Computational Fluid Dynamics

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Lecture No. # 35

Discretization of Navier Stokes Equations

Till now we have looked into the solution of diffusion type of problems and convection-diffusion type of problems. Now in solving convection-diffusion problems, we assume that the velocity field is known from somewhere and using that velocity field, other scalar fields are solved. Now question is where from the velocity field will be known? The velocity field of course, can be known by solving the fluid flow equations, that means the continuity equations and Navier stokes equations, which are to be solved together.

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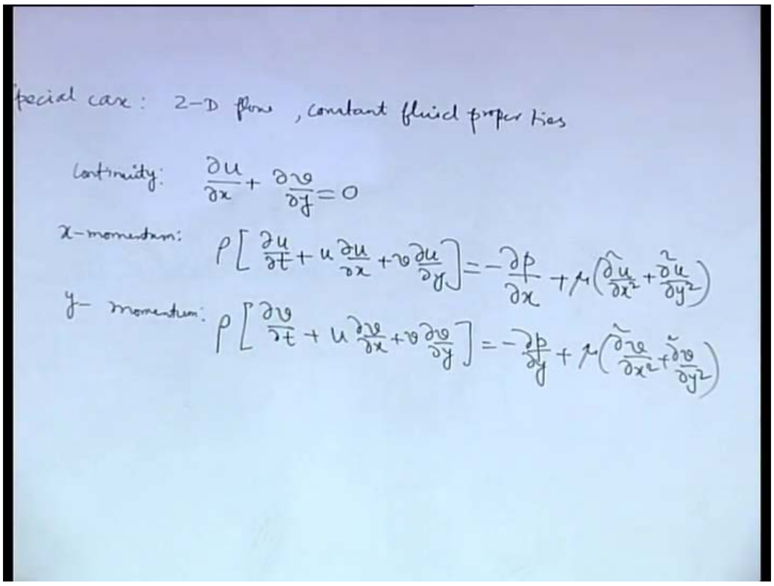

So, let us say that we are interested for a special case just to begin with of a two-dimensional flow with constant fluid properties. This is just to give an example of what can be the forms of the governing equations and what are the corresponding challenges. The same concept can be extended to more complicated flows, but let us stick to these special case.

So, you will have continuity equation as this then x momentum we can write the momentum equations in either conservative form or non conservative form, but let us just to begin with write in a non conservative form. We will use the conservative form for solving the equations in a finite volume frame (( )), but, let us consider just consider the non conservative forms to begin with. So, rho this is a x moment equation and similarly, the y momentum equation.

So, what are the features of this equation we have not considered any body force you can generalized by adding some body force that I can sight. Now what are the important features of this equation? First of all they are coupled next they are non-linear. They are non-linear because of the presence of advection terms in the left hand side. So, u del u del x like that these are non-linear terms.

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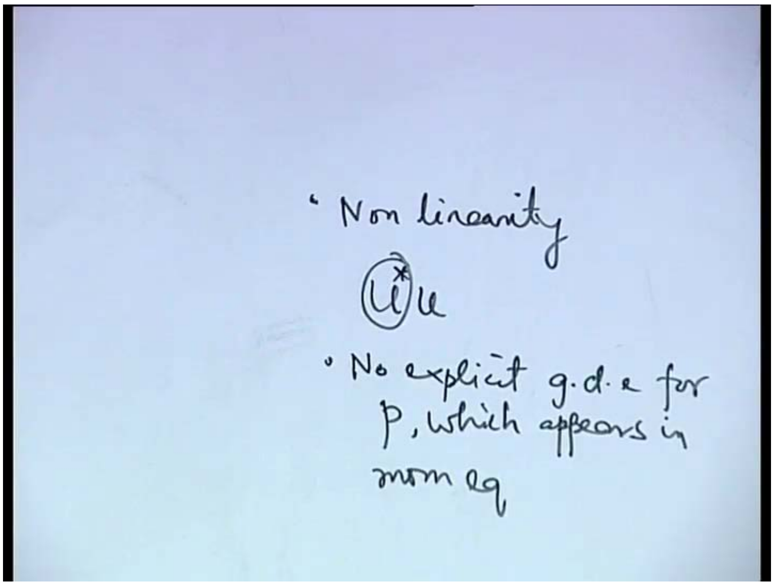

So, non-linearity is one of the important considerations. But, non-linearity is something which can be addressed to some extent in an iterative framework, which is well understood that for example, if you have u into u. So, one of the u, you can consider as the value from the most recent iterations which starts with the initial guess. So, it becomes a pseudo linear variable, where u is like a constant at each level of iteration. So, it is possible to use a pseudo linear frame work even if it is a non-linear by using a iterative scheme, that is well understood. But, the other issue is a bit more complicated that see the fundamental is a momentum equations are also convection diffusion equation. So, the question is we have already studied how to solve the convection diffusion equations. Why do we require to study separately how to solve fluid flow equation? Because fluid flow equation is also convection diffusion equation with some source term, which is a pressure radiative negative of the pressure radiative. Where this is the advection term and this is the diffusion term and this is like the source term.

