Computational Fluid Dynamics

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Lecture No. # 38

Part 1: Discretization of Navier Stokes Equations (Contd.)

Part 2: Fundamentals of Unstructured Grid Formulation

In the previous lecture, we discussed about some aspects of the simple algorithm and we demonstrated the algorithm through some illustrative examples. Now, we will look into an algorithm which takes care of some of the limitations of the simple algorithm and tries to revise it in some form. It is known as simple revised or simpler algorithm. R for revised.

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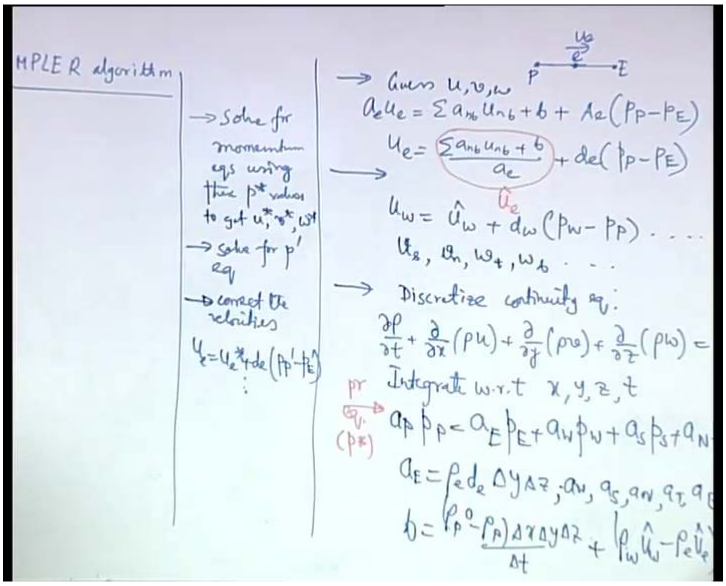

So, what is the motivation behind the revision? If you consider the simple algorithm, the heart and soul of the algorithm is the pressure correction. In the pressure correction, we have seen that the pressure correction does a good job in correcting the velocities, but pressure correction is not so good for correcting pressures, because you have omitted the neighboring velocity correction in terms for the correction of velocities. Hence, the pressure correction takes the sole burden of correcting the pressure and correcting the velocities.

When it corrects the velocities, it takes the burden of correcting the velocities even of the neighboring velocity correction, terms because those terms are omitted. It takes the burden of those terms as well. In the process, it corrects the velocities fine. But it exaggerates itself, it over expresses itself, because it has to now take the burden of another correction term by itself and in doing so, it over estimates the pressure correction. So, the pressure correction ironically is not so good for correcting pressure. Although, the name is pressure correction, but it works fine for correcting the velocities. But in the simple algorithm when it is implemented for correcting the pressure, it is not so good because it over estimates the pressure correction. So, the simpler algorithm what it tries to do? It tries to obtain the pressure through a different route, but not through the pressure correction equation. So, how does it do it?

So, if you recall the momentum equation, if you consider the grid points like this. So, a e u e is equal to sigma a nb u nb plus b plus A e into p P minus p E. So, u e is equal to sigma a nb u nb plus b by a e plus d e into p P minus p E, where d e is a capital A e by small a e. This particular term let us give a name u e hat. Why we give it a compact name is because of the fact that instead of starting with a guess value of pressure, if you start with the guess value of velocity, then this term is known. So, in the simpler algorithm, the first change from the simpler algorithm is that you start with the guess value of velocity, not the guess value of pressure. So, when you start with the guess value of velocity, then this term is totally known. So, u e equal to u e prime plus d e into p p minus p e.

Now, you try to have this velocity formula satisfy the continuity equation and that will ensure that you have obtained a pressure field that satisfies the continuity equation as well. So, how do you do it? So, you have this as a formula for u e. Similarly, u w is equal to u w, u w hat plus d w into p w minus p p. Like that it goes on. So, this formula visibly is analogous to the formula corresponding to the velocity and pressure correction relationship in the simple algorithm. Just recall u e equal to u e star plus d e into p P dash minus p E dash. So, here the pressure corrections are replaced themselves by pressure and u e star is replaced by u e hat. So, it is very very similar. Therefore, the algebra is also very similar.

