Computational Fluid Dynamics

Prof. Dr. Suman Chakraborty

Department of Mechanical Engineering

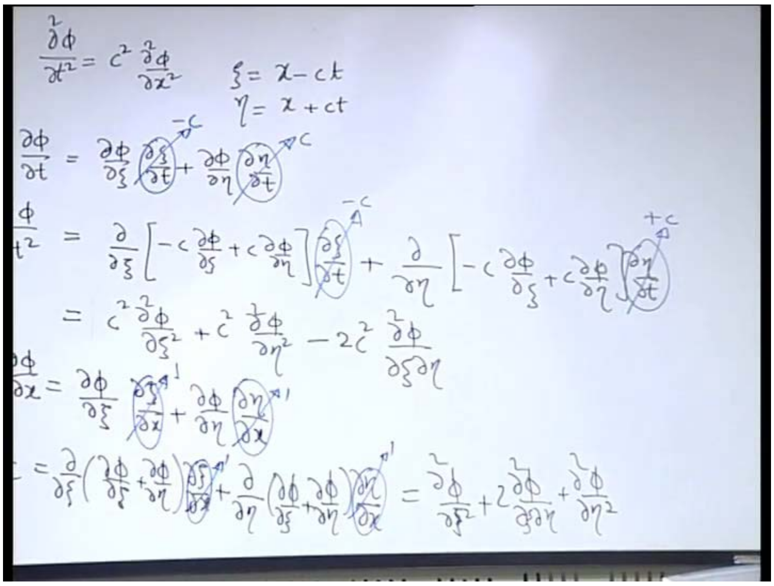
Indian Institute of Technology, Kharagpur

Lecture No. # 06

Classification of Partial Differential Equations and Physical Behaviour (Contd.)

The features of different types of partial differential equations, and the most recent example that we considered is the example of hyperbolic equation. Let us continue with that.

(Refer Slide Time: 00:30)



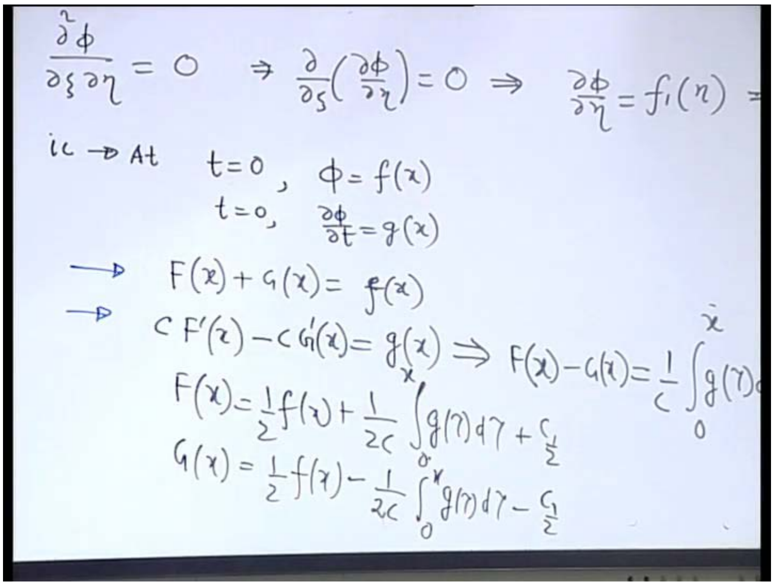
So, we could find out that x minus c t and x plus c t may turn out to be the two characteristic variables corresponding to this partial differential equation. And now, what we will show is that it is possible to write the solution of the partial differential equation solely in terms of this characteristic variable. That is the canonical or general form of the solution, where you write the solution in terms of the, this characteristic variable.

Let us give this some name say zeta and eta. So, let us try to cast the solution in terms of these variables. So, let us do it step by step. First del phi del t, remember phi is a function of x and t which should be map to zeta and eta. So, del phi del zeta into del zeta del t plus del phi del eta into del eta del t. What is del zeta del t that is equal to minus c, and del eta del t is plus c. Next we differentiate this once more with respect to t.

So, we use the same chain rule that we used for the previous term again this is equal to minus c, and this is equal to plus c. So, it becomes c square del square phi del zeta square plus c square del square phi del eta square; no this is eta square minus 2 c square, there are two such terms del square phi del zeta del eta. Similarly, let us find out the derivative with respect to position.

Del phi del x, what is that? This is equal to 1 and this is equal to 1. Second derivative. So, this is also equal to 1 and this is 1, hence it becomes these one. Now, we use the equation and substitute the corresponding derivatives.

(Refer Slide Time: 06:28)



So, what we will get… is equal to… Once we write it in this form you can very easily simplify it, because several terms in both the sides cancel. For example, these two terms cancel these two terms cancel. So, you are left with this one or… So, from here what we can conclude It is not constant it is partial derivative with respect to zeta is equal to 0, that means, it is some function of eta.

So, if you integrate this once then what is phi it is some new function of eta which can be obtained by integrating f 1 eta d eta. So, let us say that is F eta plus some function of zeta which is like a constant when you integrate partially with respect to eta. So, you can see that the general solution for phi can be written solely in terms of the characteristic variables.

So, as if it is a function of a single variable which carries the combined special temporal effect and in terms of those two single variables you are now expressing your solution so, it is not one single variable, but, two single variables, because the solution has the equation has two real characteristics. Now, if you want to get the actual solution this is the form of the solution, but, if you are interested about the solution you must use the initial condition and the boundary condition.

