Computational Fluid Dynamics
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Module No. # 01
Lecture No. # 05
Classification of Partial Differential Equations and Physical Behaviour

In the previous lecture, we were discussing about the general form of the conservation equation.

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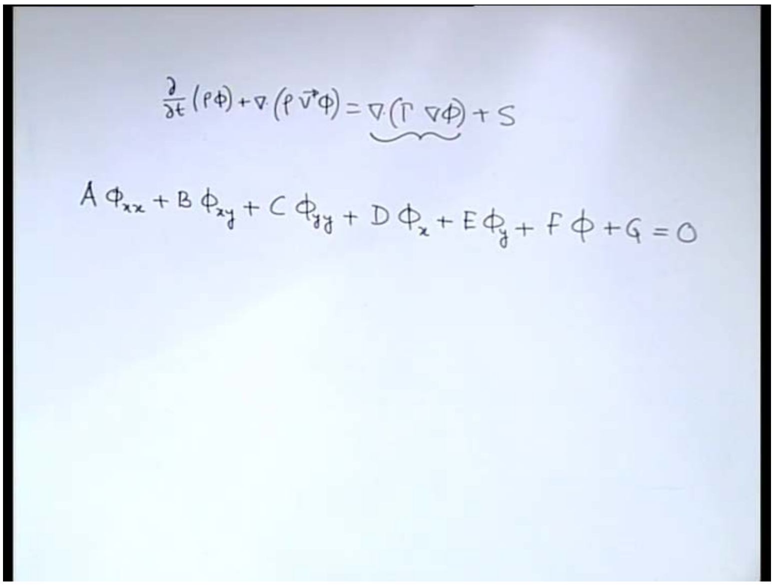

So, we came up with a form. Almost all physical problems that we will encounter in this particular course will be satisfying this type of governing differential equation. Of course, if it is a one-dimensional problem, it will eventually boil down to an ordinary differential equation, but for a general problem the prototype governing equation will be of essentially this type.

So, it is important for us to get a feel of the particular characteristics or special features of these types of equations, so that we can make an assessment of the kind of numerical method that should be applicable for solving them. Because depending on the nature of the partial differential equation, one has to design suitable numerical strategies. It is not that in this generic form you have a unique simulation strategy or unique numerical solution strategy, it all depends on what is the particular type of partial differential equation. And to understand that, of course, we can get into the broad theory of partial differential equation, but we will not try to do that, we will try to get into only some aspects of the theory of the partial differential equations which broadly can cover equations of this prototype.

So, you can understand that these equations are second order partial differential equations. Why? Where does the second order term come from? It comes from the diffusion term. So, because it is a second order partial differential equation, we will try to start or begin with a prototype form of the second order partial differential equation.

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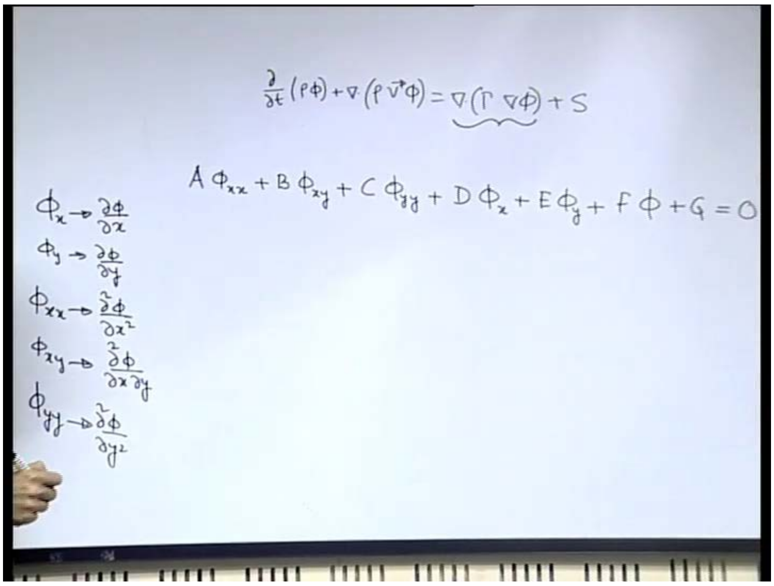

Let us say that we have a partial differential equation of this type, where the notations are like this, phi x means del phi del x, phi y means del phi del y, phi x x means del square phi del x square, phi x y means… and phi y y means this.

So, these are just short-hand notations for writing the different partial derivatives. So, this contains a maximum up to second order partial derivatives. That is why this is a second order partial differential equation. And we have retained all possible functional dependencies, remember this A B C D, these are not constants in general, these could be functions of other variables.

So, we can sort of split it into two types of terms, one type of term contains the highest order partial derivative which is the first three terms, and these terms contain the either the function or the lower order partial derivatives or the independent variables. So, the solution what you expect is that, what is phi as a function of x y, function of two variables. Now, let us see that how do we go about it. To do that, let us first try to understand that what can be different types of these partial differential equations.

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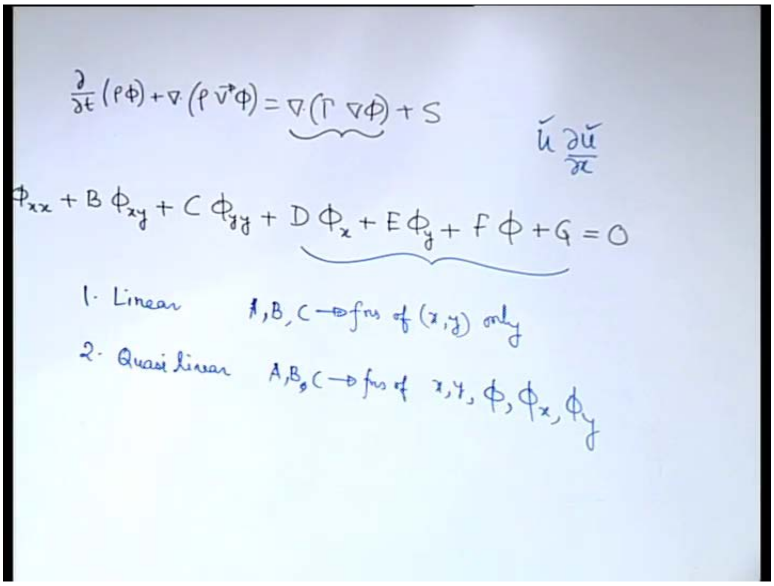

Now, the classification may be of different types. One type of classification is by classifying it in terms of linear or non-linear. So, when you say it is a linear partial differential equation, what do we mean? You have A B C, these are functions of x y only. And the remaining term which are there, these are supposed to be linear functions of phi, phi x, phi y like that. So, there is no non-linearity involved. If this is not satisfied, we say that it is a non-linear equation. But there are special types of non-linear partial differential equations which are called as quasi linear, where this A B C, these are functions of x y, but also u, u x, u y or here it is phi, phi x, phi y.

