Computational Fluid Dynamics

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Lecture No. # 27

Iterative Methods: Further Examples

Techniques and let us continue with some more examples; let us say that we have two equations.

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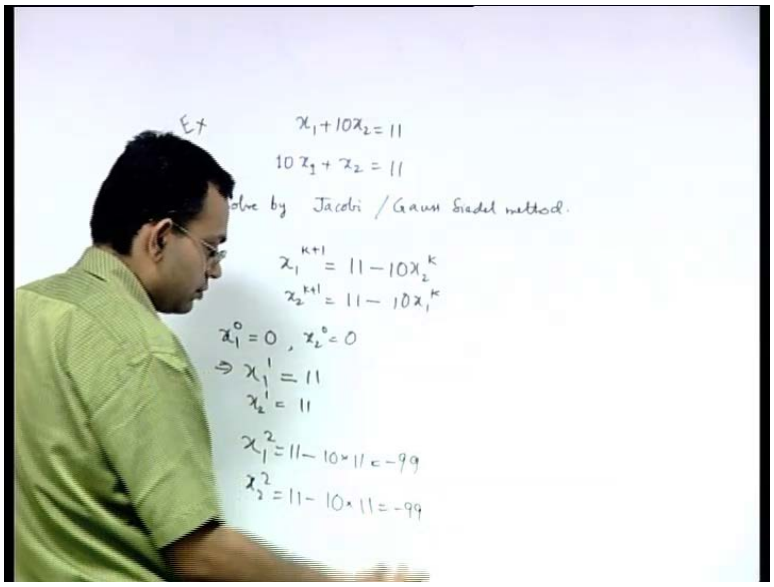

Let us say that we have these two equations and two unknowns and our objective is to solve by Jacobi and Gauss Seidel method. So, if we write the corresponding formula for iteration then x 1 k plus 1 is equal to 11 minus 10 x 2 k let us first consider the Jacobi method and x 2 k plus 1 is equal to 11 minus 10 x 1 k let us say that we have x 1 0 is equal to 0 and x 2 0 equal to 0 as the initial guesses. So, from here we get x 1 1 is equal to 11 x 2 2 equal to 11 sorry x 2 1 equal to 11.

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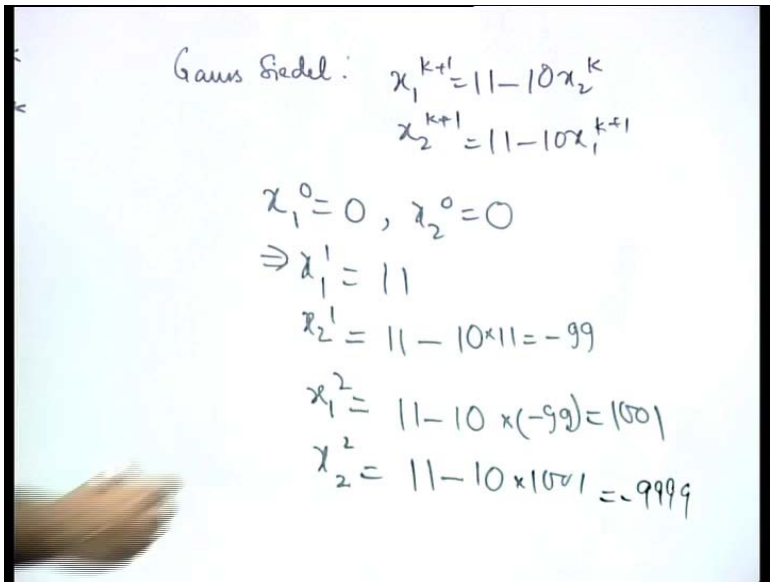

Then x 1 2 is equal to 11 minus 10 into x 2 1 11 minus 10 into 11. So, minus 99 x 2 2 is 11 minus 10 into x 1 k that is x 1 1 that is minus 99. You can see I mean, let us not go further into these steps this gives us enough insight that it is diverging let us consider the Gauss Seidel method the first step is the same x 1 k plus 1 is equal to 11 minus 10 x 2 k x 2 k plus x 1 is equal to 11 minus 10 x 1 k plus 1. So, here also if we take x 1 0 is equal to 0 and x 2 0 equal to 0 then x 1 1 what is x 1 1 11 x 2 1 11 minus 10 into 11; so, minus 99 in this way. Let us consider, let us calculate may be 1 more step x 1 2 is equal to 11 minus 10 x 2 1. So, what is this? 1001 x 2 2 is equal to 11 minus 10 into 1001; so, minus 9 9 9 9.

We can see this is diverging and unlike the case when it converges, Gauss Seidel converges faster when it diverges. It also diverges faster what we can see now of course, it does not matter whether it is a Jacobi or the Gauss Seidel method. There is one fundamental reason for which it diverges and the reason is that it is not a diagonally dominant system. So, if you find out its spectral radius of convergence, its spectral radius of convergence will become greater than 1; from the rho norm you can easily calculate it.