So, the issue is that yes this is also a convection diffusion type of problem, but, it has a source term which is a variable, that is the source term in which you have pressure as a variable, but, you do not have a separate governing equation for pressure. So, you have a situation where you are having to deal with an undetermined variable as a source term, where there is no explicit governing equation for that undetermined variable. So, no explicit governing differential equation for pressure which appears in momentum equation. So, this gives a great challenge in terms of the numerical approach of solving these equations and one has to deal with it in a particular way. So, we will try to move in a somewhat systematic way. We will try to see that what initially people thought of in terms of handling the pressure and then subsequent advancement in terms of handling the pressure term or pressure radiant term.

Now, we can see that the origin of the problem is because of the presence of the pressure radiant. So, one of the ways to overcome the problem may be to eliminate the pressure term itself. So, it is like you have headache. So, to address the problem you cut your head if you cut your head there will be no headache right you will have headache only when you have head people who do not have head have no irritations in mind because they never have headache. So, you are irritated by the presence of the pressure radiant and what you can do you can eliminate the pressure radiant to avoid this problem and that is the approach attempted in the methodology known as stream function verticity approach.

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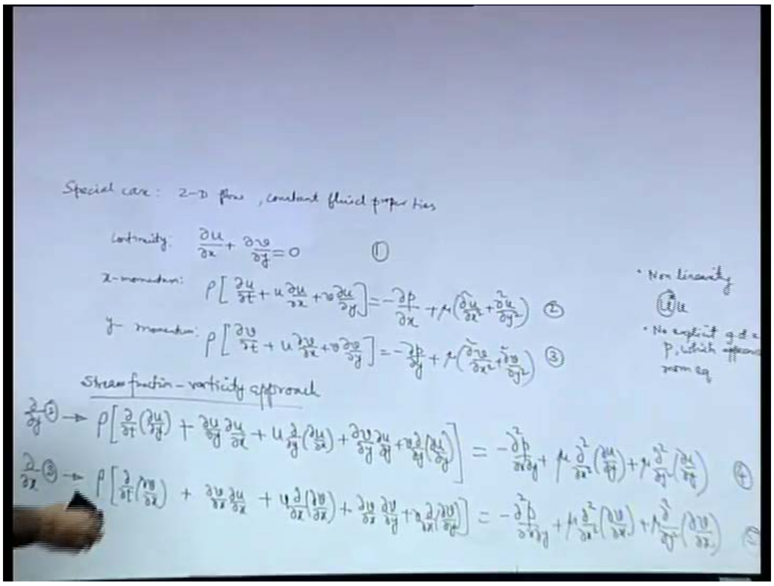

So, let us first consider the stream function verticity approach. So, the primary objective of this approach is to eliminate the pressure radiant and write governing equations in terms of the variables stream function in verticity. So, let us number these equations before we manipulate on them. So, how can we eliminate pressure radiant we differentiates equation two partially with respect to y differentiate three partially with respect to x and subtract.

So, partial differentiation of equation two with respect to y. Remember that these are continuous partial derivatives. So, you can either differentiate with respect to y first or differentiate with respect to t first. So, just for convenience in further manipulation we have written the differentiation with respect to y segment then this is the left hand side let us write the right hand side. So, the right hand side we have completed now. Let us write the corresponding terms for equation three and then we will subtract. The last term will come now. And the right hand side. So, once we have written these equations let us give to the equation some number equation four and equation five let us subtract these two.

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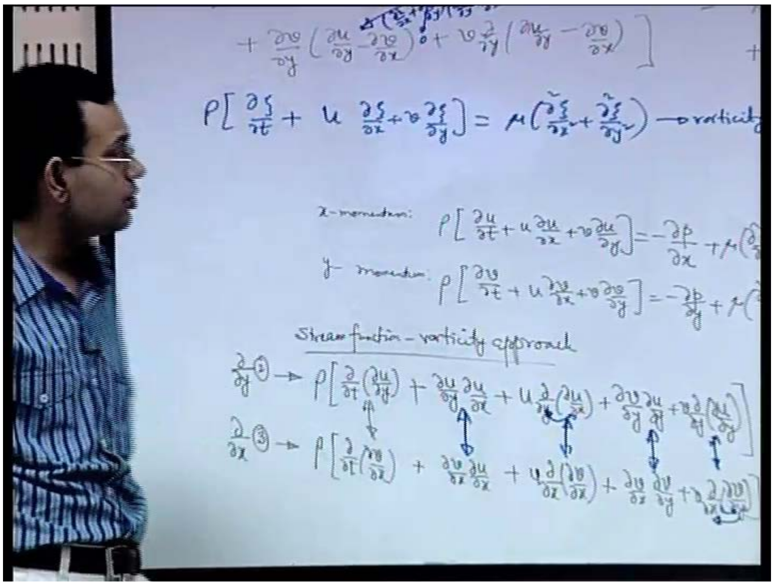

So, equation four minus equation five. So, before subtracting let us try to identify some common terms a similar terms first these two terms are similar. So, if you combine these two you will have row del del t of del u del y minus del v del x then if you consider these two terms you can take del u del x common del u del y minus del v del x. Next if you consider these terms you can take as u common and here you can swap the x and y. So, that you can write plus u del del x of del u del y minus del v del x.

