at the boundary is called as essential boundary condition. So, specifying primary variable at boundary, it is called as essential boundary condition and specifying secondary variable at boundary is called as natural boundary condition. Essential boundary conditions these are English words which are literary relevant in this context. So, essential boundary condition means this is it is it has to be essentially satisfied. So, T is specified that means the variation in T has to be 0. So, that is essential.

The other one is natural, because for that you do not have to give any extra effort that term automatically appears in your boundary term through the formulation procedure, like k dT dx, you can recall from the physical understanding that k dT dx is minus of the heat flux. So, specifying the heat flux it essentially means that you you are not having to specify it by any (( )). You are not having to specify it by any separate mechanism. The term k dT dx automatically appears in the boundary expression. So that you can substitute it there so substitute it there. So, that is called as a natural boundary condition. So, we have got an understanding of what could be the different types of variables and the corresponding boundary conditions in a variational formalism. Now, can we extend these concepts to equations which involve high order derivatives? Let us let us take an example.

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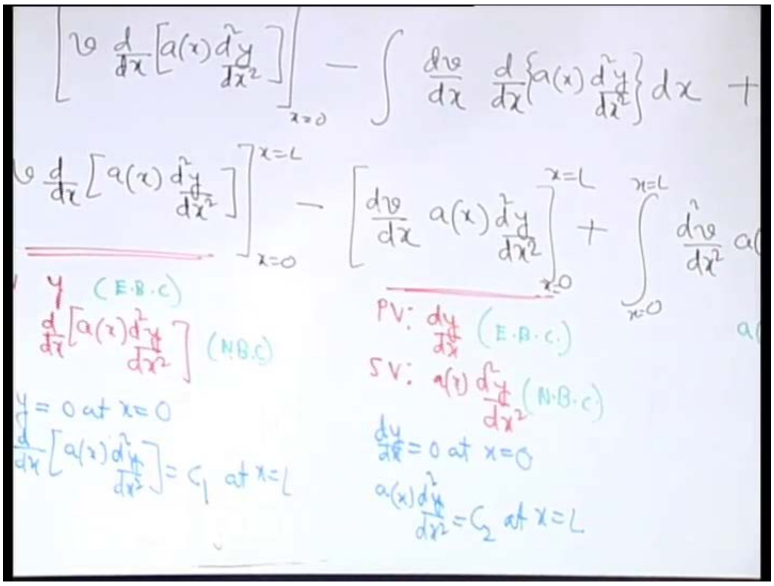

Let us say we have this differential equation. This is the fourth order differential equation that we can see, and this is many times a prototype of a very important problem in structural mechanics that is deflection of beams. So, if you want to find out the displacement, you can get similar prototype differential equations. Our objective is not to go in to such a physical problem, but see that how we can cast this equation in a similar formulation. So, what we are going to do here, we will see that how we can get a variational formulation out of it. What could be the types of boundary conditions? Let us be open and see that what could be the possible boundary conditions. So, what is the first step to get the V form from the D form? We have to multiply the equation with V and integrate over the domain. So, this is the key step to get the V form. So, we will integrate it by multiplying with v over the domain.

Next what we will do? We will integrate it by parts. So, let us just do it quickly, because we know that how how this needs to be done. So, v is the first function and the function appearing here is the second function. So, first function into integral of the second. Let us say that x equal to 0 and x equal to L at the bounce of x minus integral of derivative of first into integral of the second.

What will that what is the next step? See we will stop to the limit where will see that the orders of the derivatives in the variational term and the original term are the same. Then it will automatically become a symmetric term, the operator will be become a symmetric operator. So, we will integrate it once more by parts with dv dx as the first function and remaining as the second function. So, let us do that. Now, let us try to examine the boundary conditions and the nature of these operators. So, let us try to examine...

This will be positive right. Now, let us examine the boundary terms. In this boundary term referring to this boundary term, you tell me what is the primary variable? What is the secondary variable? Primary variable - primary variable is the variable on which the variation is taken in the boundary term y y, because v is variation in y. What is the secondary variable? The coefficient of the variation term, so whatever else remains other than v; so, d dx of this one. There is also another boundary term - this one. So, here what is the primary variable? dy dx, because the variation is taken, v is what? Delta y, so it is delta of dy dx, because v is delta y and d and delta are exchangeable. So, it is delta of dy dx. So, primary variable is dy dx. So, it is delta of what that what is the primary variable; secondary variable a d 2 y dx 2. So, what have could be the possible boundary condition that you could specify what could be your possible essential boundary conditions. So, either y specified or dy dx specified, so this could be essential boundary condition, this could be essential boundary condition.

So, you can get one important understanding that specifying the derivative is not necessarily natural boundary condition. That was a second order derivative problem. This is a fourth order differential equation problem. And specifying this is a natural boundary condition. So, these are specifications of these are natural boundary conditions. So, in the boundary either you have to specify the essential boundary condition or you have to specify the natural boundary condition.