So, in the next step if you now, similarly, you have u s u. Oh sorry, v s, v n, w t, w b like that all with similar formula and then, you discretize the, so the first step was guess velocity field u, v, w. Then, based on the velocity field, you calculate the u hat, v hat, w hat like that. Those are straight forward calculations. Once you calculate u hat, v hat, w hat, then after that you discretize continuity equation, let us consider the general continuity equation as an example.

So, what you do is you integrate with respect to x, y, z and time and there in place of the velocities, you substitute the corresponding velocity hat plus d into that pressure difference. What is the similarity with the corresponding pressure correction derivation in the simple algorithm? It was same integrated with respect to x, y, z, t. In place of u, you substitute at u star plus d into pressure correction difference. So, the discretized equation will be the same, where the stars are replaced by hats and instead of pressure correction, it is now the pressure itself, right. So, you have a p p p is equal to a E p E plus a w p w plus a s p s plus a N p N plus a p p p plus a B p B plus b, where what is a e.

Just look into your previous notes for the simple algorithm and tell from that rho e d e delta y delta z. So, you will get similar terms for a w, a s, a N, a t, a B and what will be b? Rho p naught minus rho p delta x delta y delta z by delta t plus rho w. Instead of u w star, it will be u w hat. So, rho w u w hat minus rho e u e hat into delta y delta z, then plus similar terms. It is very trivial to write it. Just replace the star by hat. So, this equation what you get for the governing equation for pressure is called as pressure equation. So, in the simple algorithm pressure was guessed here. This is actually like a guess pressure, but not obtained from a direct guess of pressure, but from the velocity guess. Through the continuity equation, you have now got some approximate or guess value of pressure.

Now, using this guess value of pressure, you can now solve for the, so this will now become like p star. With this pressure value, now you can solve for momentum equation using these star values. Momentum equation means these ones, u e equal to u e hat plus d e into p p minus p e. Now, you know the pressure values. Now, you can calculate the velocities from the momentum equation. Of course, you do not now write it as a bulk or block quantity u e hat, but you can solve for the velocities because pressure is now already solved. Although, it is an intermediate value during iterations, so using these p star values, you solve for the momentum equation to get u star, v star, w star.

Now, because this pressure is not necessarily a correct pressure, these velocities are not necessary correct velocities. So, you need to correct these velocities. To correct the velocities, you can use the pressure correction equation. We have seen that the pressure correction equation does a good job in correcting velocities, but it does not do a good job in correcting pressures. So, pressure we have used the pressure equation directly and it has no approximation. See although the formula u equal to u hat plus d e into p p minus p e is algebraically similar to u equal to u star plus d e into p P dash minus p E dash. There is a difference in terms of parodying.

Here, there is no such approximation of omitting the neighboring terms in velocities. It is an exact formula without dropping any term whereas, when you consider u e equal to u star plus d into p P dash minus p E dash, you have already neglected sigma a n b u n B dash. That is the effect of neighboring velocity corrections on the velocity correction itself and on the pressure correction itself. These effects were neglected. Here, there is no such neglecting. So, the pressure equation is expected to a give reasonably good value of, reasonably accurate value of pressure depending on how your guess on velocity is, but the velocity anyway will be updated. It will be updated based on the pressure correction equation.

So, next step will be solved for p dash equation. Pressure correction equation is exactly same as that in the simple algorithm, exactly same. So, once you solve for the pressure correction equation, then next what you do? You correct the velocities, u e equal to u e star plus d e into p P dash minus p E dash similarly, other velocities. So, with these new correct velocities, you again go back to the iteration loop, where with now these velocities will be the new guess values of velocities. Earlier, you had some initial guess. Now, that initial guess will be replaced by this updated guess. With this updated guess, you now can calculate u hat, v hat and so on. Then, you can calculate the pressure and again the momentum equation may be solved for new velocities and those velocities may be further updated by using the pressure correction and so on. So, if you summarize the simpler algorithm, you can summarize it as it is follows.