The next term you can take del v del y x common. So, del u del y minus del v del x. Finally, v del del y of del u del y minus del v del x, here also you swap the x and y in the last term. So, that is the left hand side now let us write the right hand side. First two terms cancel then mu plus mu del two del y two del u del y minus del v del x. Now you can further simplify by combining these two terms.

So, here del u del y minus del v del x is common. So, it will become del u del x plus del v del y into del v del y minus del v del x and del u del x plus del v del y equal to 0 by continuity equation. In all other terms you can see that there is a common parameter del u del y minus del v del x.

So, del u del y minus del v del x is what? It is the So, you can write say zeta is equal to del v del x minus del u del y, which you may call as verticity is just a matter of since, you are using it like a scalar variable because the verticity is with respect to an axis which is perpendicular to the x y plane in verticity with respect the zee axis.

So, it does not matter whether you take it plus of this or minus of this all the same but, let us say that we take it in these particular form. So, if you take it in these particular form say it is the same. So, it becomes row del zeta del t plus u del zeta del x plus u del zeta del y is equal to mu del 2 zeta del x 2 p plus del 2 zeta del y.

So, this equation is known as verticity transport equation it is governing equation for transport of verticity. Now, still it involves u and v all the pressure is eliminated u and v is there. So, there must be other parameters that are connected to u and v and in two dimensional incompressible flows the parameter that connects u and v is the stream function.

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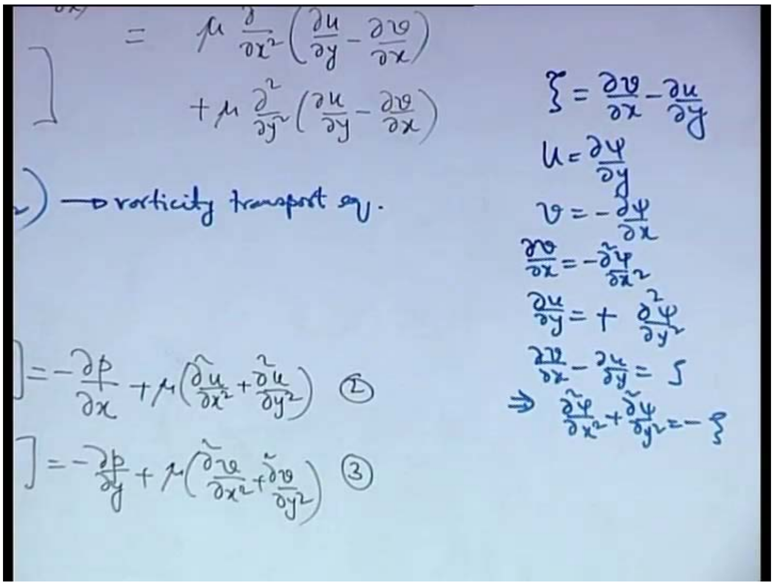

So, you can define u is equal to del shi del y v is equal to minus del shi del x again anyone of these could be a plus sign another could be a minus sign the objective is to satisfy the two dimensional incompressible form of the continuity equation. So, if we define for example, in this way then how can you write del v del x minus del square shi del x square then del u del y plus del square shi del y square. So, del v del x minus del u del y equal to zeta implies del square shi del x square plus del square shi del y square is equal to minus z.

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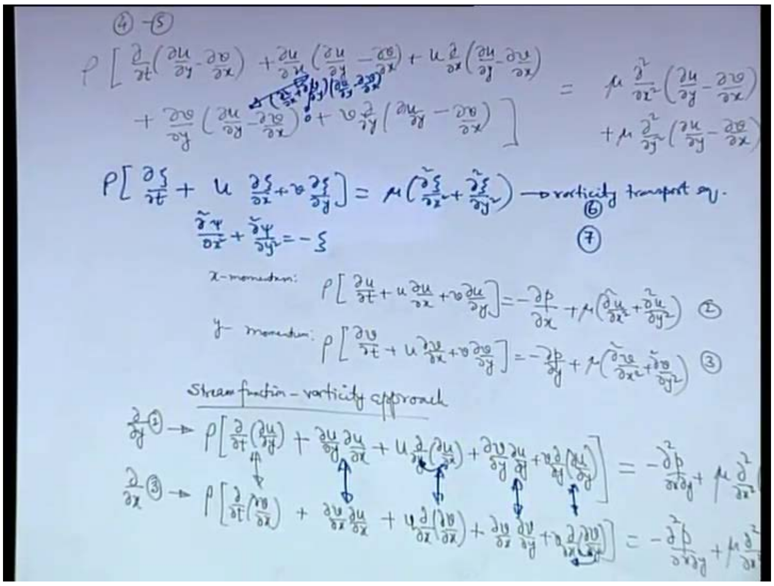

So, you can have the governing equation for stream function, which couples the stream function with the verticity in this way. So, this verticity transport equation let us call it equation number six and let us call the stream function equation as equation number seven. So, then what we can say is that these are two coupled equations and numerical solution of these two coupled equation from the basis of the stream function verticity approach. So, how these equations are numerically solved.. So, you must have the stream function field and the verticity field at time equal to 0.