Now, what we will do with this classification? It is of course important to note whether the equation is linear or quasi linear or non-linear, in general I mean, quasi linear is a special type of non-linear, but you could have other types of non-linear equations. And it is important to know this because the solution strategy will depend on which type of equation is it. And not only that, once you have a type of equation, you can attach with that or associate with that a particular physical perspective. For example, many of the equations in fluid dynamics will be quasi linear in nature.

So, for example, if you have a term of this type, u del u del x. So, you can see that you could have a non-linearity because of a multiplication of u with a differential function of u. So, you could have a non-linear differential equation which describes the fluid flow. So, when we will be describing the fluid flow equations in details, we will see that this non-linearity is one of the key aspects of the Navier Stokes equation, and how to take that into account while solving the Navier Stokes equation is one of the big challenges.

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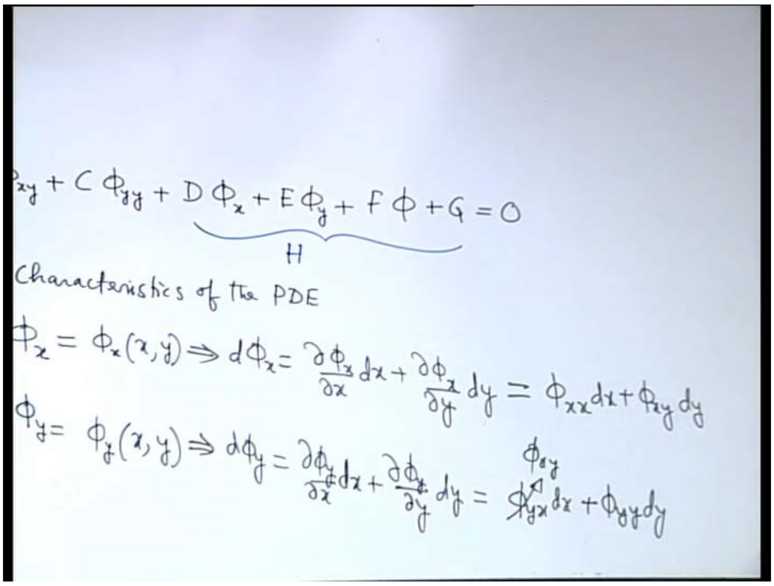

To proceed further what we will do is, we will identify certain aspects known as characteristics of the partial differential equations. The first question that we would like to answer is that, what is a characteristic? Now, when you say this partial differential equation, it involves with it the partial derivatives and there are highest order partial derivatives like phi x x, phi x y, phi y y like that, these are the highest order partial derivatives appearing in a second order partial differential equation.

So, when you have these partial derivatives, these partial derivatives may be continuous or may be discontinuous over some portion of the domain. You may not ensure that trivially the partial derivatives are continuous at all points. There may be locations or there may be certain lines across which the partial derivatives may be discontinuous, and these lines are called as characteristics of the partial differential equations.

So, why these are lines? Because you have the domain x y as may be a plane. So, on that domain, you could have certain lines across which you could have discontinuities derivatives in the highest order derivatives of the variable phi and those are called as characteristic lines.

So, characteristics of the partial differential equation with reference to this second order partial differential equations are lines across which you could have discontinuities in the highest order derivatives, here second order derivatives. Why it is important to understand or identify? See, if there are discontinuities in certain variables, your numerical method that you are using should be able to accommodate that discontinuity. So, if you do not know a priory whether there will be discontinuity or not, you cannot design the numerical method. So, looking at the partial differential equation, you must first ascertain whether such discontinuities exist or not. If such discontinuities may exist, then your numerical method should take care of, should be capable of taking care of such discontinuities.

How do we assess whether such discontinuities are there? So, to do that what we will do is, since the highest order partial derivatives are important parameters, we will club all other terms in the form of a function called as H, just we give a name of this as H and then proceed further in mathematical simplification.

First what we do is, we write phi x and phi y as functions of what? x and y. These are functions of x and y. So, from here you can write D of phi x… Similarly phi y is a function of x and y. So, D phi y is this one. Since phi y x and phi x y are the same, the order of the derivative so long as it exists does not matter. So, we can write this also as phi x y.

So, let us try to organize these equations. We have now got these three equations 1, 2 and 3. We want to organize these equations with an objective that we are interested to get the characteristics of the partial differential equation.

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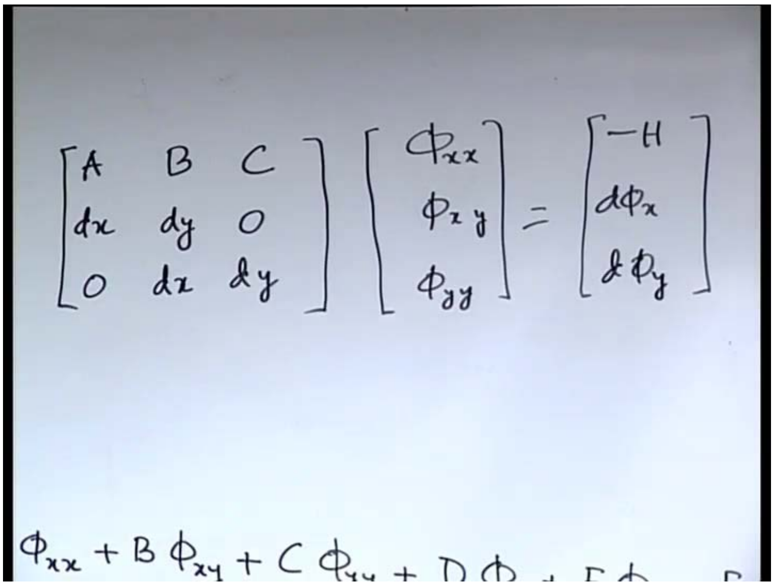

So, we arrange this in a matrix form. So, from the equation one, it is a into phi x x plus B into phi x y plus C into phi y y is equal to minus H. From the second equation, phi x x into dx plus phi x y into dy, there is no phi y y, is equal to d phi x. And the third one, this is 0, phi x y into dx and this into dy is equal to d phi y.