So, may be initial condition at t equal to 0 phi is some f x. It is having a second order time derivative. So, you require another condition with respect to time and that is maybe the initial. This is like the initial displacement of the string, you may also specify the initial velocity of the string. So, at t equal to 0 what about the boundary conditions here you can see that the boundary conditions that is the constraints with respect to x are automatically inbuilt that you can see.

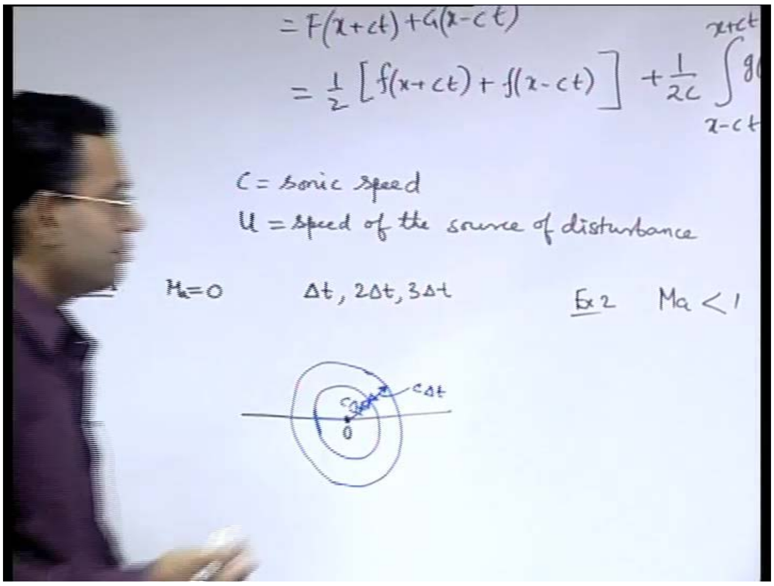
So, you do not have to explicitly separately specify that again and let us see that what is the consequence of these initial conditions, because of course, how they are automatically related x and t are related variables. So, as you have different t you have your domain of influence bounded by different values of x and we will come across such physical examples subsequently, but, let us just for the sake of completeness complete this exercise. So, at t equal to 0 you have this equal to f x.

So, remember eta is what x plus c t. This is x minus c t. So, F x plus G x is equal to f x and derivative with respect to time. So, first you differentiate this with respect to eta. So, d f d eta into del eta del t. So, c into d f d eta at t equal to 0. So, it will become c F dash x because eta is x plus c t. So, at t equal to 0 it will become x remember is this derivative is with respect to the original argument it is not with respect to x then minus c g dash x is equal to G x.

So, from here you will get F x if you integrate it minus G x is equal to someone by c you can just make a change of variable plus you can put some constant of integration. So, from these two equations you can now find out what is capital F x and capital G x. No, because in this limit this is this is not a this is not an actual constant this is a pseudo constant this is a variable. So, we are putting just the variable in the limit this is not a constant limit.

So, if you add these two together you will get F x is equal to small f x plus. So, if you add these two it will be half of small F x plus 1 by 2 c plus c 1 by 2 and what is G x that is obtained by subtracting these two. So, half of F x minus 1 by 2 c minus c 1 by 2 the solution is not F x plus g x, but f eta plus g zeta.

(Refer Slide Time: 15:07)



So, let us write the solution so, that is F x plus c t plus g x minus c t these two integrals in one case x is replaced by x plus c t and another case x minus c t. So, it will become from x minus c t to x plus c t of course, it shows that once you substitute the initial conditions how your solution depends on the combinations of x and t.

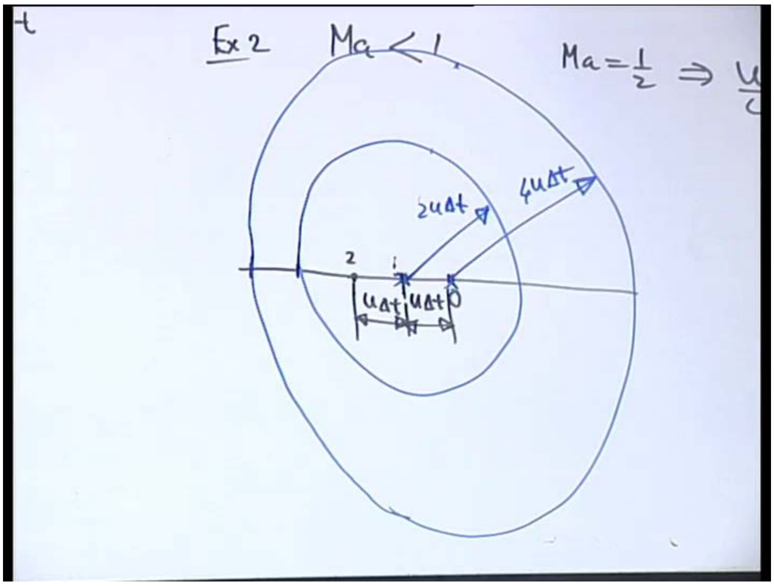
Now, this gives us a physical this gives us a mathematical feel of the solution to this problem, but, it does not give us directly a physical feel of the meaning of these characteristics. So, let us try to have a deeper assessment of the physical implication of the role of the characteristics. To do that we take the example of compressible fluid dynamics.

Let us say that there is a disturbance source which is moving through a fluid medium and the disturbance source is moving through a fluid medium with some speed. So, there is a speed at which the disturbance propagates which is the sonic speed. Let us say c is equal to the sonic speed, this is the speed at which the disturbance propagates and let us say that u is the speed of the source of disturbance or the speed of flow. So, to say if the flow is the source of disturbance we know that this u by c is known as the mach number of the flow.