Now, once these are specified, the remaining terms you have a y comma v as this one and l of v as this term. Obviously, when you substitute the boundary conditions here, the boundary conditions are not such that they are always 0. So, there will be term some term that will remain after substituting the boundary condition. For example, let us say that at x is equal to L you have, d dx of a d 2 y dx 2 is equal to some value L in some units. So, it will become v at x equal to L. So, that is not 0, because you cannot simultaneously specify v and this one. So, v in the boundary will remain. So, where will that go? That you can club up with this L term. So, the L term may contain this plus some boundary terms depending on what what are the non-zero terms in the boundary. So that that should be clear.

So, it is not that these terms are together giving rise to 0. Some non-zero terms may they are depending on what are the boundary conditions specified. And those specified values of the boundary conditions, let us let us let us take an example; let us say that let y is equal to 0 at x equal to 0, d dx of a d 2 y dx 2 is equal to c at x equal to L, dy dx equal to 0 at x equal to 0 and a d 2 y dx 2 let us call it c 1 and let us call it c 2 at x equal to l. Then let us write the corresponding terms. So, in the boundary term you have at x equal to 0 this term will be 0. Even if y was non-zero, but some specified that means v is zero.

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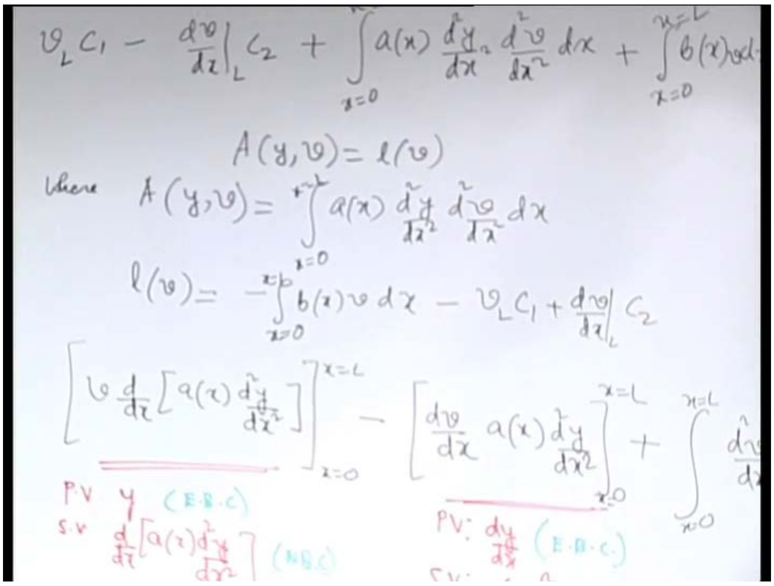

So, here it will be minus sorry this is equal to v at L into c 1 this minus zero. So, that is the boundary term that we get from here. Then minus at x equal to l it is c 2. So, c 2 into dv dx at l, then the remaining term will be 0, because dy dx is specified, so dv dx is 0. Plus a d 2 y dx 2 d 2 v dx 2 dx.

So, this you can write in a form a(y,v) equal to l(v) where a(y,v) may be let us use a different symbol say capital A because we have already used one small a in the problem, where what is capital A(y,v)? It is integral of a d 2 y dx 2 d 2 v dx 2 dx x equal to 0 x equal to l. What is l(v)? Whatever you take it on other side. So, minus integral of b v dx minus v L c 1 plus dv dx L c 2; this is a linear operator on v. So, the objective of going through this example was to illustrate that even some terms from the boundary they do remain; they also can be clubbed and put in the general form. So, here also you can see that capital A is symmetric and it is positive definite; capital A is a bilinear operator and l is a linear operator. So, with these considerations you can also convert this problem into a corresponding M form, and that I leave on you as an exercise, it is very simple, one more step can lead it to M form.

Now, we have seen that how to make a variational formulation of a differential problem differential equation problem, and not only that how can how can that problem may be converted into an equivalent minimization problem. The question is that, once we have formulated these types of problems we were intending for some solution, but never we hinted that how should we get the solution. We we were abstracting our self from the solution, we were not putting so much of attention on how to get the approximate solution, but the formulations which can perhaps lead us to the approximate solution. Now, if these formulations or the concepts of these formulations are to be implemented in practice, then there are certain mechanisms by which we can do it. So, those mechanisms we learn one by one.