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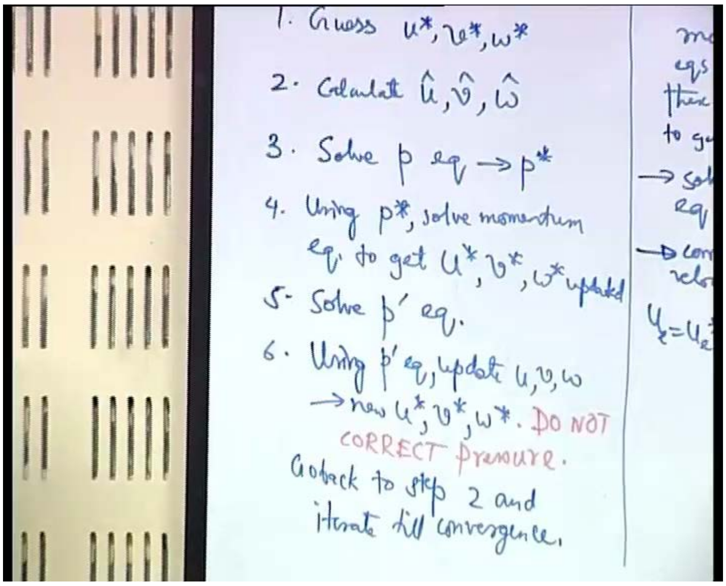

So, guess u star, v star, w star. Based on that you calculate u hat, v hat and w hat. Then, solve pressure equation. You get p star. Then, using p star, solve momentum equation to get u star, v star, w star updated. Then, solve pressure correction equation. Next using pressure correction equation, update u v w. These will be new u star, v star, w star. Important is do not correct pressure because we have obtained the pressure directly through the pressure equation. We have deliberately done so because we have marked that pressure correction is not good for correcting pressure. So, we should not correct pressure using pressure correction and then, go back to step two and iterate till convergence. So, this is the summary of the simpler algorithm.

Now, what are the advantages of the simpler algorithm we have already seen that. That seems that pressure correction in the simple algorithm does not do a good job in correcting pressure. That effect is now gone, that adverse effect is now gone. In the simpler algorithm by virtue of the consideration that you no more use the pressure correction for correcting pressure, you solve the pressure directly, but what is the cost that you have to pay. You now have to solve an additional equation which is the pressure equation. So, you have a computational burden or a computational overhead of solving one additional equation. One additional equation means, system of several algebraic equations you need to solve during each iterations. So, computational expenses go up, but at at the cost of that you may attain faster convergence.

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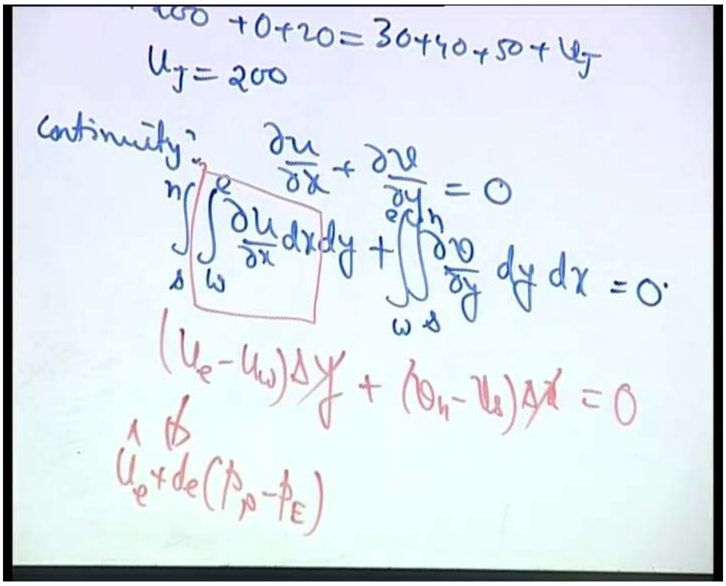