Let us consider a case, let us consider some examples first example let us consider Mach number equal to 0. So, let us consider that there is a source which is located at 0 it does not move it remains at that place so, let us consider some small time intervals delta t 2 delta t phi delta t like that. So, over a time of delta t the source will emit a signal that will go that will cover how much of a distance c into delta t. And that will essentially cover such a zone, which is a circle with this as a center and c delta t as the radius because the effect is propagated in all directions see what is the basic difference between this and the case of an elliptic equation. In elliptic equation the disturbance is propagating at infinite speed.

Here the disturbance is propagating at a finite speed which is the sonic speed of the characteristic of the medium. So, in a time 2 delta t it will be a radius of c into 2 delta t. So, this is this much of additional is c into delta t. So, in this way at a particular location it is as if generating or emitting some Spherical wave fronts. So, these are essentially the locus of points over, which or within, which this disturbance is having its effect felt, but, the disturbance is remaining stationary at a location.

(Refer Slide Time: 20:08)



Let us consider a second example, say Mach number less than 1 less than one can be. So, many. So, just as just for special case let us say that Mach number equal to half that is u by c equal to half.

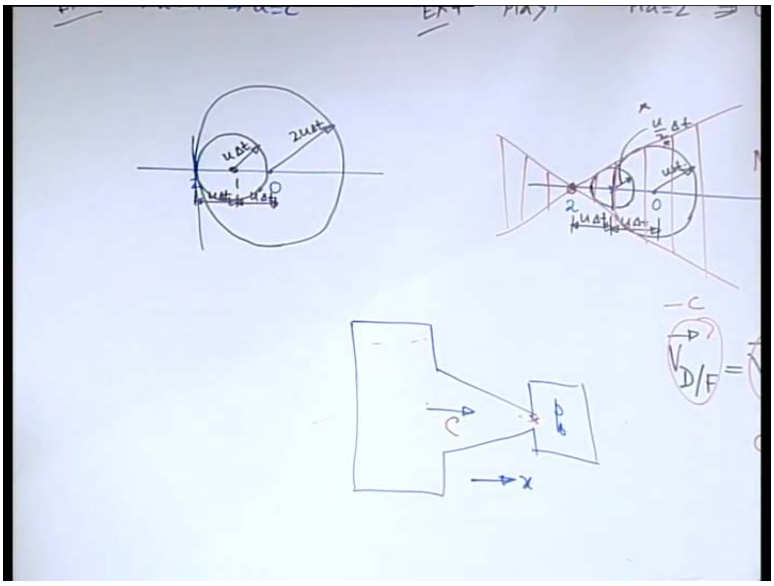
Let us try to draw a similar type of diagram for this case let us say that at time t equal to 0 the source is at 0 then at a time t equal to delta t the source will come at a point 1, which is located at a distance of u into delta t from 0 because source is moving itself at a speed of u.

Similarly, in the next instant of time it moves by another distance of u delta t and. So, on now, by this time what happens to the emission waves? So, if you consider for example, a point centered at 0. So, by the time the source has come by a distance of 2 u into delta t how far the disturbance wave has propagated c into 2 delta t and c is equal to c equal to 2 u. So, 4 u delta t.

So, taking this as a center and 4 u delta t as a radius you draw a big circle then that is the extent over, which this wave has propagated by the time this has come by u delta t any source at 1 the source at 1 has emitted a wave of what extent 2 u delta t. So, taking 1 as a center and 2 u delta t as radius is taking 0 as the center.

So, this is how you can have emission of different fronts let us consider a third example which is the more interesting one that is when the Mach number is greater than 1. Or let us take a limiting condition, when Mach number is equal to 1.

(Refer Slide Time: 23:50)



So, let us try to draw a similar diagram. So, Mach number equal to 1 means u equal to c. So, by the time this moves over u delta t and again another u delta t the disturbance moves by the same amount. So, because u and c are the same.

Similarly, from here it will emit u into delta t. So, you can see that there is a critical condition such that the disturbance wave does not propagate beyond the point, where the source is now currently located. Let us see a fourth example when Mach number is greater than 1 let us say an example with mach number equal to 2 which means u equal to 2 c. So, the source of disturbance propagates at a faster rate than the disturbance itself.

So, by the time it has come by the source has come by 2 u delta t what will be the corresponding u into delta t will be the corresponding propagation of the disturbance wave front. So, by the time from 1 to 2 it has come by u delta t the disturbance front will come by half u delta t. So, taking one at center and half of this as radius. So, in this way it is possible to draw and envelop of these zones of influence and that envelop will basically come from a common tangent to these circular fronts of course, you can extend it on either side.

So, physically your domain of dependence and your zone of influence is confined within this envelop not the current position of the source I mean it is it is very easy to prove it. If you if you draw this tangent these two common tangents they will be an intersect at this point. I mean if you know that this is u by 2 and this is u from that by geometrical construction you can easily show it is very trivial.

So, if you have. So, this is the apex of in a three-dimensional space it is like a cone and the section is of course, a straight line which is the generetrix of the cone. So, you get 2 straight lines as the cross-section in the plane and these two straight lines are nothing, but, the characteristics of the partial differential equation.

So, what basically happens is that any disturbance propagates and makes its effect felt only within this zone outside this zone is a sort of zone of silence. So, if there is some observer, who is located outside this zone will not feel the effect of any source of perturbation pressure perturbation within the domain because of the movement of the source of disturbance, but, when an the observer just enters this zone will definitely feel the influence.