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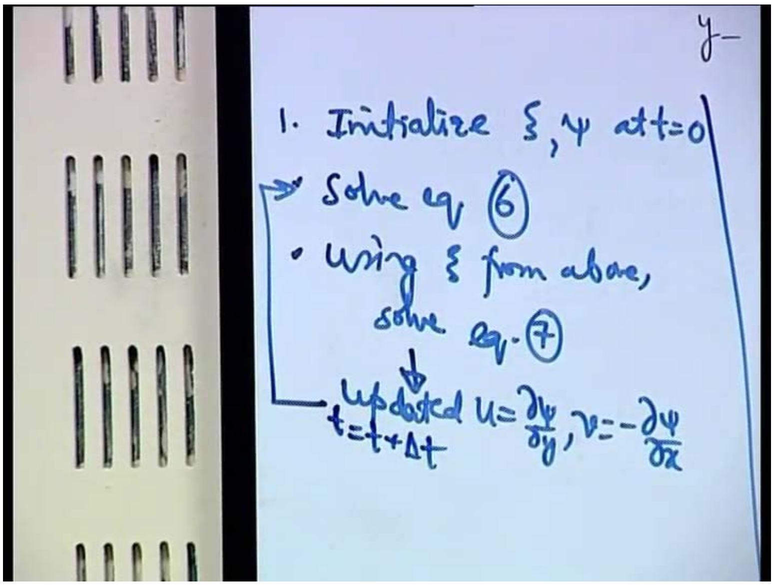

So, let us write it somewhere. So, step one you have initialize zeta psi at t equal to 0. Then once you know initial psi initial zeta that is this is have initial guess then from there you can calculate what is initial u and what is initial v. Because u is del psi del y and v is minus del psi del x. So, using that u and v now you can solve these equation six like a convection diffusion equation ,where the variable is the verticity.

So, solve equation six. With that verticity as the source term now you can solve equation seven what is the nature of the equation seven what type of the equation is this? This is the Poisson equation if the right hand side is 0 that is called as Laplace equation.

So, if it was del square psi equal to 0 that would have been the Laplace equation here it is not 0 there is something in the right hand side. So, that is called as a Poisson equation. So, that is also a standard equation standard partial differential equation in numerical science and therefore, one can elegantly solve this, using whatever scheme that one can use, one can choose it is possible to solve this with zeta the information of the zeta from equation six that can be used as a source term.

So, solve using zeta from above solve equation seven. Remember this is at a particular time, at that particular time you are solving the stream function equation if you recall the definition of the stream function here you do not have any reference of time. So, at a given time you are calculating this u v on the basis of the relationship with the stream function. So, at that given time you have to solve this equation may be by using some iterative technique.

So, using zeta from above solve equation seven once you solve equation seven you have updated u is equal to del psi del y and v is equal to minus del psi del x. Then based on this updated u v you go for the solution of the zeta again to equation six. So, again you go and solve equation six for the next time. So, you have t equal to t plus delta t, new time.

So, there is some outer iteration where the iteration variably staying and within that there is some inner iteration, where you have the iterative solution of this equation as well as this equation at a given time. So, you first you freeze time at a given level of time you have special variables those special variables you solve by an iterative technique and then you go for the next level of the time.

That is how you can march in the outer iteration with time and the inner iteration you have this is like an elliptic equation so you it is not a marching problem. So, you have to solve by any of the iterative schemes or elimination schemes whatever is convenient, but, you cannot march in time for this equation because this is not a marching type of problem.

Now, it is straight forward to destroy the solution of stream function verticity equation, but, we have not yet discuss about the corresponding boundary Conditions, the boundary conditions on velocity pressure these are physical (( )) variables. So, the boundary conditions on these are intuiting like you can say that velocity is something pressure is something, but the boundary condition on stream function in verticity stream function and verticity these are somewhat derived quantities.

So, these are not the fundamental variables of fluid flow, fundamental variables of fluid flow are velocity and pressure. So, these are derived from the fundamental variables and therefore, it is not so, trivial to impose a boundary condition on stream function in verticity. So, let us see that how we can give a boundary condition on stream function.