Why have we arranged it in this particular form? Because we are interested in the variables in the highest order derivatives. So, phi x x, phi x y, phi y y, these are the variables for which we are interested, along the characteristic lines these will be discontinuous. So, we are interested about the values of these variables. So, as if it is a pseudo linear algebraic form, a x equal to B, where a is this matrix, x is this one and B is this one. We are interested about the case, when the solution of this x; that is, phi x x, phi x y, phi y y does not exist.

If it is discontinuous, then we can interpret it in certain way, either solution does not exist or infinite number of solutions exist, that is if you have a discontinuity of a variable across a line, so you may interpret that solution does not exist because the variable is discontinuous or it may undergo a transition through infinite number of possible values, no matter in whatever way you interpret. Let us try to interpret it in a way that solution does not exist. So, let us try to see an analogy of this with an algebraic equation, so that our understanding will be clear.

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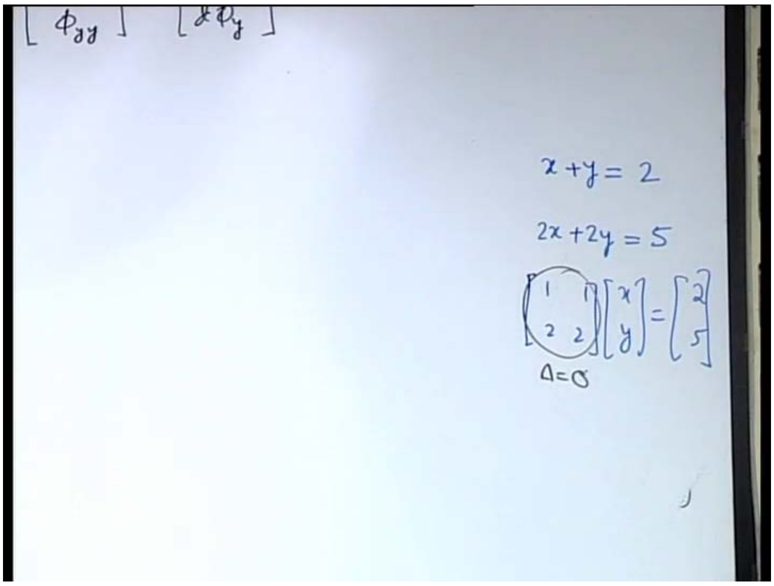

Let us say that we have two algebraic equations, x plus y equal to 2 and 2 x plus 2 y equal to 5. So, what will be the nature of the solution of this? Solution will not exist. So, how can we assess that the solution does not exist in a more formal way? Here of course, by visual inspection we can do that, but this visual inspection gives us a clue. What kind of a clue? If we write it in a matrix form, we can see that these determinants is equal to 0. This determinant is equal to 0, which essentially stems from the fact that you have this row that can be written as some linear multiplier of this particular row.

So, the second row is a linear multiplier of the first row. So, it is essentially the same coefficient just multiplied by a fixed parameter. And then you can sort of write this matrix in terms of rows, where two rows are identical with a multiplying factor, and then that means that the determinant will be 0.

Of course, you can generalise it to a system of equations with n equations and n unknowns. So, how can we borrow this knowledge from here to here? How can we do that? So, here we want that solutions, we want to get the locus of those points across which you will have discontinuities in phi x x, phi x y and phi y y, that means the solution will not exist. And that is possible only if this determinant is 0. So, for phi x x, phi x y, phi y y to be discontinuous, you must have this determinant to be equal to 0.

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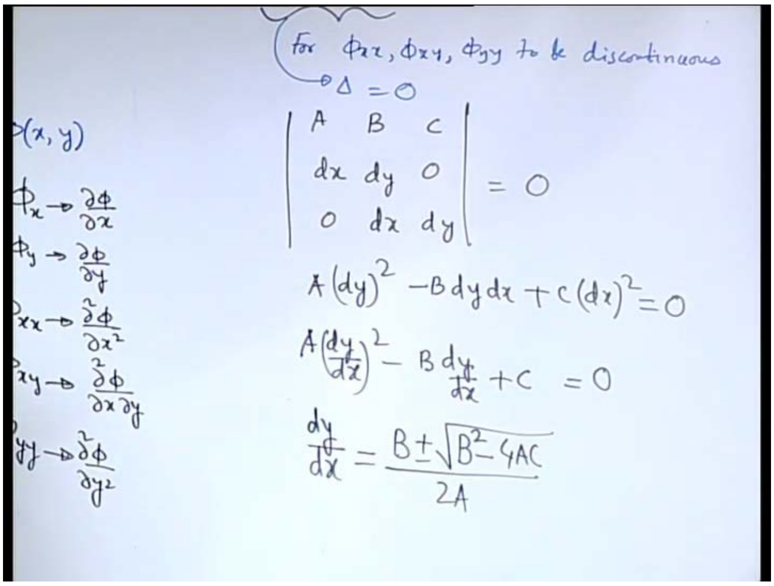

So, let us expand that determinant and see that what is the corresponding condition. One important and interesting thing you can see from here is that, it is the highest order derivative in the equation that is mattering. So, you have only A B C these coefficients appearing in the condition, but not D E F G like that. So, coefficients of lower order derivatives, those do not matter in terms of the characteristics of the equation. Only the coefficients of the highest order derivatives, they do matter.