So, there is some sort of discontinuity across these. So, discontinuities of what discontinuities of actually the second order partial derivatives of phi these are sort of weak discontinuities and these are the characteristic lines across which these discontinuities are there. You can see sometimes you have seen such physical example, if you have seen jet planes moving in the sky at a very high speed. So, these are super-sonic speeds and at those high speeds you will see that they will emit emissions which sort of will look like this particular I mean they will have their boundary like this in a conical fashion.

So, this is in compressible flow jargon is known as Mach cone this is like a cone imaginary cone within which you have your disturbance propagation effect felt. So, question is why do you have such discontinuities in compressible flows or strongly compressible flows the reason is that. If you have hyperbolic equations, which are which are valid in cases of highly compressible flows example super-sonic flows then what happens.

If you have a disturbance the disturbance propagates at a speed which is not matching with the source propagation itself the source propagates at a very rapid rate the disturbance cannot match with that. So, disturbance tries its best to move at par with the source, but, source moves at a faster rate than the disturbance. So, disturbance is not able to get dissipated in that easy manner as it could have been possible for a case, where the compressibility effect is negligible.

When the compressibility effect is negligible disturbance propagates at infinite speed. So, automatically the effect gets dissipated throughout the medium at a very rapid rate however for highly compressible flows that is not possible. So, the disturbance gets accumulated. So, to say so, you can have the disturbance that is getting propagated of course, along a certain direction which has a spaceo temporal regime associated with it, but, the disturbance is propagating at a finite rate.

The implication is that these are time dependent problems with significant amounts of with dissipations not that rapid or not that significant as the case that could have been possible if the propagation of disturbance was at infinite speed. So, you have insignificant amount of dissipation as compared to the case with infinite speeds. So, the disturbances will accumulate and once they have accumulated too much there may be a rapid discontinuity or a very strong discontinuity over, which these disturbances are released.

So, to say so, you can have strong discontinuities of properties that is you can have jumps in properties of the flow those do not take place across these characteristic lines, but, those take place across other lines which are called as shock fronts and the corresponding waves of propagation are called as shock waves.

So, since you have these accumulations of disturbances somehow they have to be relieved and shock waves are mechanisms by which these disturbances are relieved. So, you can have possible discontinuities one is this weak discontinuities across these characteristic lines plus strong discontinuities across the shock fronts and once you have discontinuities, that means, you have jump type of boundary conditions across these lines. So, your numerical method must be able to account for such discontinuities. So, that is why it is very important to know the characteristic of the equation because if you know then you can plan for a numerical scheme that will capture these discontinuities otherwise, you did not have special provisions in the numerical method to capture discontinuities.

In fact, shock capturing is one of the very key issues in the numerical solution of the compressible flow equations wherever shock is possible that is you have a change from super-sonic to a sub-sonic flow, abrupt change from a super-sonic to a sub-sonic flow. So, we have seen some examples and these examples tell us that there is a very important role played by the finite speed of propagation of disturbance in a hyperbolic type of equation environment, let us consider another physical example.

Let us say that you have a converging nozzle you have a supply reservoir and you are maintaining a back pressure. So, that fluid is coming from the supply reservoir along the positive x direction. So, what you are basically doing say you are trying to increase the flow rate. So, what you do you reduce the back pressure in this chamber. Once you reduce the back pressure in this chamber this supply reservoir will know that now I have to send fluid at a higher rate so, it will start sending fluid at a higher mass flow rate. So, this back pressure regulation is like giving a pressure perturbation or a pressure disturbance into the flow medium that propagates upstream and then gives a message that well now I have reduced my pressure it is now your responsibility to respond to that and send more mass flow rate. But interestingly that does not happen perpetually.

So, what happens is that if you reduce the back pressure this mass flow rate comes it increases, but, it comes to a maximum when it comes to the sonic condition that is the Mach number becomes equal to 1 and then you cannot increase the mass flow rate further by reducing the back pressure. This is a well-known physics in compressible flow question is why does it happen. So, you can explain it in a very simple manner by considering the velocity of the propagation of the disturbance. So, let us say that by d symbolic notation we represent d as disturbance and a f as flow. So, velocity of disturbance relative to flow is velocity of disturbance minus velocity of flow. Let us say that the flow is moving with a along the positive x direction with velocity c then the corresponding Mach number is equal to 1 u equal to c. So, velocity of flow is plus c. Velocity of disturbance relative to flow or what is the velocity of the disturbance the velocity of the disturbance? So, it is velocity of disturbance is created here.