So, if you want to solve this system by using these same methods, what you have to essentially do? You have to reorder the equations; so, you reorder. So, these will become equation 1 and this will become equation 2; accordingly let us write the formula.

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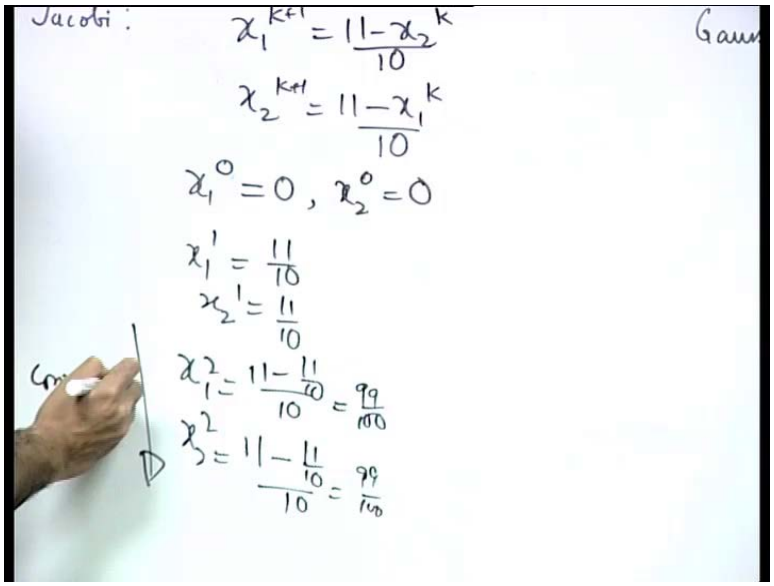

So, x 1 k plus 1 is equal to 11 minus x 2 k by 10 and x 2 k plus 1 is equal to 11 minus x 1 k by 10. Now, if you take x 1 0 is equal to 0 and x 2 0 equal to 0 then x 1 1 is equal to 11 by 10 x 2 1 is equal to 11 by 10.

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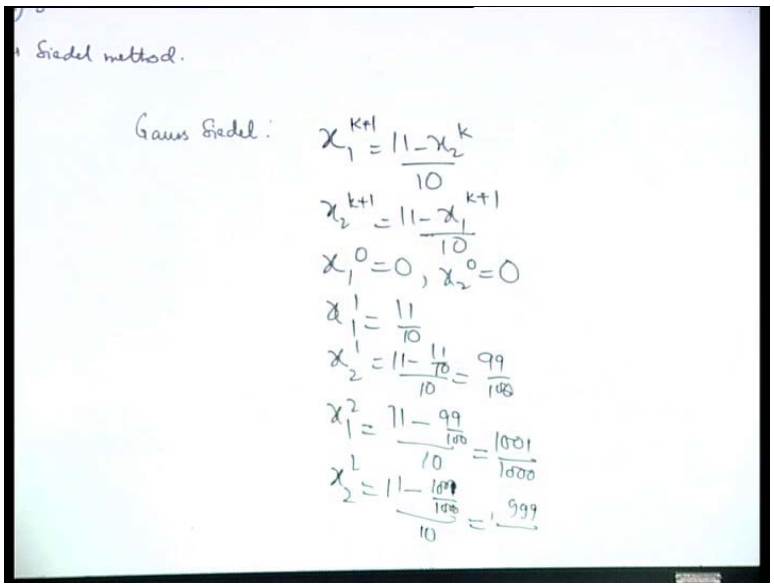

Next step x 1 2 is equal to 11 minus x 2 1 that is 11 by 10 by 10. So, 99 by 100 then x 2 2 is equal to again, this is the same 11 minus x 1 1 by 10. That is 99 by 100. So, this is converging to 1 and 1. Let us consider the Gauss Seidel method. So, x 1 k plus 1 is equal

to 11 minus x 2 k by 10 and x 2 k plus 1 is equal to 11 minus x 1 k plus 1 by 10; here also we may start with an initial guess of 0 0.

So, x 1 1 is equal to 11 by 10 x 2 1 is equal to 11 minus 11 by 10 by 10. So, ninety nine by hundred then x 1 2 is equal to 11 minus ninety nine by hundred by 10 what is this thousand 1 by 1000 and x 2 2 is equal to 11 minus 1001 by 1000 by 10 what no no no 49 is right by ten thousand. So, what we can see from here is that this is also converging to 1 comma one, but at a faster rate. So, it is quite clear that what are the important convergence characteristics of the Gauss Seidel method and the Jacobi's method through this very simple example.

Now, in our previous lecture we have discussed extensively about the sufficient condition for convergence and it is always a question whether a method will automatically satisfy the sufficient condition for convergence or not. In many cases for some special matrices fortunately the method always satisfies the sufficient condition for convergence.