Let us try to work out one example to illustrate the use of the simpler algorithm. There is a square grid like this with various flow inlets and outlets are marked as AB. Then, you have C, D, E, F, G, H, I, J, K, L and the corresponding main grid points are marked as 1, 2, 3, 4. It is assumed that the flow is steady two-dimensional, density is constant and it is a square grid. Square grid is divided into four equal parts. So, for each part, you have delta x equal to delta y. The boundary velocities are given as V A equal to 30, V B equal to 40, U C equal to 100, U E equal to 50, U H equal to 200, U J equal to 210, V K equal to 0, V L equal to 20. It is given that amongst these given numbers, there is some doubt about the correctness of the value of U J, doubt about correctness of U J. If other numbers are correct, what is U J? That is the first part of the question.

Then, the second part. The momentum equations for the velocities of the interior grid points are given U D is equal to 70 plus 0.5 into P 1 minus P 2, U I is equal to 10 plus 0.7 into P 3 minus P 4, V F is equal to 30 plus 0.5 into P 3 minus P 1, V G is equal to 18 plus 0.8 into P 4 minus P 2. Then, your objective is to derive the pressure equations and hence, solve P 1, P 2, P 3, P 4. Hence, solve them to obtain P 1, P 2, P 3, P 4 and using that, you obtain U D, U I, V F and V G, ok.

So, let us proceed step by step. Let us mark the values which are given. So, you are given V A equal to 30, V V equal to 40, U C equal to 100, U E equal to 50, U H equal to 200, U J equal to 210, U V K equal to 0, V L equal to 20. There is some doubt about the correctness of U J. So, how will you find out what is the correct U J? You have to satisfy the continuity equation. It is a steady two-dimensional flow with constant density. The net rate of inflow must be same as the net rate of outflow. So, summation of rate of inflow is equal to summation of rate of outflow.

So, what is the rate of inflow? 100 plus 200 plus 0 plus 20. Rate of outflow is 30 plus 40 plus 50 plus U J. Remember that the corresponding areas of the phases through these velocities are flowing basically assuming to be delta x equal to delta y. Those get cancelled out. Those multiplying factors of the surface areas, they get cancelled out. So, from here you can find out what is U J. What is U J? 200. So, this 210 is not correct. It should be replaced by 200.

The next step will be to derive the pressure equation. That is what is given. Momentum equations are already given in some derived form. So, to derive the pressure equation, let us say that you have two-dimensional control volume P, E, W, N, S. You have delta x, delta y as the dimensions. In this example, delta y equal to delta x. So, what is your continuity equation? Del u del x plus del v del y equal to 0. Now, what is the next step? Integrate with respect to x and y.

So, here if you integrate it by considering the first inner integral, it is u e minus u w that times delta y. So, u v minus u w into delta y plus v n minus v s into delta x equal to 0. Delta x and delta y are equal. In place of u e, you write u e hat plus d e into p P minus p E, right.

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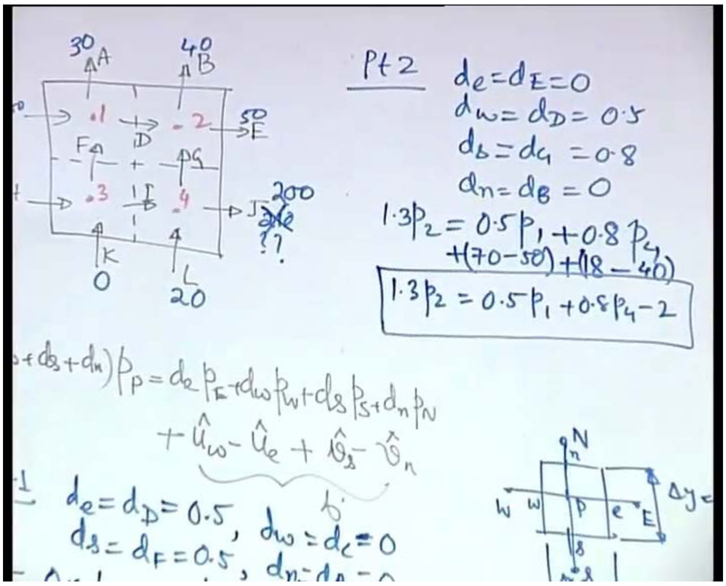