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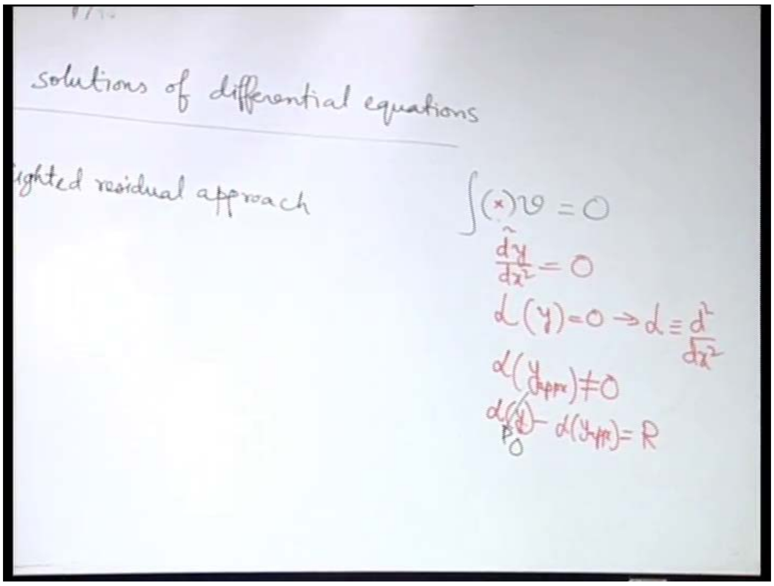

So, the objective of the next part of our study will be to learn the art of approximate solution of differential equations through the principles that we have just learnt. So, when we say approximate solutions of differential equations, the situation is that we can have various techniques really a lots, a lots of techniques for using which we can get approximate solutions. We will try to cover as many techniques pertinent to CFD as possible in this particular course, but we will start with those techniques which directly follow from our previous discussion. Because that gives us a general collection of methods, so to say with a name weighted residual approach. This approach gets a clue from the variational formulation. What is the clue? In the variational formulation, we are multiplying that differential equation with a weighting parameter which is nothing but the variational in the in the variable itself. So, v we are multiplying the differential equation with v and integrating over the domain and setting it to 0. Only thing is that this v is an arbitrarily small variation. Taking clue from that the weighted residual approach, it was defined it was introduced in this way.

So, here in the differential equation form you have an operator which is may be a first order derivative operator, second order derivative operator, whatever, a differential operator. Now, the differential operator contains a dependent variable like say y, which you want to solve as a function of x. So, let us take an example; let us say that you want to solve d 2 y dx 2 equal to 0, one of the simplest possible equations. So, we call it a different linear differential equation where L(y) equal to 0, where L is the linear operator d 2 dx 2 just a notation. Now, we could get a variational formulation of this one very easily by multiplying this with v and integrating it over the domain; that is very simple. Question will remain that - what is that v in a non-abstract sense? In an abstract sense, of course we know that it is the variation in y, true, but if you want to implement it in practice, if you want to use a function in place in place of, this after all in the function place it is the function; if you want to implement a function in in place of this one what could you do?

So, the relaxation to the abstract understanding is that what we could do perhaps is to use some special function in place of this v, which could achieve the same purpose as that of this original introduction original meaning of v in the introduction. Then the question remains that - what will be this y? You want to solve for this y; this y you do not know. So, you did some approximation to this y which you want to substitute in this integral. Otherwise this will remain a differential equation. We want to convert it into some non differential form, may be an algebraic form. So, if you want to convert it to an algebraic form and why do you want to convert it to an algebraic form, because we know that how to easily solve algebraic equations. And in fact, one can use numerical tools for solving algebraic equations. So, to get an algebraic form what we will do is we can substitute y as an approximate polynomial. So, if you substitute y as an approximate polynomial then d 2 y dx 2 will be at will be another polynomial which is the approximate polynomial differentiated twice with respect to x. So, that approximate polynomial will be approximate, and because it is approximate d 2 y approximate d y 2 or L(y) approximate will not be equal to 0 right. Because the exact function if it was there, now if you are very lucky or if you have such a great inside that you know what is the final solution and you substitute that as an approximate solution it will be 0 directly. So, that can be valid for only simple cases, but not in complex cases, which you really want to solve numerically.

So, in general if you substitute y approximate in the differential equation it will not be 0. So, when it will not be 0 you have L(y) minus L(y) approximate that is R - residual R which is non-zero. So, this residual is your error. L(y) and L(y) approximate if these two are both equal to 0 then the residual would have been 0, because L(y) itself is 0, your L(y) approximate is the residual. Your objective is to minimize the residual in an integral sense over the domain. So, for that what you do is you multiplied with a weight function and make the weight function in such a way that in an integral sense over the entire domain, the error incurred because of substitution of an approximate y in place of the actual y is minimised. So, that approach in a mathematical formalism is called as weighted residual approach.