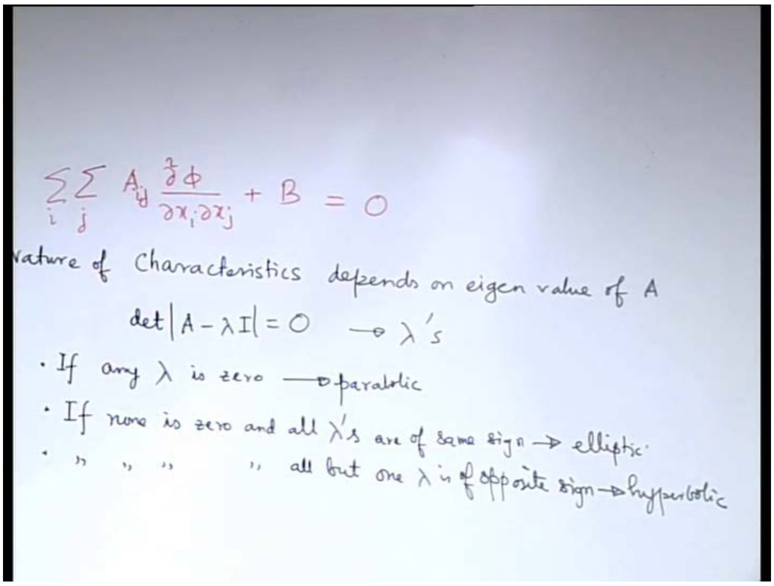
The disturbance is propagating in this direction how it is propagating it is propagating relative to the flow. So, relative to the flow it is moving along the negative x direction with a velocity c. So, velocity of the disturbance relative to the flow, that means, velocity of the disturbance is 0. So, disturbance cannot propagate further. So, you are what you are doing is you are reducing the back pressure, but, what happens is that that message is not able to get propagated. So, this supply reservoir does not know that you have further reduced the back pressure. So, it cannot respond to that by increasing the mass flow rate.

So, I mean of course, this is not a very complicated example, but, if you go to more and more complicated examples you can find out similar cases the important lesson that we learn out of it is not this example or the examples. That we have considered earlier, but, the moral of the example that we learnt is one simple thing that in hyperbolic type of equations you have the finite speed at, which the disturbances propagate and when the disturbances propagate at finite speed, that means, what are these disturbances say pressure.

So, a pressure wave propagates at a finite speed the source of disturbance on the other hand can move further ahead of that one and that can give rise to discontinuities in the flow medium which has to be taken into account while designing the corresponding numerical solution.

Now, we have seen the classifications of the partial differential equations, but, a partial differential equation that contains only two variables two independent variables. That is the generic form with, which we started, but, it can also have more than two independent variables it is still a second order partial differential equation, but, you can have more than two variables.

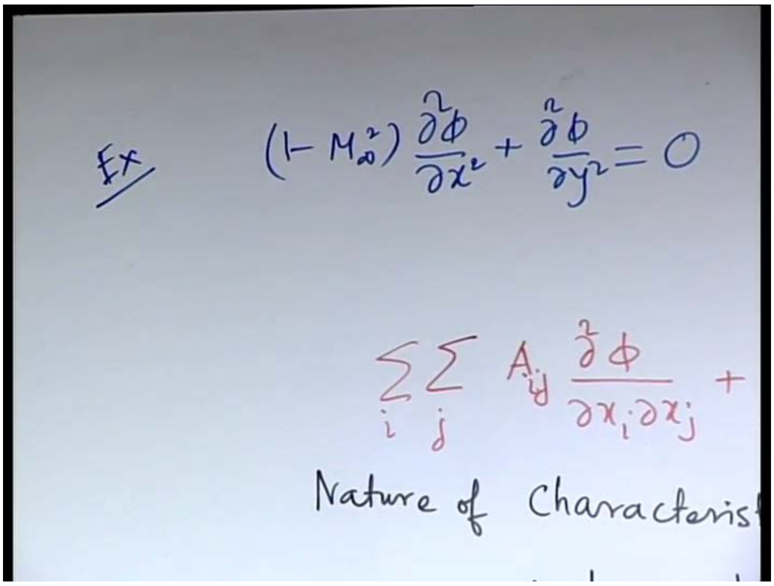
(Refer Slide Time: 39:48)



So, the general case can be represented in this way…Where B contains several functions we have clubbed all together we have seen that only highest order derivatives are important and that is why we have just kept that as a specific term and other terms are just kept generically. So, here in this case this is a more general example than the special cases that we have considered with two variables. So, here depending on the values of i and j you can have many variables, but, still the highest order derivative is second order partial derivative that is how this as still a second order partial differential equation.

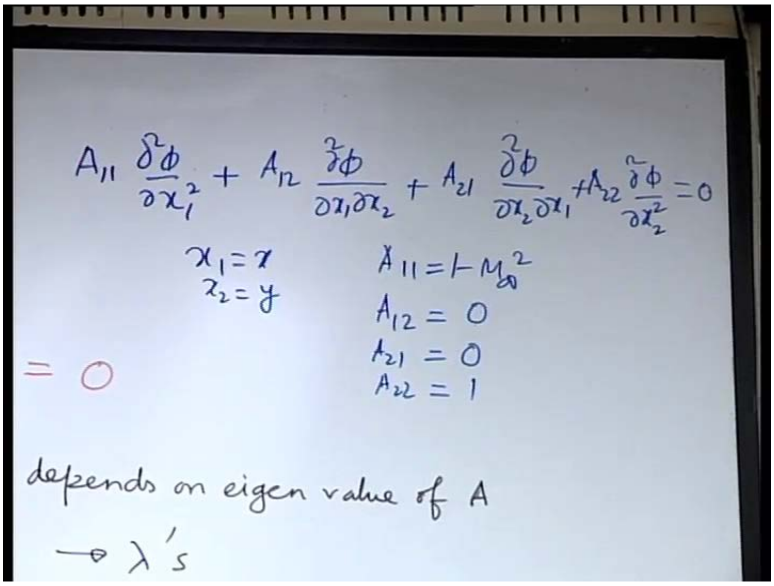
Now, the characteristics of this equation these depend on the Eigen values of the coefficient matrix a. So, characteristics or nature of characteristics rather depends on. So, how to get the Eigen value of a you find out determinant of a minus lambda i equal to 0. So, that will give you the lambdas. If any lambda is 0 out of all the Eigen value if any one Eigen value is 0 then it is a parabolic equation if none is 0 and all lambdas are of the same sign then it is elliptic and if none is 0 and all, but, one lambda is of opposite sign. So, one at least is of opposite sign than the all other lambdas then it is hyperbolic so, let us consider an example by, which we will try to illustrate this principle of classification.