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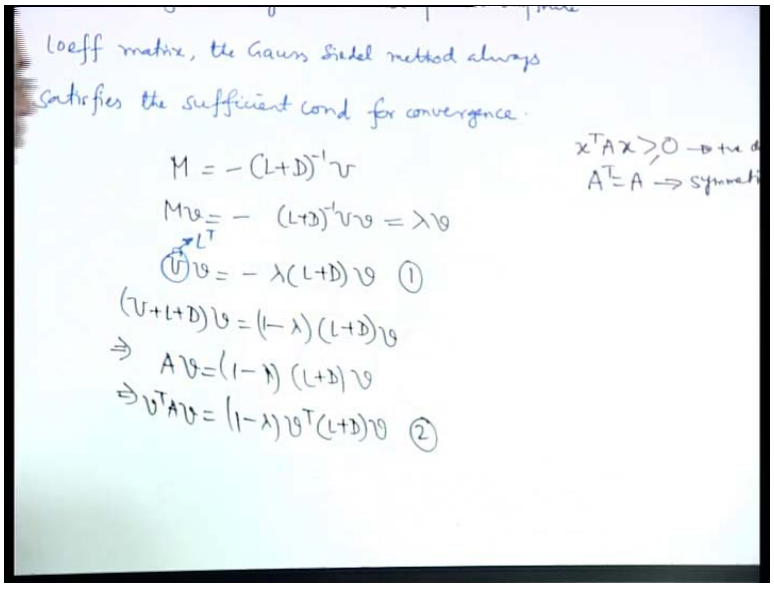

For example, let us consider this as a problem. Show that for symmetric and positive definite coefficient matrix, the Gauss Seidel method always satisfies the sufficient condition for convergence. I will partially work it out for you and leave the remaining on you as an exercise. So, the situation is that we are interested for the Gauss Seidel method. What is the m matrix for the Gauss Seidel method minus l plus d inverse u? to show that

it satisfies the sufficient condition for convergence for a matrix coefficient matrix a which is symmetric and positive definite, we have already seen that what do we mean by a symmetric matrix and a positive definite matrix.

Positive definite matrix has many interesting properties; we will come into that later on. But, just to refresh that if you have a vector x if you have a positive definite matrix x then x transpose x is greater than 0 or you can put a greater than or equal to 0. So, this means it is positive definite. Positive definite matrices have positive diagonal elements and positive principal minors. They have certain, several interesting and important properties.

Now, if some of the diagonal elements appear to be negative you should suspect that it is not a positive definite matrix and symmetric matrix A means A transpose equal to A for symmetric. Now, if v is an eigen value then we have m v is equal to minus lambda l plus d inverse u v. why we are bringing the eigen value and eigen vector into the picture? Because, it is the mod of lambda max of that will detect whether it is satisfying the sufficient condition for convergence. We have to bring the lambda into the picture now. So, this is equal to, sorry, this is equal to lambda into v (Refer Slide Time: 18:11).

So, now you can write u into v multiplying p multiplying both sides by 1 plus d is minus lambda 1 plus d into v because it is a symmetric matrix. u is nothing but equal to 1 transpose because A is a symmetric matrix. It is symmetric with respect to the diagonal. So, u and 1 are related by u equal to 1 transpose. So, this is one thing that we will keep in mind; let us call it equation number 1.

Also, u v equal to this one; now you can write it in terms of A v. here it is u into v; if you want to write it in terms of A into v you have to add 1 plus d into v because 1 plus d plus u is A. So, u plus 1 plus d v is equal to minus lambda then plus 1 you add 1 plus d into v both sides which means a v is equal to 1 minus lambda 1 plus d v. So, if you want to calculate what is v, transpose A v.

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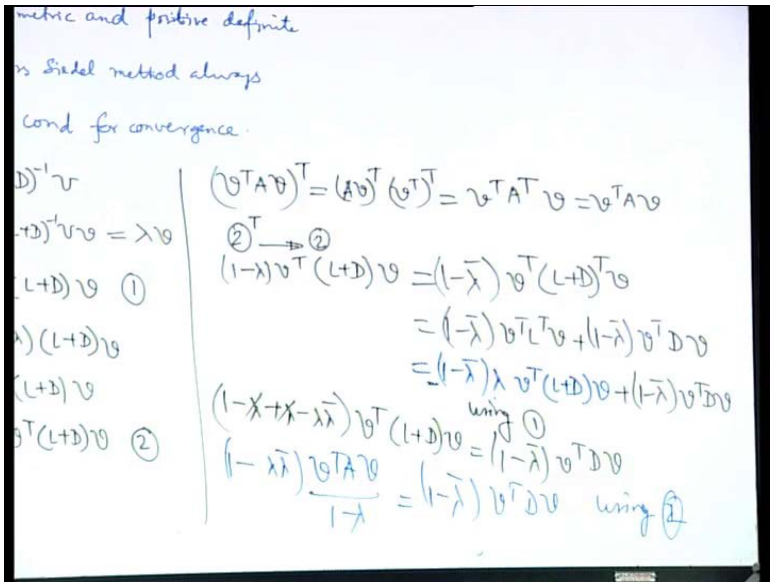

Now, let us calculate the transpose of v transpose A v. If you calculate v transpose A v transpose then it is as good as A v transpose into v transpose. Transpose it is we have written A into b transpose equal to b transpose into A transpose. So, this is equal to v transpose A transpose and v transpose; transpose is v it is same because A and A transpose are the same.