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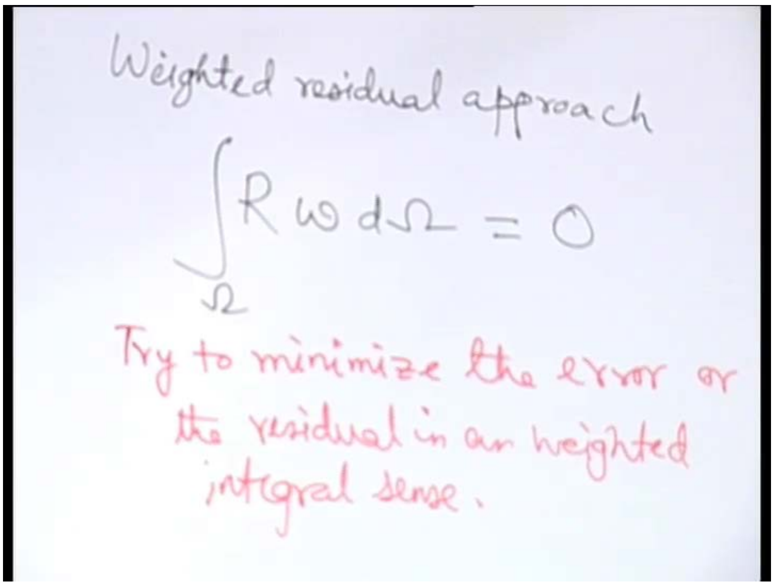

So, what it is basically trying to do? You are writing the residual, you are multiplying the residual with a weighting function w which is a non abstract function which is not like the variation in y, but it is some function, we will later on see in the next class that how to choose these functions and integrated over the domain. So, let us call it d omega where d omega is the elemental part of the domain and set it to 0. So, by this what we try to do? Try to minimize the error or the residual in an weighted integral sense. So, what you are basically doing? You are multiplying it with a weighting function.

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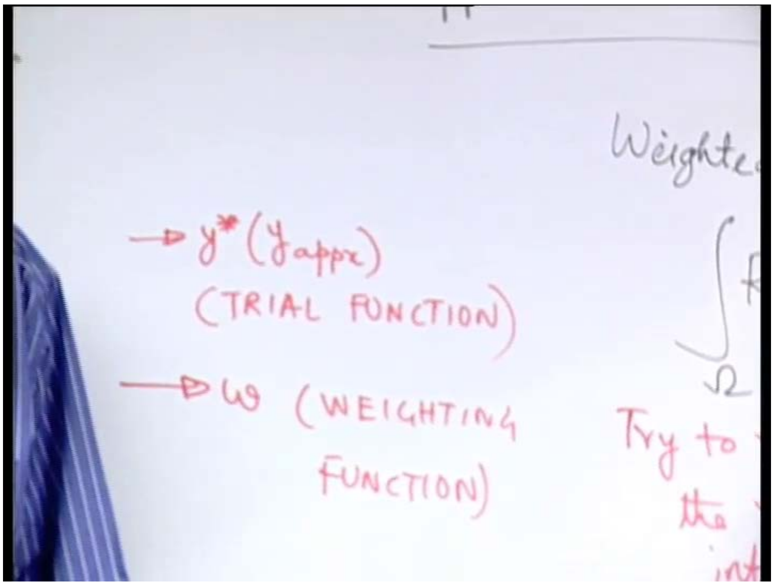

So, you have got now two types of functions; one is the y star or y approximate, this we call as trial function. Why it is a trail function, because you do not know what is y. This is a just as a trial you have substituted. And this trial function may have unknown coefficients which you want to evaluate by imposing this constraint. So, you have another function w which is you call as weight function or weighting function. So, you choose some trial function, you choose some weighting function with an effort that your total error in an integral sense over the entire domain is 0. So, this is the physical meaning of this mathematical equation that you try to you make an attempt to minimize the total error incurred over the entire domain, because of substitution of an approximate function instead of the actual function you have incurred an error, and you are attempting to minimize the total error in an integral sense over the domain, and that you do by setting this integral equal to 0.

So, that is as if your effort is to make a zero error in an integral sense. And for that you required these two functions, one is the trial function another is the weighting function. And this concept of this entire formulation is borrowed from the previous variational formulation, because it is as good as writing the original differential equation with just approximate function substituted in place of the actual function that is one approximation, and the weighting function not the variation, but some arbitrary weighting function. So, these are the two relaxations that we have made. We have not substituted the actual function, we have substituted the approximate function instead, and we have not substituted variation in the variable, but we have substituted at arbitrary weighting function instead.

But otherwise this formulation has a great similarity with the variational formulation, one advantage of this is that here you do not required the functions to be satisfying the special conditions like symmetric, positive definite all those things. Because we will not in general be simplifying this by using integration by parts. So, these functions will require some other special requirements that we will see in the next class. But we can see that these functions are not as rigorous as that of the original functions, original solutions and the variation, and therefore, it will lead us to the approximate solutions of the differential equations. How it can lead us to that? That we will see in the next class. Thank you.