(Refer Slide Time: 43:15)



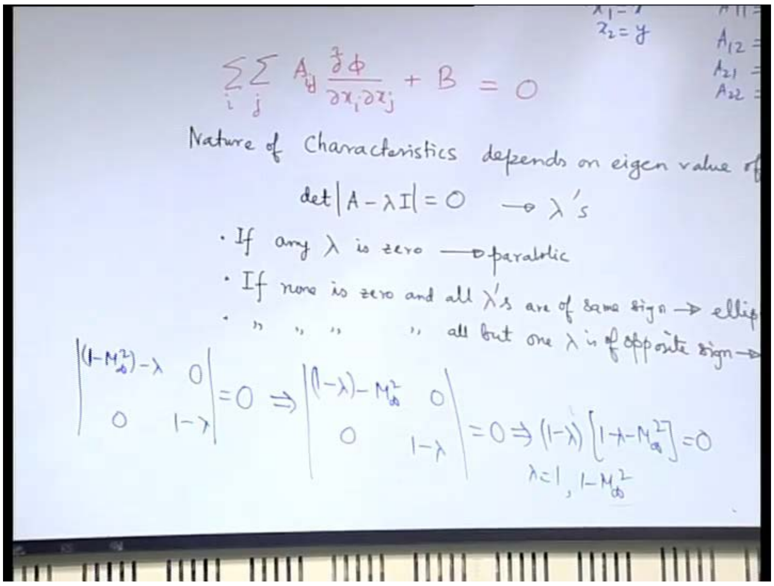
This is an equation relating the velocity potential for isentropic in visit compressible flows, over slender shaped bodies, that is bodies, which are having a slender shape. So, from that we have to identify the characteristics of the corresponding partial differential equation.

(Refer Slide Time: 44:14)



So, let us try to use this rule. So, you have A 11 This is the particular form of this one where x 1 is x and x 2 is y. So, let us identify the values of the coefficients what is A 1 1 1 minus M infinity square what is A 1 2 0 what is A 2 1 0 and A 2 2 1.

(Refer Slide Time: 45:33)

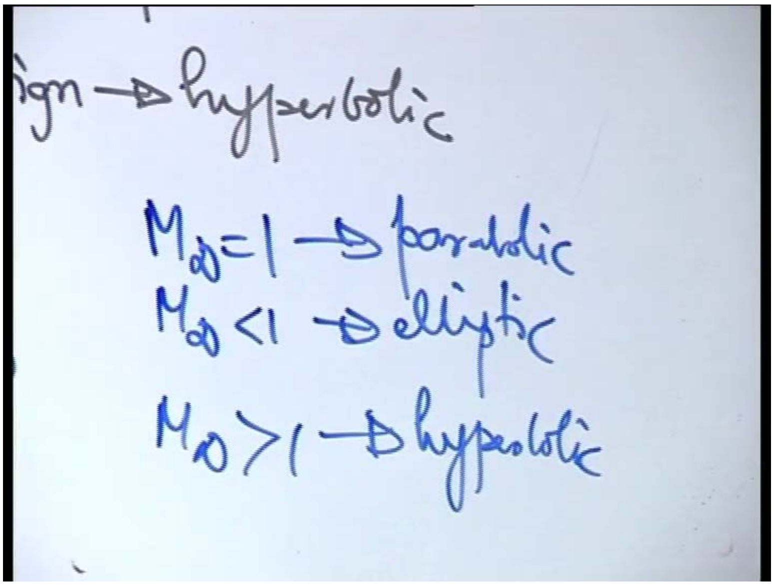


So, with this understanding we will try to find out the Eigen values of a. So, it will be 1 minus M infinity square minus lambda. This is the characteristic equation. So, this is basically 1 minus lambda minus M infinity square 0 0 1 minus lambda. So, 1 minus lambda into 1 minus lambda minus M infinity square equal to 0. So, lambda equal to 1 and 1 minus M infinity square. So, now let us find out what are the characteristics of this equation or what are the natures of this equation first? So, here no Eigen value is 0.

It can be 0 one of the Eigen values can be 0 if M infinity is equal to 1, then what happens. So, one Eigen value equal to 0 means it becomes parabolic. When M infinity is less than 1 then both are non-zero and of the same sign, so, that is elliptic. So, that you can see from this rule whereas, if M infinity is greater than 1 then 1 of the Eigen values is positive and another Eigen value is negative. So, it becomes hyperbolic.