So, if you take transpose of 2 then what you can write one minus lambda v transpose 1 plus d. So, transpose of 2 is same as 2; that means, 1 minus lambda v transpose 1 plus d v is equal to its transpose. So, when we take a transpose of a matrix and then find the eigen value then, what can be the eigen value? Does it remain the same? It may remain the same, but we can generalize it a little bit like for example, let us consider a 2 by 2 matrix say a b c d. So, if you want to find out its eigen values what you do you? Write a minus lambda b c d minus lambda determinant; say, determinant equal to 0 if you have the transpose of this one the same equation governing the eigen value will be valid; but it is a quadratic equation it has 2 roots. So, it could be that for the original matrix one of the roots was the lambda and for the transpose the corresponding conjugate root was the lambda.

So, in general you could write that the transpose will have, may have an eigen value which is lambda bar which may be either the same or the conjugate of that. If it is a complex number it can be a complex conjugate. So, we just generalize it by giving it a

name lambda bar keeping in mind that it can also be same as lambda. So, 1 minus lambda bar then the transpose of this one. So, heat transpose into 1 plus d transpose into v.

So, this is nothing but equal to 1 minus lambda bar v transpose one transpose v plus one minus lambda bar v transpose d transpose and d are the same it is a diagonal v transpose d v and from equation 1 we can write 1 v l transpose v in terms of 1 plus d into v. So, this from equation 1, we can write 1 minus lambda bar in place of... So, we multiply equation 1 by v transpose. That becomes minus lambda into v transpose l plus d v and then the original term this is using 1.

You can see in the left hand side and in the right hand side, you can take v transpose 1 plus d v as common in the first two terms. So, in the left hand side and in the right hand side, the first term you have v transpose l plus d v common if you take that as common then you have 1 minus lambda plus lambda minus lambda into lambda bar into v transpose l plus d v is equal to 1 minus lambda bar into v transpose d v.

So, next is we can write, see we we are interested to write everything in terms of A and d because if A is a positive definite matrix then v transpose A v is greater than 0 that we know and also v transpose d v is greater than 0 because the diagonal elements are all positive. So, if we want to write v transpose l plus d v in terms of v transpose a v then, we need to use equation number 2. So, we can write 1 minus lambda into lambda bar in place of v transpose l plus d v it is v transpose a v by 1 minus lambda is equal to 1 minus lambda bar v transpose d v this is using 1, sorry using 2.

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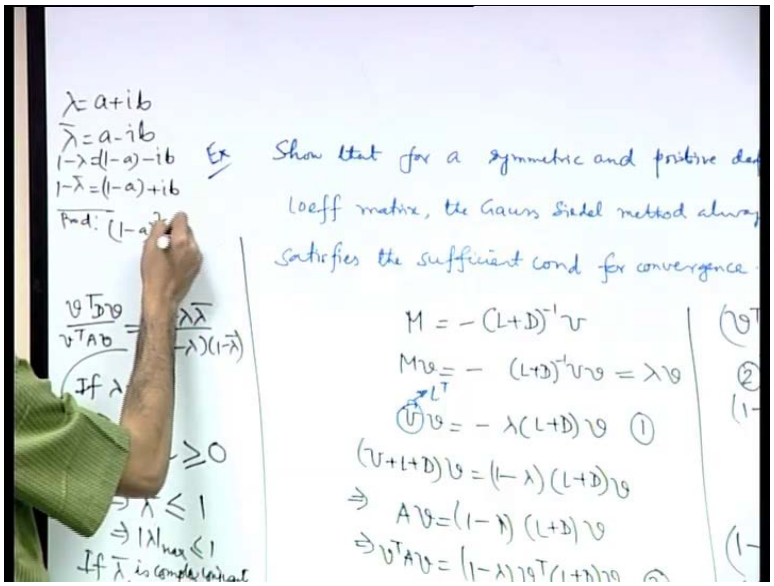

If you consider lambda and lambda bar as complex conjugates first of all, let us write this one as v transpose d v by v transpose a v is equal to 1 minus lambda lambda bar by 1 minus lambda into 1 minus lambda bar. Let us take the simple case when lambda bar is equal to lambda. We have seen that it is possible that by transposing you get the eigenvalue which is same as the eigen value without transposing. So, if lambda is equal to lambda bar then, this becomes 1 minus lambda square by 1 minus lambda whole square.

Now, v transpose d v is greater than 0 v transpose a v is greater than 0 for positive definite a therefore, this is greater than 0. In general, greater than equal to 0; that means, lambda square is less than or equal to 1 or mod of lambda max is less than equal to 1 denominator is always positive. So, if the eigen value remains exactly the same. It is very trivial to show that it satisfies the sufficient condition for convergence. if on the other hand, if lambda bar is complex conjugate of lambda that is another example.