Similarly, you write for the other 3 terms. Once you write that, then what is the equation that you get. So, you get de plus d w plus d s plus d n into p p is equal to d e p E plus d w p W plus d s p S plus d n p N plus u w hat minus u e hat plus v s hat minus v n hat. That is the term b. Now, let us apply this to the grid pints 1, 2, 3 and 4. There are four control volumes here. You can see there are four compartments. So, for grid point 1, what is d e? D e is nothing, but d of capital D, right. What is the value of this? Look into the momentum equations given u d. It is just like u d equal to u d hat plus d d into p 1 minus p 2. So, d d is 0.5. Then, d w. Why is d w 0? D w is equal to d c, but u c is given. So, there is no correction for, so there is no, so u c is equal to u c given. That is no additional term that is necessary for that. So, d C is equal to 0. Then, d s that is equal to d F is equal to 0.5 and d n is equal to d A. That is equal to 0 because v A is given. So, what you get is 0.5 plus 0.5 that is p 1 is equal to 0.5p 2, d w is 0, d s is 0.5. So, plus 0.5p 3 and then, d n equal to 0 plus u w hat. U w hat is same as u w. So, that is 100 minus u e hat. U e hat is u d hat, that is equal to 70 plus v s hat is equal to v f hat, that is equal to 30 minus v n hat equal to, what is v n hat. V n hat is equal to v a, that is equal to 30. So, if you simplify, you will get p 1 is equal to 0.5p 2 plus 0.5p 3 plus 30. This is the governing equation for p 1.

Similarly, you will get equations for p 2, p 3 and p 4. Let us try to write those bit quickly because we know how to write that. So, for 2, what is d E? D E is 0 because u e is given. Then, what is d w? That is d D that is equal to 0.5, d s is equal to d g equal to 0.8 and d n equal to d B. That is equal to 0. So, 0.8 plus 0.5, that is 1.3p 2 is equal to 0.5p 1. The W grid point is 1, the corresponding D is 0.5, 0.5p 1. Then, 0.8 p 4 plus u w hat, that is u d hat that is 70 minus u e is 50, u e hat is 50 plus v s is v g hat 18 minus 40. So you have, if you simplify this equation, it would be 1.3p 2 is equal to 0.5p 1 plus 0.8p 4 minus 2. So, with these grid points as example, you have now got the idea how to derive it. So, let me just write the corresponding final expressions for the points 3 and 4 which you can check at home.

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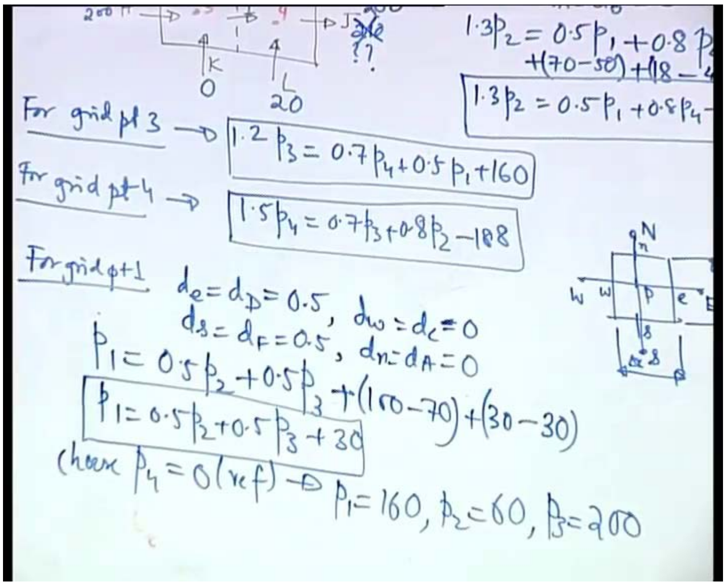