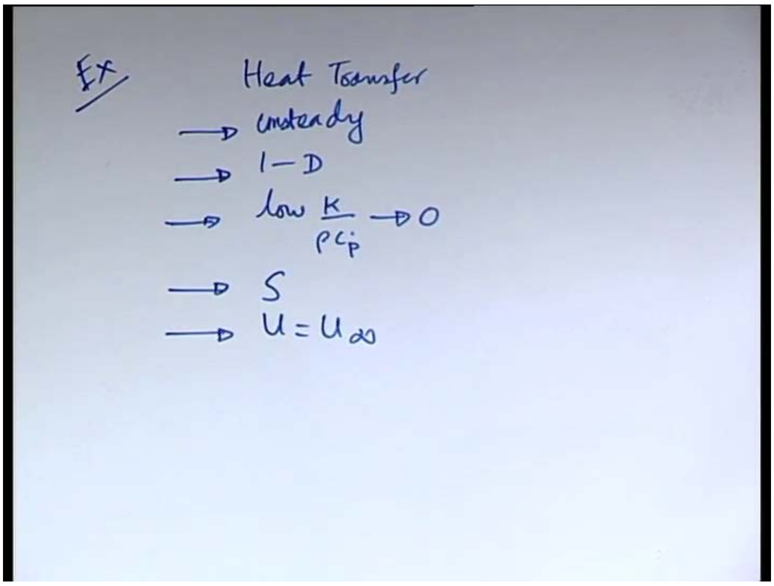
So, you can see the same generic equation depending on the free stream Mach number can either be parabolic or elliptic or hyperbolic that is what the important role that is being played by Mach number in compressible flows. So, you cannot trivially say that this equation is parabolic elliptic or hyperbolic without regarding to the mach number you have to refer to the mach number and based on the mach number you have to make this conclusion.

(Refer Slide Time: 48:08)



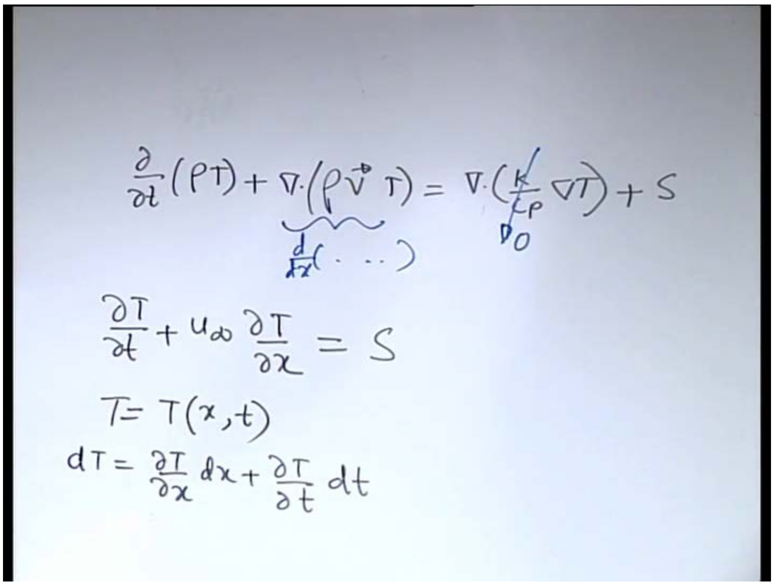
That when M infinity is equal to 1, it is parabolic; when M infinity is less than 1 it is elliptic and when M infinity is greater than 1 it is hyperbolic. Now, till now we have considered classification of second order partial differential equations, but, in many examples we may come across first order partial differential equations, which may be applied to examples in thermo-fluid sciences also. Let us consider one such example and find out, how do we determine the nature of such an equation? So, let us consider a problem through which we may illustrate that.

(Refer Slide Time: 49:21)



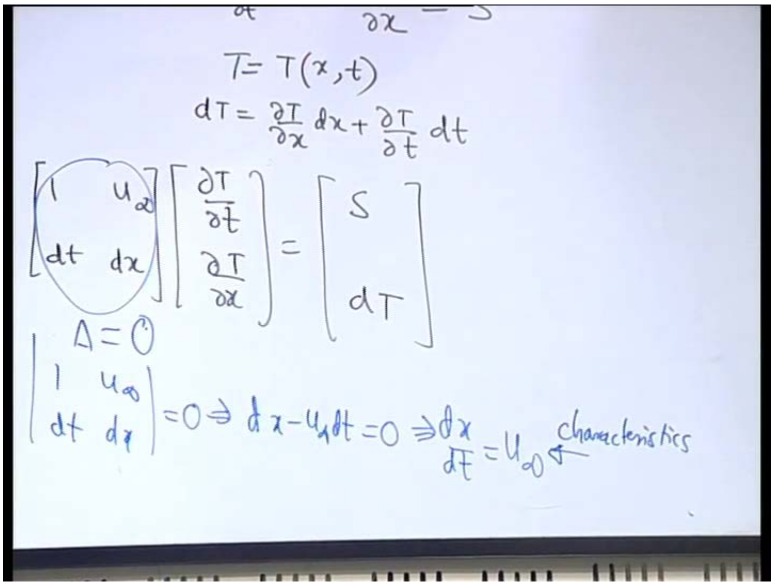
So, let us consider a heat transfer problem Unsteady, one-dimensional, low k by rho c p, which almost tends to 0 of the flow medium, there is some heat source and u equal to u infinity approximately. So, these are the assumptions corresponding to the physical problem unsteady flow one-dimensional flow low value of k by rho c p for all practical purpose it may be may be taken to be as good as 0, some constant heat source and u is equal to u infinity. So, with this let us try to formulate the governing equation.

(Refer Slide Time: 50:33)



So, let us write the most general form of the conservation equation… plus some s. So, let us consider the simplifications. So, this will have only a d d x component because it is one-dimensional this term will go away and if you use the continuity equation together with this you will get equation of this form because u is equal to u infinity one-dimensional only 1 x component of velocity. Let us try to find out the characteristic of this equation. T is a function of x and t. So, you can write d T as del T del x into d x plus del T del t into d t.