So, if we simplify this… So, we can organize the equation in the form of a quadratic equation in dy dx. So, from here you can get the solution. How many real characteristics of this will exist? It depends on whether B square minus 4 A C is greater than equal to 0 or less than 0. So, just like a quadratic equation, here also this discriminant is what that is mattering. So, there can be certain special cases.

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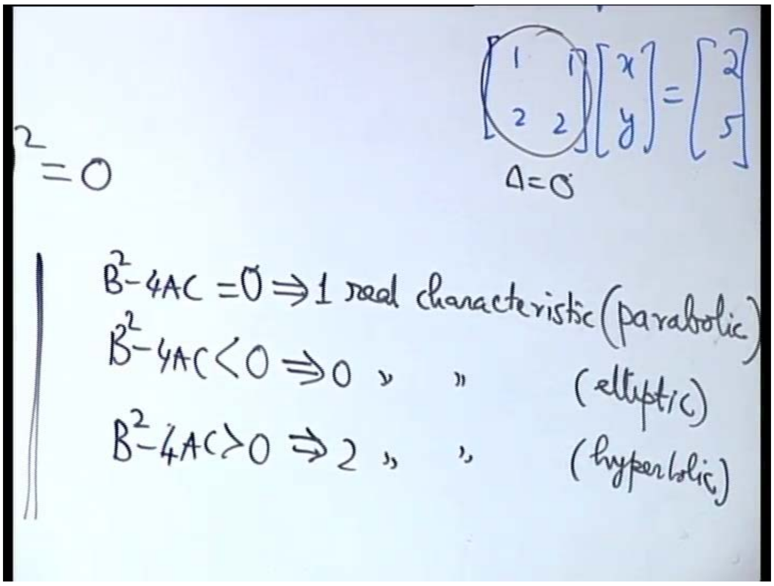

So, if you have B square minus 4 A C is equal to 0, then how many real characteristics will be there? Only one real characteristic will be there because this will have only one root, dy dx will be something, so y will be some function of x. So, only one solution will come out of it. So, this implies one real characteristic. Such an equation is called as a parabolic partial differential equation. So, a parabolic second order partial differential equation will have only one real characteristic.

Then, if you have B square minus 4 A C less than 0, then there is no real characteristic. So, this is called as an elliptic partial differential equation. And B square minus 4 A C greater than 0 will have two real characteristics, this is known as the hyperbolic partial differential equation. Remember, this classification is only for second order partial differential equation, so do not try to generalize it for higher orders. Because we are mostly confined up to second order partial differential equations in the context of computational fluid dynamics, so this is very much pertinent for our discussion.

Let us take some examples. What we will try to do is, very intentionally what I am trying to do is, we are first trying to understand these thing in a mathematical manner. Next what we will try to do is, we will try to understand each and every aspect of these equations from a physical perspective, and try to understand that how the mathematical characteristic of a partial differential equation reflects the corresponding physics. How does the physics come out from mathematics, that is what is very important. But first we should learn it mathematically and then try to relate that with the corresponding physics through certain examples.

So, first we will go through some mathematical exercises, very simple exercises to identify the natures of certain partial differential equations. So, we will give one example each of these equations, which are very common in fundamental science and engineering applications. So, let us take an example.

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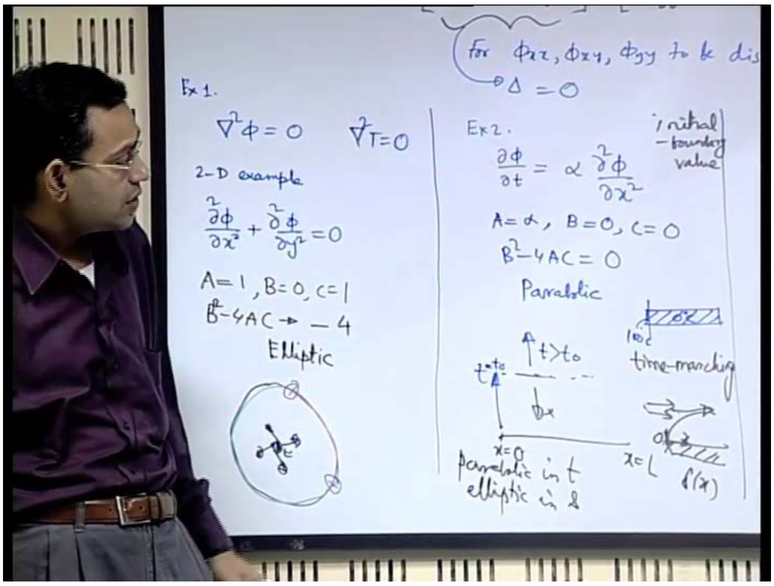

Del square phi equal to 0. This is perhaps the most simple, but most commonly encountered partial differential equation in mathematical physics, which is known as Laplace equation. So, this Laplace equation is a prototype of many physical problems. Like let us try to give some example. Like one of the possible examples is if you have steady state two-dimensional heat conduction. So, you have instead of phi, you have the temperature as the variable. So, say let us consider there is no heat source and the material has uniform thermal conductivity.

So, then it is del square T equal to 0. Or you could have electrical potential distribution within a chart system. So, that also may be governed by the Laplace equation. There are many examples. For example, velocity potential distribution in a rotational flow, that also is governed by the Laplace equation. So, there are different fields, different applications, physics changes, but what remains the same is the prototype, the Laplacian of some variable that is equal to 0. So, this is a very important equation.

Let us try to assess that what is the particular nature of this equation, is it parabolic, is it is it elliptic or is it hyperbolic. So, let us consider a two-dimensional example. So, if you try to figure out what is its particular nature, you have A equal to 1, B equal to 0, C equal to 1. So, what is B square minus 4 A C? So, minus 4, so that means it is elliptic equation. The big question is elliptic equations, so what? We understand that this equation is elliptic, but what physical meaning does it carry? Let us try to answer that. So, let us try to take an example of a heat conduction problem. Let us say that there is a domain, throughout the domain there is a uniform temperature, say the entire domain is at a temperature of 30 degrees centigrade.