Then we can write lambda into lambda bar is mod lambda square, right? If lambda and lambda bar are complex conjugate to each other then, lambda into lambda bar is mod lambda square and 1 minus lambda into 1 minus lambda bar is positive like for example, let us say lambda is equal to a plus i b then lambda bar may be a minus i b complex conjugate. So, 1 minus lambda is 1 minus a minus i b and 1 minus lambda bar is 1 minus a plus i b. So, the product is 1 minus a square plus b square that is positive. So, in either

way, it can be shown that mod of lambda max is less than 1 less than equal to 1 so that it satisfies the sufficient condition for convergence. Anyway, I thought that I will leave it to you as an exercise, but I completed this. So, I will leave it leave something else on you as an exercise.

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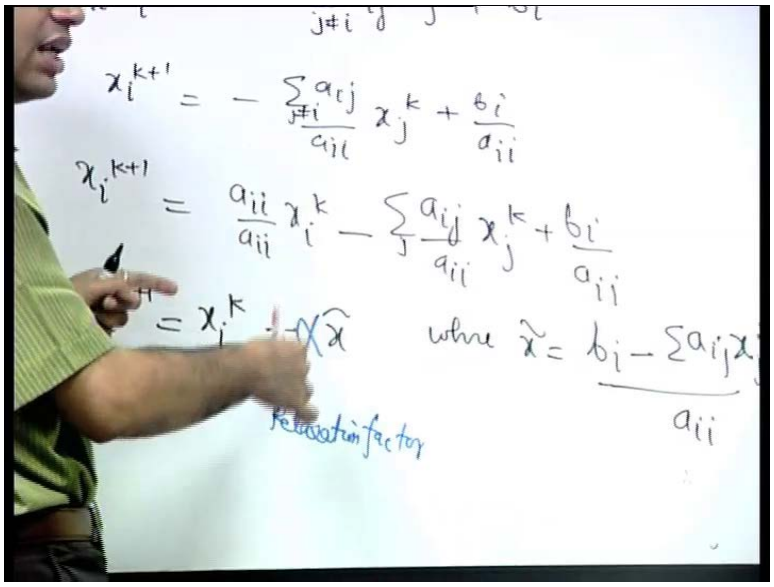

Now, the next issue is we have seen the convergence characteristics of the iterative methods. Are there any means by which you can improve on the convergence characteristics? There are certain ways of doing that and those methods are called as relaxation methods. The whole idea of relaxation method is to improve the convergence characteristics of of particular iterative method.

Let us say, let us consider the Jacobi's method. So, in the Jacobi's method if you write it in terms of the indices of the matrix it is a i i x i k plus 1 is equal to minus summation of a i j x j j is not equal to i plus b i k right. So, remember it is l plus d plus u which is the matrix a this corresponds to d and this corresponds to l plus u in the Jacobi's method you have l and u together with the same level of iteration.

So, x i k plus 1 you can write minus summation of a i j j not equal to i by a i i x j k plus b i by a i i let us try to write these x i k plus 1 in terms of x i k. So, what we do is we write these as a i i by a i x i k then we also have to subtract a i i by x i a i k. So, if we combine that with this negative term then what we get minus summation of a i j by a i i, but now j can be equal to i. So, 1 extra term we add this is clear to everybody. So, what we have

done we have added 1 term which is a i i by a i i x i k that we have subtracted and when we have subtracted that means, we have also considered j equal to i. So, that j not equal to I, we have relaxed.

So, i plus 1 is equal to x i k plus x tilde where x tilde equal to b i minus summation of a i j x j k by a i i. So, in the relaxation method what we do we multiply this x tilde with a factor alpha which we called as relaxation parameter or relaxation factor. This relaxation factor can be a number which is different from 1. If it is equal to 1, it will mean the original Jacobi's formula; if it is different from 1 it will be a different formula.

Now, the big question comes that how can we tamper a formula like this. So, if you make this as alpha which is not equal to 1 you are tampering a formula which we have derived from certain fundamentals still you expect that you will get the final solution why because you expect something to occur which is very very intuitive that if convergence occurs then x i k plus 1 will be equal to x i k and then the choice of this alpha is irrelevant. So, what it basically does is it can take you to the same convergence, but through a different route of iterations and the route of iterations is decided by this relaxation factor. So, this is called as a relaxation method. So, depending on its value it can sometimes accelerate the convergence. Sometimes the the value is not chosen very judicially; it can slow down the convergence process. So, it has a control over the rate of convergence.