So, for grid point 3, you will get 1.2p 3 is equal to 0.7p 4 plus 0.5p 1 plus 160 and for grid point 4, it is 1.5p 4 is equal to 0.7p 3 plus 0.8p 2 minus 188. Now, you have to remember one thing that pressure is a relative quantity. So, it is not the absolute value of pressure that you are looking for. Why? See at no point you are given any value of pressure. You are given the values of velocities and these similar velocities maybe achieved by a corresponding pressure difference. So, that pressure difference remaining the same, absolute values of the pressure may be different. So, for example, when you are writing p 1 minus p 2, it is not important what is p 1 and what is p 2? Of course, if you specify p 1, then p 2 is also specified to have a fix p 1 minus p 2, but you have neither specified p 1 nor specified p 2 nor p 3 nor p 4. So, if you specify anyone, then the remaining ones will follow. Whatever you specify anyone that will act as a reference, just as a reference. So, different choices of pressures at some reference point will give you different values of the pressures at the remaining points, but the pressure differences will remain the same and those are which are responsible for the momentum equation solution.

So, if you choose p 4 equal to 0 just as a reference, then from these equations, you can solve for p 1 is equal to 160, p 2 equal to 60, p 3 equal to 200. These are the values which you can check later on. You have solved for the pressure equation to get the values of pressure. So, we have come up to step 3 in the simpler algorithm, where we have used the pressure equation. What would be the next step? Using the pressure equation now, that is using the solution of the pressure equation, solve for the momentum equation to get u star, v star, w star updated. So, let us do that.

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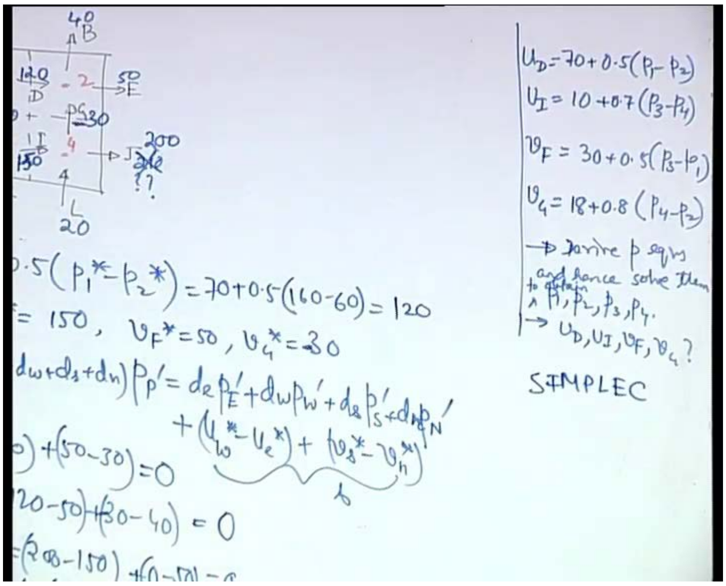

So, U D star is equal to 70 plus 0.5 into p 1 star minus p 2 star. So, 70 plus 0.5 into 160 minus 60. So, this is 100 into 0.5, 50 plus 70, that is equal to 120. Similarly, let me just write the other results. U I star is equal to 150. Then, V F star equal to 50, V G star equal to 30. How do you obtain this? By using this momentum equations and substituting the values of pressure as p star. Once you substitute pressures as p star, you will get corresponding U D star, U I star, V F star and V G star. So, let us put those values that we have solved in this figure. So, U I star equal to, where is I here is I. U I star equal to 120. Then, U D star is there, sorry this is 150. This is 150. U D star is 120. Then, V F star is 50, V G star is 30.