Now, suddenly you heat some location of the domain, so that as if there is some heat source which is existing at some point of the domain and you are activating this heat source, maybe it is an electrically heated situation, so that the domain gets heated up at that location. That means, what you are doing? You are disturbing the initial temperature profile. So, once you are disturbing it, this disturbance will propagate to other points in the domain.

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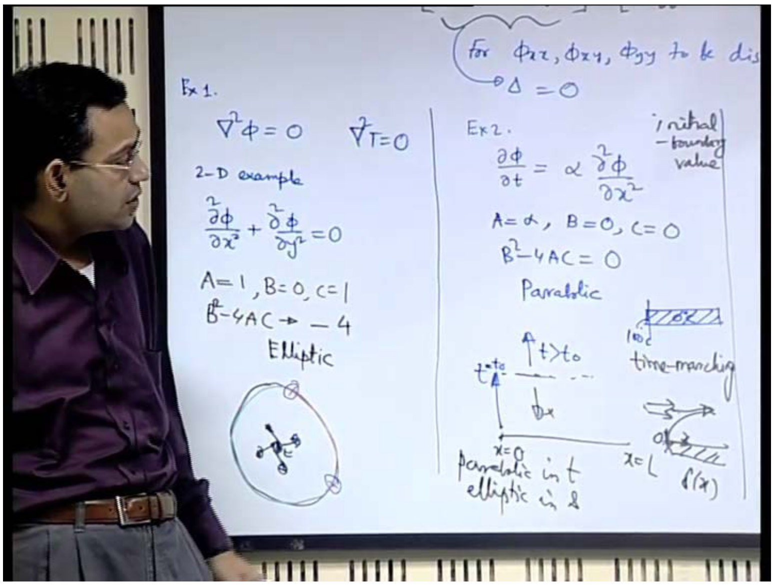

And when it is elliptic, this disturbance propagates at all directions inside the domain towards all directions inside the domain and it does so at infinite speed, that is very important.

So, this is one of the important physical meanings that is being conveyed by the Fourier's law of heat conduction, that when you have a thermal disturbance, the thermal disturbance propagates towards all possible directions at infinite speed. So, that means, therefore, what does this disturbance propagation try to do? It tries to nullify the difference in temperature between different points in the domain. So, it tries to bring the domain to a homogenous temperature. Because there is a gradient of temperature, there is a messenger that goes from one point to the another point in terms of a disturbance and that disturbance essentially tries to nullify the temperature differences between different points of the domain.

So, if you have disturbance at a point, that would be felt anywhere and everywhere. So, if you are considering a numerical solution strategy, then you have to be careful that if you have a particular point, then this point should be, temperature at this point should be influenced by all its neighbors because the disturbances propagate in all directions.

So, if you are writing a numerical system of equations where you have some neighboring points and you have a point under concern, then the point under concern should be influenced by all the neighbors, there is no exception to that. The other important thing is that for this elliptic type of problems, you have to specify boundary condition over the entire part of the domain.

So, the boundary condition may be of different types, we will come across the different types of boundary conditions subsequently, because numerical solution of equations is a strong function of the boundary conditions. So, you could have different types of boundary conditions.

Let us say that at over some part of the boundary you have a particular temperature and over the other part of the boundary you have the different temperature. That is possible. Say, one part of the boundary you keep immersed in steam at 100 degrees centigrade, another part of the boundary you keep immersed in ice at 0 degrees centigrade. So, they are at different temperatures. That means, you have some discontinuities in the boundary condition, but the solution in the solution domain is continuous. So, despite the possibility of discontinuity in the boundary condition, you will always have for an elliptic problem, the solution continuous inside the domain. So, there is no possibility of a discontinuity of the solution inside the domain, that is one blessing for selection of the or for designing of the numerical method, that you do not have to account for any possibility of discontinuity in the solution domain.

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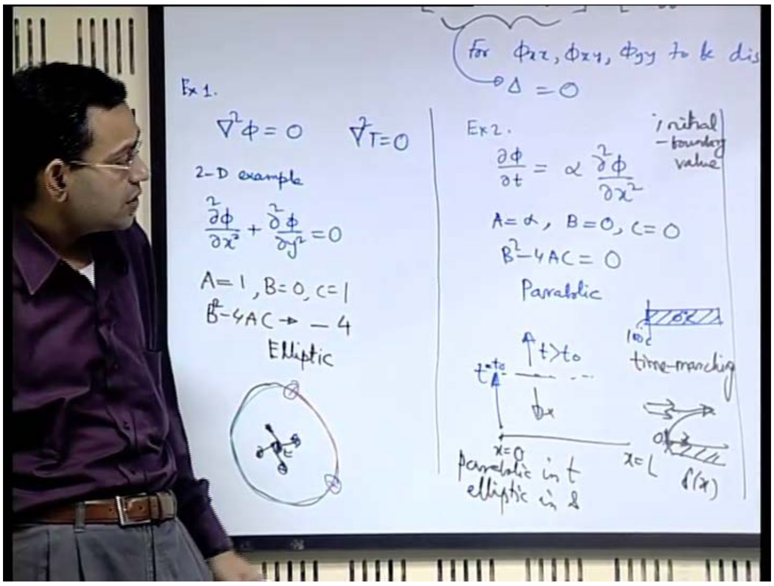

Why does it happen in such a way? Because the disturbance propagates at infinite speed in all possible directions, so that whatever is the discontinuity imposed by the boundary condition, that is very easily being counted by a very rapid disturbance message propagation from one point to another point to smoothen out any possibility of discontinuity. And this type of problem is also known as a boundary value problem, because you have to specify the boundary condition over the entire domain boundary to solve this problem.

We will later on see that what types of boundary condition you could possibly impose, but first and foremost it is important to appreciate that you need to specify boundary conditions over the entire domain boundary. So, this is one example. Let us consider another example. This is an example where we have physically put two variables, but only one highest order partial derivative term. This is an example of a one-dimensional unsteady state heat conduction. If you put temperature in place of phi and this alpha is nothing, but the thermal diffusivity.