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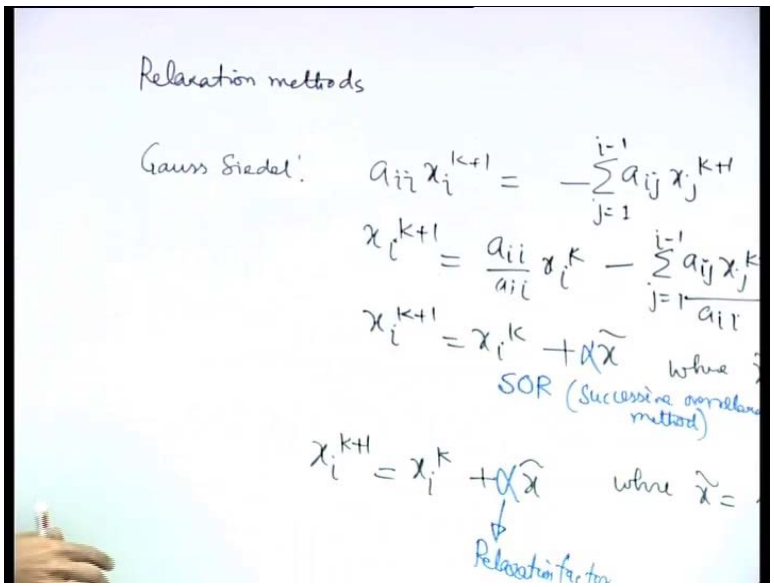

Now, let us try to write the same thing for the Gauss Seidel method. A i i x i k plus 1 is equal to now there are 2 parts minus summation of a i j x j k 1 is k plus 1, another is k. now, let me just write j is equal to 1 to i minus 1 does it belong to k plus 1 or k this is the l part right. So, l plus d that gets combined with k plus 1. So, this is d with k plus 1 this is the l part the lower triangular matrix part that gets coupled with k plus 1 then. So, this is for j equal to i plus 1 to n plus b i.

So, you can write x i k plus 1 is equal to just like the previous case you add a i i by a i i x i k minus summation of a i j x j k plus 1 j is equal to 1 to i minus 1 minus summation of a i j x j k now j equal to i to n because you have added 1 a i i plus b i.

So, you have x i k plus 1 is equal to x i k these are divided by a i i i i plus x tilde where x tilde is equal to b i minus summation of a i j x j k plus 1 by a i i j is equal to 1 to i minus 1 minus summation of a i j x j k j is equal to 1 j is equal to i to n by a i i b i by a i i also right. So, here also you can introduce the relaxation parameter alpha and this particular method when it is derived originally from the Gauss Seidel method with a relaxation parameter this is called as S O R method or successive over relaxation method.

Now, can you identify that? What is the significance of this x tilde? This x tilde is equal to 0 when the convergence has occurred and this x tilde it is nothing but equal to b minus a x during the iteration. So, it is a residual. So, x tilde is a normalized residual; it is b minus x normalized with respect to the diagonal coefficient. So, this is a representation of the residual and we know that when it converges technically the residual should vanish and therefore, when it converges it does not matter what alpha you have taken, but the path by which it will travel during iterations for convergence that will be strongly dependent on the choice of the parameter alpha. This is one very interesting and important inference and one can choose alpha in in a way to accelerate convergence very nicely in the successive over relaxation method.

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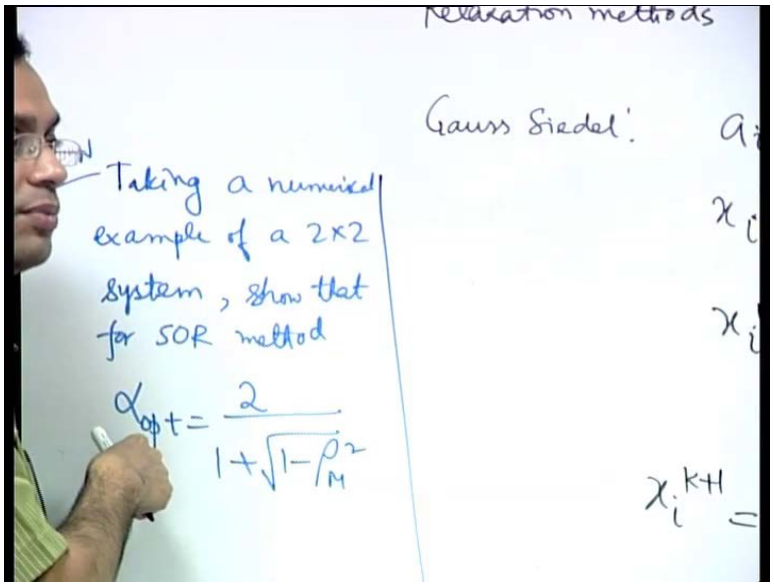

Now, I give you something as a homework to try. Taking a numerical example, this is a homework problem; taking a numerical example of a 2 by two system, show that for S O R method alpha optimum is equal to two by one by 1 minus the square of the spectral radius of convergence for the Gauss Seidel method for the matrix m ok.