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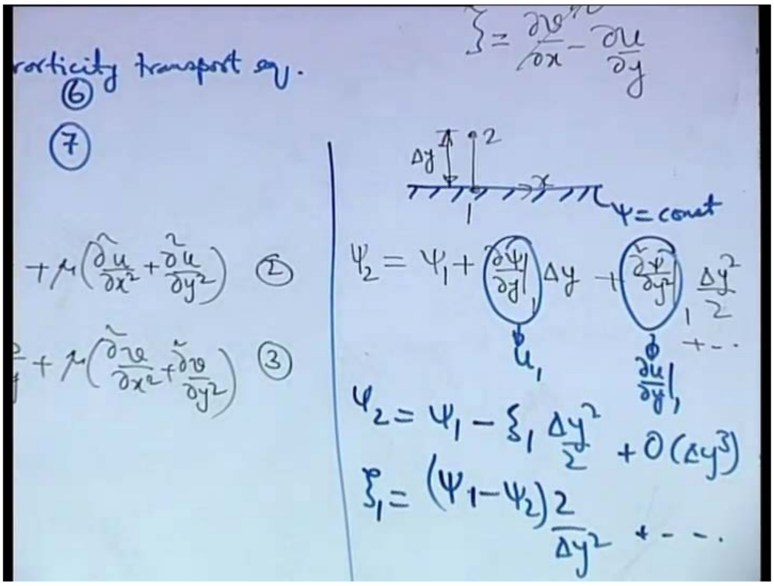

Now, let us see how we can give the boundary condition on stream function in a bit more straight forward way and then the boundary condition on verticity. We can give boundary condition on stream by noting that the solid boundary is like a stream line. So, if you recall that what are the stream lines?

Stream lines are lines are lines such that flow is always tangent to it at any point the velocity vector is always tangent to it at any point; that means, you do not have any flow normal to the streamline, when you have impenetrable solid boundary by no penetration boundary conditions. There is no penetration of flow no flow perpendicular to the solid boundary and therefore, the solid boundary across which there is no penetration acts like a streamline.

So, you it will have a particular value of the stream function. Because we know that the stream function is constant along a streamline. So, if you have a solid boundary like this, so, you can call it psi equal to constant like this. Now, what would be the boundary condition on that verticity that you can give on this boundary? So, let us try to look into that because verticity boundary condition is something again, which you need to derive at it need not follow straight forward from the description of the physical problems and these aspects were.

So, important when people started using the stream function verticity method that there were research papers was some papers totally devoted on how to give verticity boundary conditions for some complicated problems. Just to describe how to give a boundary conditions because these were not trivial at least, when this method was first establish.

Now, let us not go into such complications let us considered as a simple case where you want to give verticity boundary condition at this particular boundary. So, to do that let us considered that there is a grid point one at the boundary and grid point two, which is located at a distance of delta y perpendicular the line one two is perpendicular to the solid boundary. So, we can write psi 2 let us expand the stream function at two in terms of stream function at one by using the tailor series expansion. So, psi two is equal psi one plus del psi del y at one into delta y plus and so, on. Now, it is possible to write the stream function in terms of the velocity. So, you can write this as u at one this is del u del y at one remember that we have defined zeta has del v del x minus del u del y ,what is a value of v at one, zero. So, you do not have any del v del x because v is constant along x at one So, there is no variation of v along x at one. So, it is a uniform zero v velocity along the wall.

So, there is no x this is x direction there is no x radiant of v on the wall. So, you can write this as psi 2 is equal to psi 1 what is u at one it is zero by (( )) boundary conditions what is del u del y at 1 it is equal to minus zeta 1 plus order of delta y cube so, on. So, you can write zeta one as psi one minus psi two into two by delta y square then of course, the added term.

So, using the Taylor expansion you can artificially generate a boundary condition for the verticity in terms of the stream function. So, the stream function at the boundary psi one you can set to 0, if you want or some constant value this one will come from the iterations. So, once the value of psi 2 is known then only based on that you can calculate zeta 1 once when it is not known initially it is still prescribed by some initial guess.

So, that you can see that you have to go to an iterative scheme because you must be knowing both psi one and psi two to define the boundary condition zeta one. In an iterative framework even if you have not solved psi two you have some recent guessed value of psi two and psi one is the properly prescribed boundary condition. So, you can see that it is not very trivial to give the boundary condition for the verticity.

Now let us try to modify this situation little bit and let us say let us workout a problem, where we considered that instead of a noisily boundary condition there is a slip boundary condition. So, if there is a slip boundary condition then how you write the stream function verticity relationship as a boundary condition.

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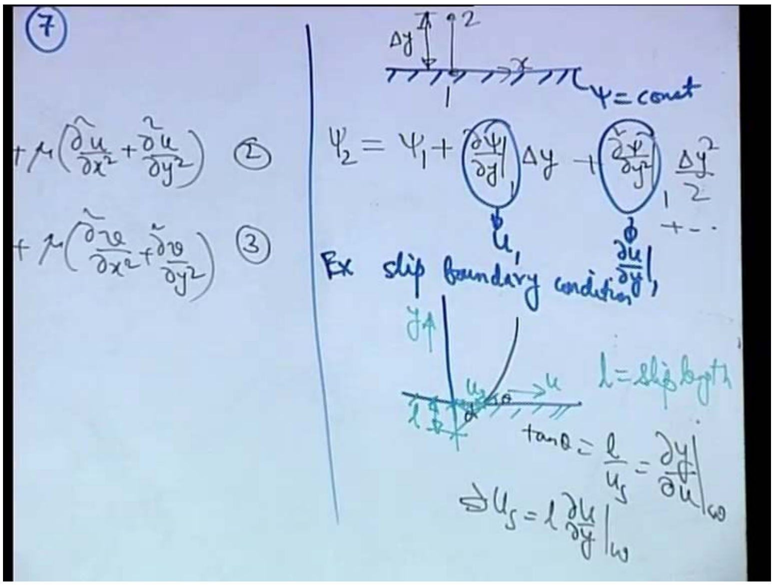