So, let us try to understand what are the features of this partial differential equation. So, here only the highest order derivative will matter. So, A is equal to alpha, B is equal to 0, C is equal to 0. So, B square minus 4 A C equal to 0. So, it has only one real characteristic, that means it is parabolic. Again the question comes, so what? So, what does it physically signify? Let us consider this t and x as two physical variables and try to appreciate or analyze the situation from that perspective.

So, let us consider there is a domain. This is x equal to 0, this is x equal to l and we plot time along the vertical axis. So, in this domain we are interested to see that how the temperature varies as a function of position and time. If you see that we are having an event at a particular time, say right now an event is occurring, some event is occurring, say at some temperature, at some time, say t equal to t 0, you are creating a thermal disturbance.

So, as an example, say that you have a one-dimensional rod which is at a uniform temperature, at time t equal to t 0 you are suddenly creating a disturbance by maintaining one end of the rod at a different temperature than the other end. So, initially say the entire rod is at 0 degrees centigrade.

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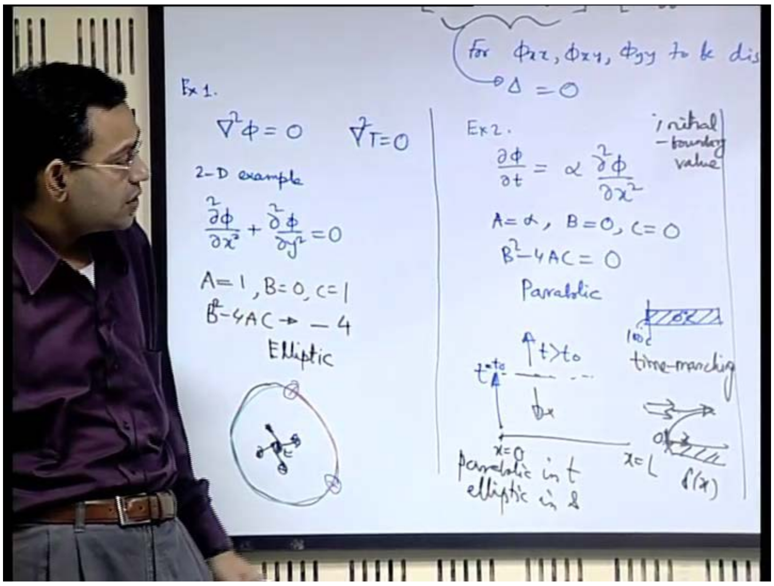

Now, suddenly you put, you subject this boundary at x equal to 0 to a temperature of 100 degrees centigrade. It is an abrupt discontinuity at time t equal to t 0. So, this is a disturbance, when this boundary temperature was not changed everything was at 0 degrees centigrade, now you are creating a disturbance, you expect the disturbance to propagate. In what direction it will propagate? It will definitely always propagate in a direction forward in time. So, what it will do? It will influence what will happen in the subsequent times. At t equal to t 0, if you have created a disturbance, that disturbance will influence what will happen for t greater than t 0, but it cannot influence that what has already happened some time back.

So, time is like a one-way coordinate system. So, there is a special characteristic of the time coordinate system, it is a unidirectional coordinate system. We will later on see that if you combine time with space that need not anymore be a one way coordinate system. Or we will also see certain example where space can also behave like a one way coordinate system, it is not always necessary that time is the only example of a one way coordinate system, but it is intuitively more obvious to consider it as time as a one way coordinate system.

How we can understand it? We can simply tell it in this way, that whatever is happening now can only influence what will happen in the future, but it cannot influence what has already happened in the past. So, whatever has happened in the past that cannot be changed, that cannot be influenced by what we are doing now, but we can always influence something which will happen in the future by what we are doing now.

So, that is how time is always a forward marching type of coordinate. So, these types of problems we call as time-marching problems. So, time-marching problems will have only one type of discontinuity, that is a discontinuity with respect to that reference time at which the disturbance is imposed. Because before that time and after that time is physically a totally different type of domain.

So, how these are related? If you have a disturbance at a point and if you see that that disturbance is propagated at different locations, so you have some domain of disturbance. And this domain of disturbance has its influence felt at certain locations. So, domain of disturbance and domain of influence these are two very important domains.

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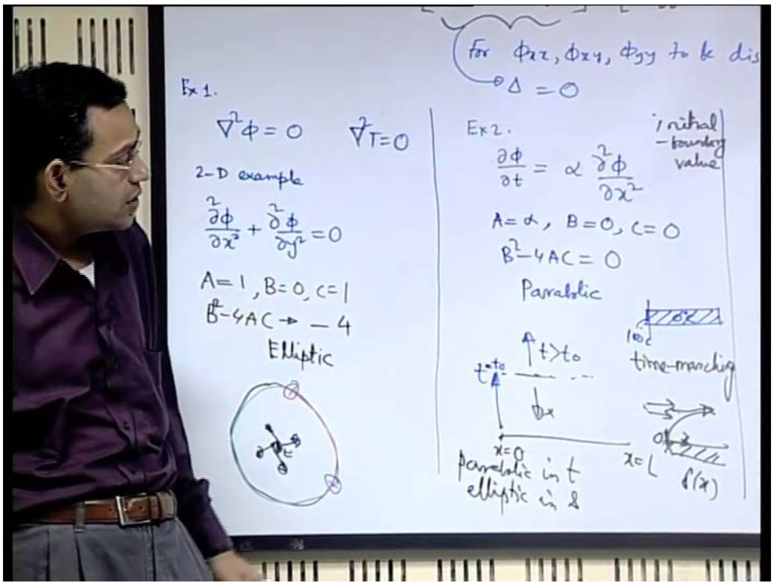

So, domain of disturbance is where you are imposing the disturbance, where you can impose the disturbance and domain of influence is where the disturbance may be felt. So, like in this elliptic problem. the entire domain is domain of influence. Whereas, when you have a time-marching problem, when you have t equal to t 0 is the time over which the disturbance is imposed, t greater than t 0 will signify the domain over which this influence will be felt.