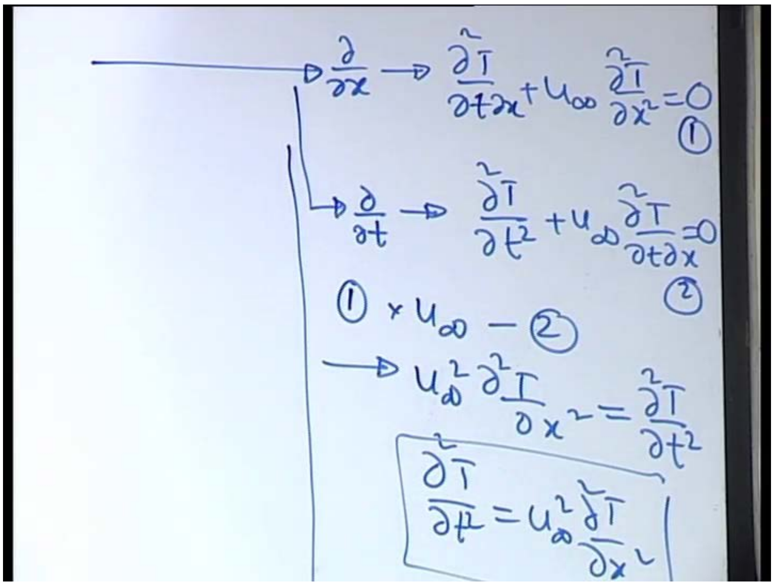
(Refer Slide Time: 52:23)



So, just like what we did for a second order partial differential equation for first order what are the variables for, which we should look for the discontinuities across the characteristic the first order partial derivatives for second order partial differential equation we look for discontinuities in the second order partial derivatives for first order we should look for discontinuities in the first order partial derivatives. So, let us write those for which we are looking for the discontinuities. So, for the discontinuity to exist we must have this determinant equal to 0. So, that means, 1 u infinity d T d x.

So, the equation of the characteristic is given by d x d T equal to u infinity. What is the nature of the p d e parabolic, elliptic or hyperbolic see never learn a subject by using magic formula the most of the formula that we have described by this time this, these formulas are for second order partial differential equation. So, if you now just conclude that well I have one real characteristic. So, it should be parabolic, is it. So, let us try to assess if you are not confident it is possible to convert this into a second order p d e and see let us try to do that. Because you know the classification for the second order p d e let us try to do that and it is very easily that can be done.

(Refer Slide Time: 54:24)

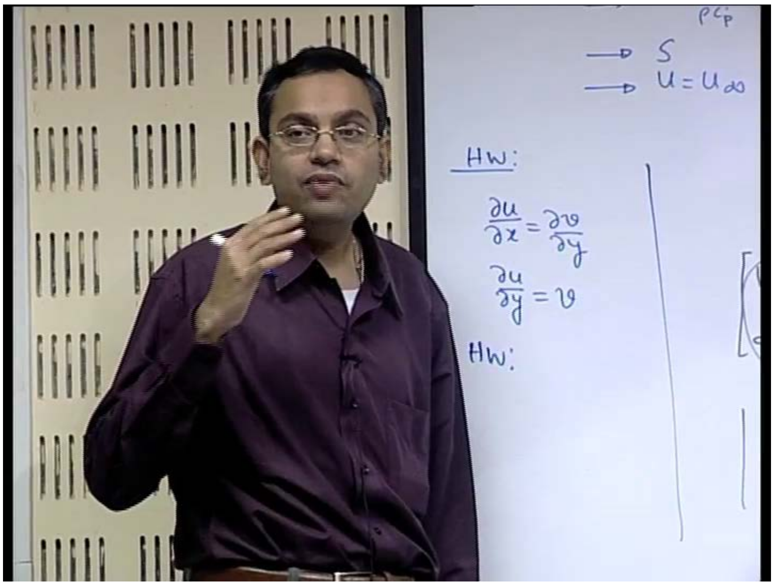


First you differentiate this partially with respect to x once. So, what you get sorry S is a constant. So any derivative will be 0. Next you differentiate the same partially with respect to T…

Because it is a second order partial derivative we are writing we are assuming it to be continuous. So, does not matter whether you write t first or x first. How you can eliminate the cross derivative let us say this is equation one this is equation two, you multiply equation one with u infinity and subtract from equation two. So, equation one into u infinity minus equation two; so u infinity square.

Now, can you tell, what is the nature of the equation hyperbolic? So, it is a hyperbolic equation apparently elusive it is giving only one characteristic, but, you should not be distracted by that, because it is a first order partial differential equation from which you are getting the conclusion. So, it gives a fair idea to you how to find out the characteristics of different partial differential equations.

(Refer Slide Time: 56:39)



So, I will give you a homework exercise; let us say that you have a system of partial differential equations, home work.

So, using these two equations, which are coupled equations, you find out that what is the nature of the partial differential equation parabolic, elliptic or hyperbolic; of course, it is quite easy to do, you can eliminate one of the variables either u or v for example, you can differentiate the second equation with respect to y, and from that you can eliminate v from the two, and you can easily find out the characteristic by using the rules. And what I will suggest you to do is to do another homework, make a chart, where you write the different features of parabolic, elliptic and hyperbolic equations for yourself that nature of the characteristics, how many number of characteristics are there, what about the zone of influence, what about the zone of disturbance, what about the speed of propagation of disturbance; based on these features, you make a table for different types of equations parabolic, elliptic and hyperbolic equations. So, these home works, you try to complete by the next class. So next time, when we meet we will move on to our next topic.