So, example slip boundary condition. So, the slip boundary condition may be possible because of different physical reasons we will not go into all those physical reasons here. This is not in this particular numerical modeling scope, but we will just consider that there is some physical mechanism by which no slip boundary conditions may be violated at the surface. This is common in many cases for example, for rectified gases, where you have not sufficiently dense pack packing of the gas molecules.

So, then the gas molecules can easily escape from their positions which could be occupied in the contact with the solid boundary and there can be a relative motion between the gas molecules and the solid boundary in a relatively ratified condition. There could be several other possible mechanism of slip now, once you have such a slip you have the velocity profile which does not show 0 velocity at the fluid solid interface.

So, then what you do, you draw a tangent to the velocity profile and that extra polite will meet the vertical line at 0 velocity at a distance l from the wall. So, as if the wall has been shifted by if the wall is shifted by length l then that would have represented a noisily boundary condition this l is called as a slip length. So, this is u this is y this is the slip velocity at the wall u s.

So, from the simple trigonometry we can say that if this angle is theta this angle is also theta what is this tan theta. Tan theta is equal to l by u s and that is equal to or if you just tried u as a function of y only it is d y d u at the wall. So, we can write u s is equal to l d u d y at the wall. So, then in this boundary condition u one is not 0, but u 1 is equal to l into d u d y at one.

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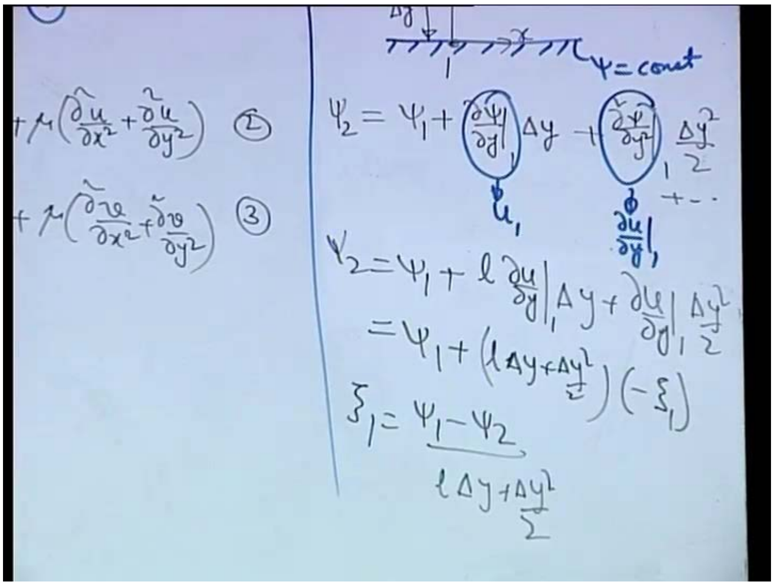

So, psi 2 will be equal to psi one plus one l u d y at one into delta y plus d u d y at 1 into delta y square by 2 and the added terms. So, it is equal to psi one plus l delta y plus delta y square by 2 into d u d y at 1, which is equal to minus zeta at 1. So, zeta at one will be psi one minus psi two by l delta y plus delta y square by 2 this is most general then the case with no slip for the case with no slip l equal to 0 and if you put l equal to 0 in this formula you will get back the corresponding verticity stream function relationship at the boundary for no slip condition.

Now, the next question is once you know the verticity and the stream function can you get back the pressure. See you eliminated the pressure because it created problem, but at the same time you may need to get back the pressure because pressure is essential flow variable in many of the fluid flow problems. So, how do you recover pressure? To recover pressure should once you know u and v you can set up a governing equation for pressure how you can do that, let us say that you differentiate now equation 2 with respect to x and 3 with respect y and add together.