So, we have come up to which step? We have solved for the pressure correction equation. How we solved for the pressure correction equation? Not yet. So, you can solve for the pressure correction equation now. What will be the pressure correction equation? It will look same as that of these momentum equations. So, it will be p dash equations U D. So, d e plus d w plus d s plus d n p P dash is equal to d e p E dash plus d w p W dash plus d s p S dash plus d n d small n p capital N dash plus u w star minus u e star plus v s star minus v n star. This is the term b. Again, very similar to the pressure equation. Thus, just switching between the hat and the star. So, this we have already derived for the pressure equation.

So, just using the analogy, we have changed the pressure to pressure correction. We have changed the hat to star. So, now solving the pressure correction equation, in addition may require to have a look at term b. So, if the term b is 0, that means it has converse because the continuity equation is satisfied for each control volume because this is the inflow minus outflow for each control volume. So, let us check it for the first control volume that is the top left. What is the total inflow? Inflow minus outflow along x, 100 minus 120 and along y, 50 minus 30. So, this is equal to 0. This is for the top left control volume, top left c v. Then, top right c v. We are calculating the terms b. B equal to 120 minus 50 plus 30 minus 40.

Seems it has, so this you have 60. We need to check the calculations also once. The numbers that I have given I have representative ones, but you need to check and figure out. So, then the next ones. Some of these may be plus or minus, but just let me check. Sorry v g is minus 30, v g star. Yes this is the one of the answers that I have given. V g is equal to minus 30, it is not plus 30. Just rectify it because I was wondering that, I mean it should come out to be 0. Continuity satisfying velocity field should be obtained here. So, it is just a matter of doing the correct algebra. You should come up with v g equal to minus 30. Please correct it. So, with this minus 30, you come up with 70 minus 70. This is 0, bottom left control volume b equal to 200 minus 150 plus 0 minus 50 that is 0. So, in the figure we show v g in in this direction fine, but we write it as minus 30 to indicate that it is actually the downward direction. Then, bottom right control volume, 150 minus 200 plus 20 plus 30 that is 0.

So, for all the control volumes, the mass source term is 0. That means the problem has converged. So, we have already got the converged velocity fields. We have got the pressure corrections. The pressure corrections are 0. Even if the pressure corrections were non 0, that is even if the velocity field was not converging at this stage, you still should not use the pressure correction to correct pressure. That is very very important, but in this particular problem in one step it has converged. So, no such possibility arises.

Now, there are several improvements of the simple or simpler algorithms and those have been reported in the subsequently literature. I mean there is an algorithm which is called as simplec algorithm so and so forth. In this particular course, our objective is not to go describing all these algorithms. Our objective is to understand the basic philosophy behind these algorithms and for that the simple algorithm which is the basic one happens to be the most important and the simpler is just a revision to that, one of course with proper objectives in mind. That is why such revision is necessary, but perhaps the simple algorithm is one of the very key algorithms in modern day cfd because using that as a basic algorithm, you get a clue of how to tackle pressure. Although, you do not have a separate governing equation for pressure.

So, how to use the continuity equation to address the changes in pressure or corrections in pressure or how to correct pressure to come up with the continuity satisfying velocity field? That particular concept, it is a concept that is important more than the description of the algorithm. That particular concept is brought out from the simple algorithm and all other derivatives of the simple algorithm follow that particular concept. So, that concept is that what needs to be highlighted while we are discussing about the solutions of the Navier Stoke equations.

Now, so far we have discussed the solutions of various equations, diffusion type of equations, convection diffusion equations and eventually fluid flow equations, using a grid system which we call as a structured grid. So, what is a structured grid? Instead of giving a formal definition, let us try to understand it physically or conceptually.