So, any event which is t less than t 0 will be the domain of disturbance. Like that especially here, it is t equal to t 0 where you have the location of the disturbance initiation. And that initiation of disturbance will be propagated in a direction forward in time. So, you have only one discontinuity here and that is how one only one real characteristic. So, only one discontinuity is with respect to the time equal to 0 or reference t 0 condition which we call as an initial condition. So, you have at t equal to t 0 some condition, say at t equal to t 0 you have the entire temperature of the rod at 0 degrees centigrade. Now, with respect to this there is a discontinuity at t greater than t 0, because of this disturbance imposed at one of the boundaries. So, that is parabolic type of problem, but it is not as simple as that, if you just consider it to be of parabolic nature and forget about the other type of behavior.

If you consider a case of a heat conduction problem where time tends to infinity, and when you say that time tends to infinity there are many problems which at large times reach a steady state. So, maybe there is transience over the initial period of time. So, if you allow this system to evolve with time, the system can attain a steady state. When it attains a steady state, there is no more a derivative of temperature with respect to time that ceases. So, once that ceases it will be just d square phi dx square equal to 0. So, then that will be just like a Laplace equation prototype and it will become a boundary value problem or an elliptic problem.

So, we can say that this problem also has some elliptic nature inbuilt with it in the steady state limit. So, it is better not to say that it is a parabolic equation, a more correct way of specifying this type of equation is that, it is parabolic in time and elliptic in space. So, this nature, which is like initial boundary value problem because you have to specify the initial condition, and over and above that initial condition is the occurrence of the discontinuity because of the disturbance that is being imposed. But when the disturbance is imposed, it still propagates in all directions at infinite speed, so that elliptic nature is also retained. So, it is an initial boundary value problem. So, combination of initial and boundary value and it is basically parabolic in time and elliptic in space elliptic. Elliptic in s.

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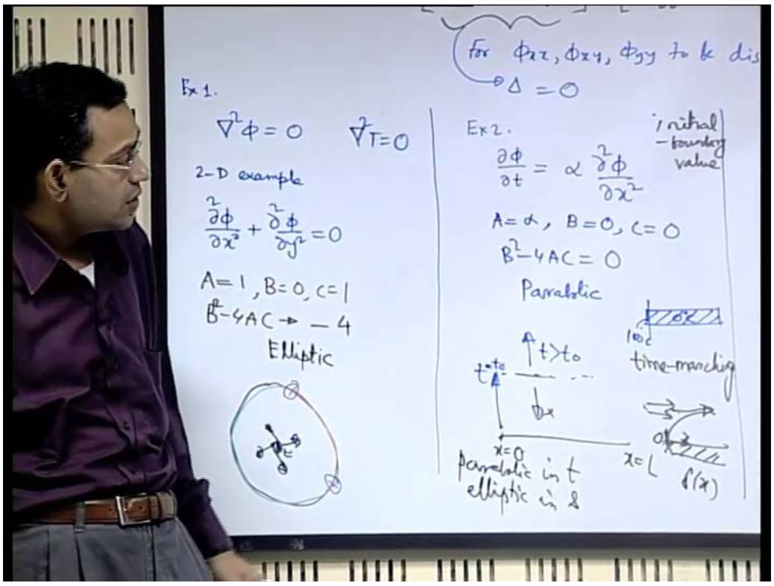

So, it is essentially a time dependent problem physically with a significant amount of dissipation. So, the dissipation is occurring at a very rapid rate, so that the system will try to homogenize any disturbance that is being imposed on it over some boundary or at some interior point. Now, regarding the time-marching or a one-way type of coordinate that the time essentially manifests itself. It is also possible that space itself manifests like a one-way coordinate. Let us take the example of the boundary layer development over a flat plate. So, if you consider that there is a flat plate over which you have the development of a boundary layer. You can clearly see that if you use the boundary layer coordinate, then along the x direction you have the growth of the boundary layer, delta which is a function of x or any other variable, it is a function of x.

So, it is just like a marching problem, that as you march along the axis of the plate, it grows and it is a open towards infinity in one direction. So, if you have x equal to 0, you have your domain from x equal to 0 to infinity theoretically. Of course, practically the plate is bounded by a finite limit, but in principle you can go up to infinite length along x, but you cannot go to negative x. So, your origin of the problem is starting from x equal to 0 towards open infinity at x tends to plus infinity.

So, that is like a space marching problem, that you are marching only along positive x and whatever is happening at some x, is not influencing what is happening in the back. And why that is possible? It is possible because this boundary theory you are developing for high Reynolds number. So, inertia force is very strong. So, essentially what happens? Viscous force is a force which tries to propagate its message in all possible directions. So, if you have a momentum disturbance, that momentum disturbance tends to propagate in all possible directions at infinite speed, that is the effect of viscous force. Whereas, inertia force is like a unidirectional effect, because inertia force is very strong for boundary layer types of flows, so what you are having, because these are high Reynolds number flows, so there will be predominant direction of propagation of disturbance in the forward x direction along which it will march.

So, whether there is some viscous effect, there is definitely some viscous effect, but viscous effect will not be propagating this entire flow towards the other direction, until and unless there is boundary layer separation. So, if there is boundary layer separation, then this entire discussion is not relevant, but until and unless there is boundary layer separation, it is a unidirectional way in which the message is propagating from the left towards the right and that is a sort of a space marching type of situation.

So, when you are designing a numerical scheme, here you can treat this x like time coordinate, where you are marching along positive x and for each and every new positive x, it is influenced by what was the event in the old x. But the new x event will not influence back what is going to happen in the old x, because of a predominantly unidirectional nature of the flow with a high inertia. Let us consider a third example.