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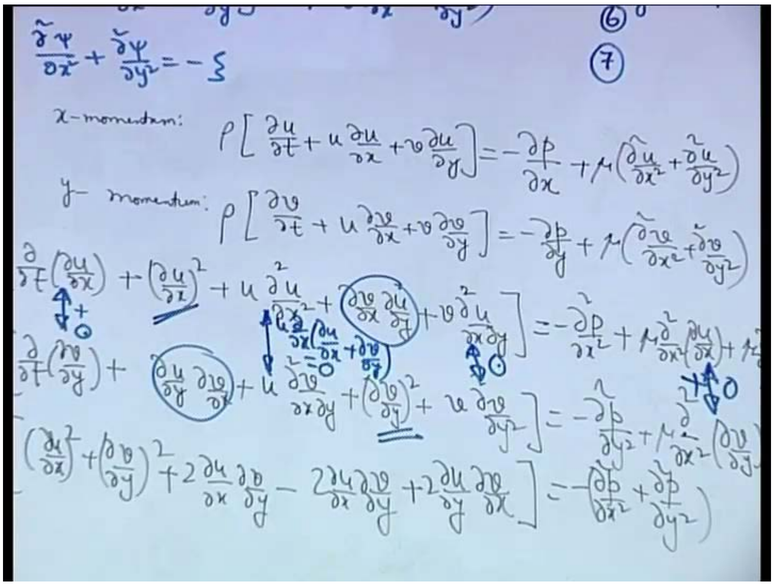

So, del del x of 2. This is what you get if you differentiate with respect to x. Similarly, you differentiate with respect to y equation (( )) 3 plus u del square v del x del y plus del v del y whole square plus v del square v del v del y square (( )). Then what we can do we can add these two. So, if you add these two this will be del del t of del u del x plus del v del y by continuity that will be 0. So, the unsteady term is gone. So, one of the objective that this manipulation satisfies is that unsteady term is eliminated then the remaining terms are there you can simplify it or you may not also simplify it if you the right hand side it can get simplified very conveniently.

So, it is mu del 2 u del x to into del u del x plus del v del y this is also 0 by continuity equation if you add. So, let us give some numbers to this equation let us this is 8 this is 9. So, we are now adding 8 with 9. So, this addition will be 0, this addition will be 0, this addition will be 0 by continuity equation ,left hand side you can simplify in several ways you can write rho del u del x whole square plus del v del y whole square plus you can write it as a a plus b whole square formula plus two del u del x del v del y. Then you subtract again 2 del u del x del v del y. So, that means, you have considered these terms and these terms then you have 2 del u del y del v del x from these two terms then you can take u del del x as common here that will be.

So, if you consider these two terms you can take u del del x as common with del u del x plus del v del y. So, that is 0 similarly, if you combine these two terms that will be 0. So, you are left with minus del square p del y x square plus del square p del y square . So, this part is 0. Because this is del u del x plus del v del y whole square.

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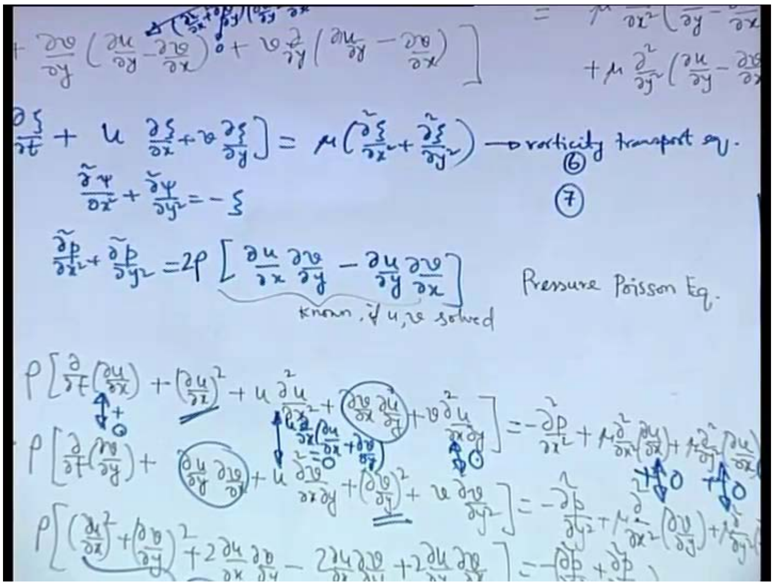

So, you have the governing equation for pressure as del square p del x square plus del square p del y square is equal to 2 rho into del u del x del v del y minus del u del y del v del x once u and v have been solved from by using the stream function verticity approach this is the right hand side which is known if u v are known if u v are solved.

So, then what type of equation is this, this is a Poisson equation, but, now the it is a governing equation for pressure not stream function. So, it is called as a pressure Poisson equation. So, you can recover you can get back the pressure by solving this pressure Poisson equation after solving the velocity is. So, that is the total scheme of the stream function verticity approach.

Now, what are the demerits of this approach what are the limitations of this approach. So, one of the limitations, that we have seen while working out the approach itself is the artificiality in the verticity boundary condition. So, the verticity boundary condition is the not something, which is coming from basic physical arguments it is we are writing it from the tailor series expansion of the stream function.

So, it is a mathematical derived quantity, but, not a fundamental input physical parameter that is one, but, still it would have been ok the mode general problem is something like this see if you are very careful you have seen that very deliberately we have walked out this particular method by considering a two special case two dimensional problems the reason is that the stream function is defined in this way for a two dimensional problem. So, you cannot easily extend this concept to a three dimensional problem.