So, what you have to do you have to take any numerical example or take a simple form of a two by two matrix some of that entries you may take 0, but you you may may not also take that. So, from that you write the corresponding m matrix, find out the corresponding spectral radius of convergence and with different choices of the spectral radius of convergence. That means, the different choices of a you show that out of all possible choices, the optimum alpha that that will give you the best possible convergence is these two.

Of course, this can be even shown analytically, but it is much more convenient to show it through some numerical examples. So, you can work out a few numerical examples with a few choices of the coefficient matrix a and you can show this by showing that you require the minimum number of iterations to achieve convergence when the value of alpha is equal to this one. So, that will give you the fastest rate of convergence.

Now, we have learnt some iteration methods and some elimination methods by now. Question is, what do we prefer in c f d applications? Out of these two, if you have a choice very commonly we use the iteration methods as a preference. We will see later on

that it is not always true. Sometimes, a combination of these is what we look for. But, in general, iterations are preferred and why first of all. So, some preferential characteristics of the iterative methods the first and foremost reason is that the iterative methods can very easily handle non-linearities in the system can accommodate nonlinearities very easily.

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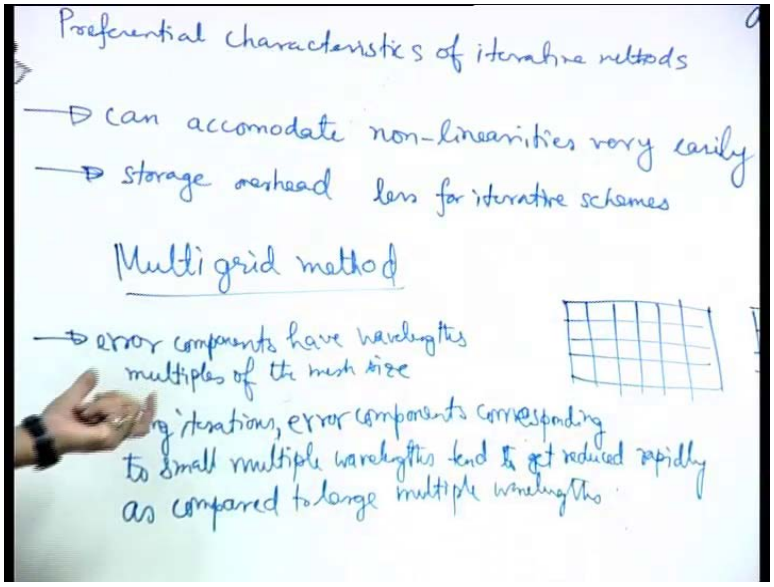

Just let us take an example; it need not be a example of a differential form just let us consider that you are interested to solve a x's square plus b x plus c equal to 0 this you can write as a x into x plus b x plus c equal to 0. Let us call it x k plus 1 and this x as x k there are some arbitrary iterative formula.

This is just for illustration do not take it as a very rigorously derived formula, but what you can see from here is that you can express x k plus 1 explicitly in terms of x k and it is a linear formula in x k plus 1. You can even write these as x k plus 1 for the coefficient b in place of x k you can write it x k plus 1 still it remains linear in x k plus 1 although the original equation was non-linear. So, what you are doing is in you are breaking xs square as x into x in 1 of those xs you are using the old value.

See we can write a formula in some mathematically rigorous way and but it is very important to appreciate that. Essentially, what we are doing we are replacing the variable with some, we are assigning the variable with some value. What is that value? a part of that it is some old value. So, that when you write it in this way it is a pseudo linear form

in terms of x k plus 1. In an iterative environment you can very easily accommodate non-linearities and storage overhead is also less for iterative schemes. Storage means computer storage overhead for the data is less for iterative schemes.

But are all aspects of iterative scheme. So, good we have to assess one of the important assessments is that if you refined the mesh spacing your rate of convergence decreases in our previous lecture we worked out an example where we decrease the mesh size from say 10 to the power 3 by 10 to the power 3 we coarsen it to 10 to the power two by 10 to the power 2 and we showed that that rate of convergence increases.

That means, you have… if you use a refined mesh and many times we use a refined mesh to get more accurate solution that is the natural tendency, but at that cost what we sacrifice is that now we have reduced rate of convergence. So, we will be getting more accuracy hopefully, but with a much reduced rate of convergence which perhaps we are not able to afford. What we do with that? Simply we cannot get rid of the method because the method is otherwise good, but we may have to adapt this method for two types of systems: for one final mesh and for a coarser mesh and then transform the method from one mesh to the other back and forth and this particular method is known as multi grid method.

We will try to develop an outline of the multi grid method, but before doing that we should appreciate certain things. How do the errors propagate in an iterative scheme? So, in an iterative scheme you have the mesh size like this. This may be a fine mesh and this may be coarse mesh. As an example, you can refined a coarse mesh very easily; let us say this is a refined version of a coarse mesh.