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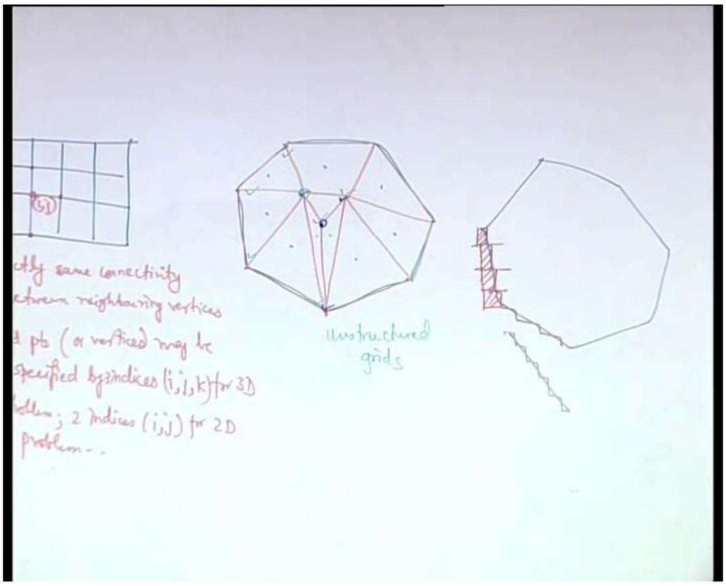

So, if you have set of control volumes like this, then there are certain aspects or characteristics of such layout. What are the aspects or characteristics? If you see that if you consider these internal boxes as rectangles, then each vertex is connected with 4 other vertices, neighboring vertices. Then, exactly the same repeatable manner for all points except, of course the boundary points. So, connectivity between the neighboring vertices is same. So, strictly same connectivity between neighboring vertices.

That is the first thing. The other thing is that because the grid lines are laid along respective coordinate directions, it is possible to give an index to each vertex or even a grid point as i, j for a two-dimensional problem and by using 3 indices for i, j and k for a three-dimensional problem. So, grid points or vertices maybe specified by indices, 3 indices i, j k for 3D problem, two indices i, j for 2D problem and so on. So, this is one grid layout. Now, let us say that you have a domain like this.

Let us say that somehow you have divided the domain into a number of sub-domains like this. The sub domains, it is a perfectly valid one. Nobody has told us that sub-domain has to be a rectangular one. So, you can use different shaped sub-domains. We will see later on that what shaped sub-domains are important for what, but one of the important things that we need to remember is that, it can be of any shape. Now, once you have whether it is a valid domain discretization or not, that we will discuss later on, but the first thing to observe here is that if you consider a vertex, here you can see this vertex is connected to how many neighboring vertices. 3 neighboring vertices. Let us consider another vertex. Let us consider say, this vertex, it is connected to how many neighboring vertices? 1, 2, 3, 4, 5, 6. So, it is not strictly the same connectivity between the neighboring vertices. Not only that, see the grid points which may be representatives of each of the control volumes, say the centroids of each control volume, they are located at arbitrary locations in a space in a way that you do not have them located in a particular laid out grid lines. Neither along x nor along y nor along z. That means you cannot use structured indices i, j, k to specify a particular location of a grid point. It has to be specified by a single number with some other connectivity's amplifying that number, but not by indices i, j, k like that. So, this type of layout of grid is known as unstructured grids.

Why the unstructured grids are necessary? Most of the times these are necessary to hanging the complicated nature of the domain boundary. So, if the domain boundary is regular, you will always prefer to have a structured grid layout. You can of course use a structured grid layout even for a regular domain boundary. When structured grid was not there, what people use to do?

People use to represent these each steps like this, where there will be certain counter column which are blocked off, so that the domain is represented whatever is a line like this is represented by steps and whatever part is blocked off, they are artificially some other properties are. For example, very high viscosity so that there is no flow. So, these types of representations are given. So, what we can see here is that the unstructured grid is important for representing the domain bounded in a very realistic form without going for such approximation and the philosophy of discretization in unstructured grid is something which is not a straight forward extension of philosophy of discretization for structured grid. It is not a straight forward extension what it has a lot of resemblance with that one. So, what is that similarity, what is that dissimilarity we will look up into that matter in our next class. Thank you