So, this puts a lot of restrictions on the solution methodology not only you lose track of pressure you introduce artificial boundary conditions for verticity those are these are still peripheral points, but, to me the most important restriction is that it is not generally applicable to a three dimensional problem.

So, if we want to have a method which is generally applicable to a three dimensional problem we should try to address the problem of keeping the pressure as it is in the governing equations. So, if there is a headache try to go for a good medicine rather than cutting the head all together and that is what we will try to attempt and that attempt is known as primitive variable base solution approach. That is you are trying to go for a solution based on velocity and pressure by themselves without eliminating pressure directly going for solution of velocity and pressure from the governing equations. So, that we will do next.

If we are interested about the primitive variable approach there are certain restrictions there are certain obstacles we have to face one of the important obstacle is the calculation of the pressure radiant term. So, the pressure radiant term if you recall acts as a source term in the momentum equation. So, let us see that how we can approach towards that now our objective is not to eliminate the pressure radiant term retain it and see that what challenges we face if we retain it.

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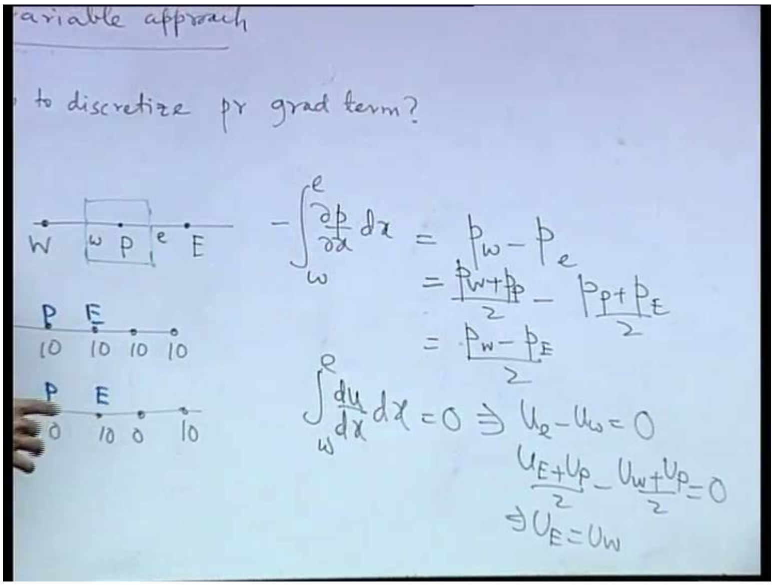

So, primitive variable approach. So, let us say the we are interested to descretize how to descretize the pressure radiant term let us say that you have point p with the neighboring point E W like this and we are interested to calculate the descretize form of the pressure radiant term for this control volume. So, what will be the corresponding representation minus del p del x d x from small w to small e. So, that is equal to p small w minus p small e that is the integral form of the source term, which we require in the finite volume formulation.

Now, when you have p w minus p e small x small w and small e if you do not define the variables usually we define the variables at the main grid points. So, you need to interpolate the value of the variables at this point if all the point are equal distance then this is nothing, but, p w plus p capital W plus p p by 2 minus p capital p plus p capital E by 2. So, it is p w minus p E by 2 remember that the pressure radiant terms acts as a body force per unit volume. So, it is a driving force for a fluid force to take place across a control volume. So, this is the forcing parameter.

Now, let us consider 2 examples in one example we have the pressure field like this 10 10 10 10 10 10 10 like this in another case we have a pressure field like this 10 0 10 0 10 0 10 like this let us try to calculate the body force in these 2 cases. So, if you consider this as W ,this as E and this as p you can see that the body force term is given by p w minus p. So, that is 10 minus 10 0 here also it is given by p w minus p. So, it is given as

0, but, these 2 cases are not physically the same here you have a wavy type of pressure field, where the pressure alternates between 0 to 10 and 0 to 10 like that. So, here you have a net pressure radiant the pressure radiant is not 0.

So, physically here the pressure radiant is 0 here the pressure radiant is not 0, but, because of the artifact of this interpolation both the cases are handled in the same way both the cases are interpreted in the same way that is really wrong here the pressure radiant is 0 we can understand, but, here the pressure radiant is clearly not equal to 0. So, this is a big problem even if you want to integrate the continuity equation you still get the same problem.

Let us try to do that. So, if you are having the continuity equation del u del x say one dimensional continuity equation d u d x equal to 0 so, that means, you have u e minus u w equal to 0. So, if you write it in terms of capital letters. So, what you will get is u e is equal to u w. So, instead of velocity field let us considered this instead of pressure field let us considered this a velocity field.

So, if you see that this velocity field satisfies u v equal to u w and this is a uniform velocity profile in one dimensional d u d x equal to 0 here this is not d u d x equal to 0 because u is a function of x still it is interpreted as u v equal to u w and it gives a false idea that as if the continuity one dimensional continuity is satisfied. So, there is something wrong with treating the first order derivative terms in the source term of this equation or even in the continuity equations. So, how we get rid of this problem that we will take up in the next lecture. Thank you.