Now, if you make an make a spectral analysis of the errors then errors in an iterative scheme have wavelengths which are multiples of the mesh size. So, let us note that first. So, error components have wavelengths multiples of the mesh size. What is the largest wavelength possible that is the domain size itself? If you have a refined mesh, let us say that you you have a refined mesh then what is the maximum multiple of the smallest mesh size that is possible for the wavelength you have four multiples of that if you had a coarse mesh you could have maximum only as two multiples.

So, what happens is during iterations small multiples small multiple wavelengths errors corresponding to small multiple wavelengths errors components basically error

components corresponding to small multiple wavelengths small multiple wavelengths 10d to get reduced quickly get reduced rapidly when we say rapidly we mean more rapidly than the large multiple wavelengths as compared to large multiple wavelengths.

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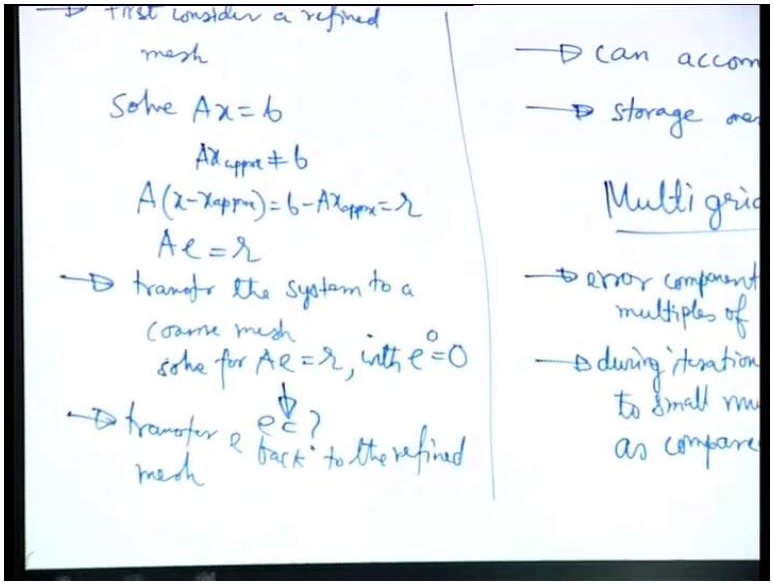

So, if you are solving a problem in a fine mesh with iterations you are taking care of the small multiple wavelengths error corresponds corresponding to that, but the large error corresponding to the large multiple wavelength they 10d to remain and that is why you require more number of iterations to achieve the convergence.

Now, in multi grid method what is done is first we consider a refined mesh; first consider a refined mesh. So, in the refined mesh you solve a x equal to b you have to keep in mind that because of the non absorption of the large wavelengths large multiple wavelengths errors will remain. So, what you get is a approximate which is not equal to b. So, a x minus x approximate will be equal to b minus a x approximate equal to r which is the residue that is a e equal to r.

Now, to absorb the errors in the large multiple wavelengths you now are interested to solve for this a e equal to r by transferring the system to a coarse mesh not a refined mesh then transfer the system to a coarse mesh how do you transfer it to a coarse mesh from a fine mesh may be you do averaging let us say that you have coefficients like this and when you have coefficients like this say or let us say you have coefficients like coefficients at this points now you are interested to find out coefficients at which points

in the coarse mesh coefficient at at this point. So, that you may consider as an average of the coefficients between these two points something like that. So, particular point you can consider coefficient as an average of the neighbors information of which now is lost.

So, transfer the system to a coarse mesh then in the coarse mesh you solve for a e equal to r with e 0 equal to 0. So, you have a new e a new a which you have formed by using the refined mesh you are interested to solve for e for that initial guess is 0 and the residual you have already got from that refined mesh and that you can average just like how you did average for the coefficient a. So, from here you can get what is e; then transfer this e back to the refined mesh.

So, when you transfer this e back to the refined mesh then, you have to use some interpolation techniques because you have ‘e’s for the coarse mesh points. So, you have ‘e’s for this; you have e for may be this point, you have e for this point, but not for this point and not for this point because, these were not the points of agenda for your coarse mesh, but now when you refined mesh. You have to calculate e here, you have to calculate e here. So, for that you can use some interpolation technique may be a linear interpolation; so, transfer. When you transfer here, you require some averaging technique and when you transfer here, you require some interpolation technique and then, you can write that if you know the error then you have new x is equal to x old plus error x old was x approximate this is x old because x minus x old is error ok.

So, you can update your x by considering the previous x plus the error. So, what you have done essentially is that you have taken care of the large wavelength error by transferring it to a coarser mesh the small wavelengths were already taken care of during iterations in the final mesh, but large wavelengths were not taken care of that is why you have to transfer it to a coarser mesh and take care of the large wavelength error and then bring it back to the refined mesh.

So, this in this way you are using grids of multiple sizes to get an accurate solutions and this technique that is why is known as a multi grid technique. So, there are there are similar such advanced techniques, we will not go into more details of that and we will stop here today. In the next class, we will try to look into some techniques where we combine the iteration and the elimination methods and then we will go for the gradient search method; thank you.