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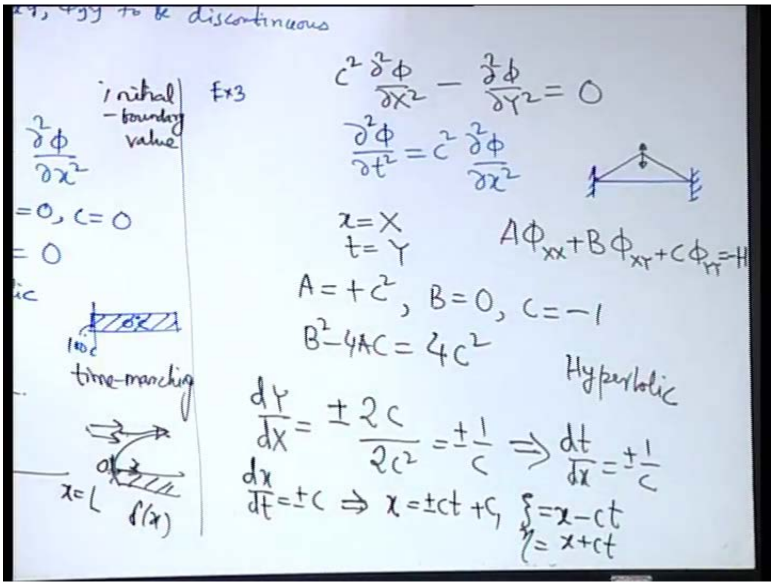

So, what is this example about. Just we considered one example where it was a steady state heat conduction, another example where unsteady state heat conduction, this is an example of maybe a wave propagation or we can say that physically an example, where you have a string which is tied at the two ends. Now, you create an initial disturbance by giving some initial displacement and initial velocity to it and then release it. Then what will happen? The stream will vibrate and its displacement will be governed by this equation as a function of position and time.

Now, the question is that what type of equation is it. Let us try to assess that. We will assess that subsequently. So, to do that, let us first try to identify the nature of the characteristics of this equation. It is possible to write this in this way. Let us say that we give a name small x is mu capital X and small t is mu capital Y, just so that we can formulate it in this form equal to some function.

So, what is A here? Minus C square. What is B? 0. What is C? No you have to bring all these terms in one side. So, it is not one, it is minus 1. C square this one minus this is equal to 0, that is the equation. A equal to plus C square and C equal to minus 1. Then what is B square minus 4 A C… So, what is the characteristic, dy dx, B is 0, so plus minus 2C by 2C square. So, plus minus 1 by c. What is capital Y? Capital Y is t. So, you can write this as dt dx is equal to plus minus 1 by c. That means dx dt is equal to plus minus C. So, if you integrate it you will get x is equal to plus minus C t plus some C 1.

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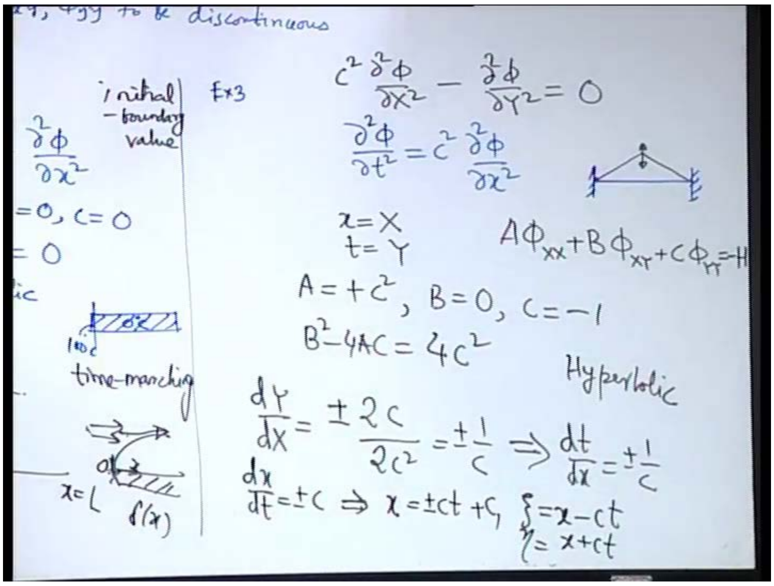

So, you can find out the characteristics, first you can forget about this C 1 because it is just an addition of the constant, it will shift only the line by a constant, but the main functions are x minus C t and x plus C t, these are the two characteristics. So, one is x equal to plus C t another is x equal to minus C t x equal to plus C t plus C 1 and x equal to minus C t plus C 1. So, but if we forget about that C 1 the functional dependencies one is x with plus C t another with x with minus C t.

So, these are the two characteristics. So, you can see that here the effect is not a function of time only, it is not a function of position only, it is a combined spatiotemporal effect. Where you have a new variable, let us give it some name say zeta and eta, these are called as characteristic variables across which you have possible discontinuities. And these zeta and eta are combined functions of x and t, not only that, it is possible to write the solution of this equation entirely in terms of this characteristic variables, zeta and eta.

So, the role of the characteristic is very interesting. The characteristic not only shows the possibility of discontinuity across certain lines. So, across the lines x, family of lines x minus C t and x plus C t, there can be possibilities of discontinuities.

In the next class, we will see through examples that how such discontinuities may be physically originating and how that relates to the physical property of these equations. Of course, because you have two real characteristics this is a hyperbolic equation and hyperbolic equations have certain interesting features. These features are not evident in the parabolic or elliptic equations. So, we will understand these features very carefully because some of these features are very important in high speed compressible flows and these have lot of relevance in fluid mechanics numerical simulation.

So, we will look into that carefully, but more importantly what we will try to understand first and see that we can write the entire solution in terms of the two characteristic variables. So, the characteristic variables play such roles that in terms of those you can write the solution and those variables signify locations of certain lines, across which you could have possible discontinuities.

When we say discontinuities, in the next class we will try to understand that what is the origin of that discontinuity, where from that discontinuity comes and are those the only lines across which you have discontinuities, these are important questions that we have to clarify. Because if there are lines beyond these characteristic lines across which there are discontinuities, we have to accommodate that also in the numerical method. So, we will identify that these are not the only lines across which you could have discontinuities. You could have discontinuities across certain other fronts, which we call as shock fronts or the corresponding discontinuity is known as a shock wave incompressible flows.

So, we will try to understand where from physically that comes, and for each and every case we will try to relate that physics with the corresponding mathematical behavior of that partial differential equation. So, we stop here today and we will continue with this in the next class. Thank you.

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