Computational Fluid Dynamics

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Lecture No. # 20

PART 1: Mid-Semester Assessment Review (Questions and Answers) (Contd.)

PART 2: Finite Volume Discretization of 2-D Unsteady State Diffusion Type Problems

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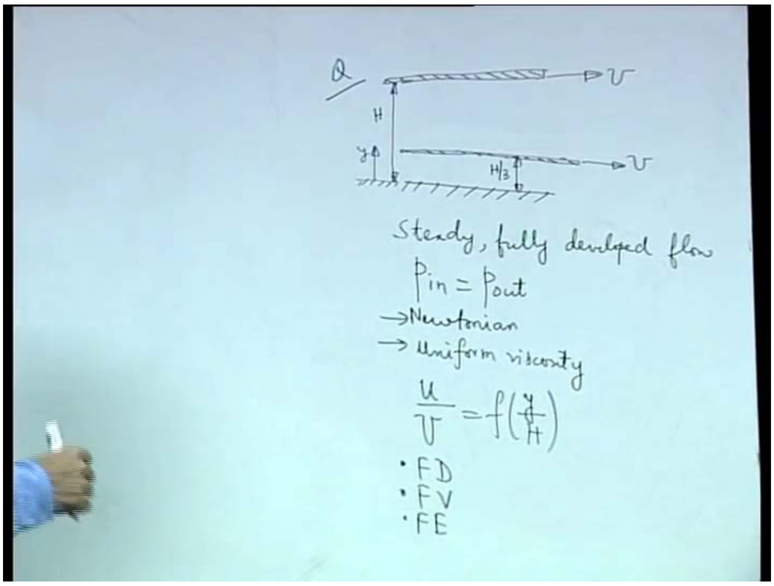

We continue with our review of the mid-semester questions, and let us proceed with that. So, the question that now we will address is as follows. You have three parallel plates, and the y axis having its origin at the bottom plate. The top plate moves towards the right with a velocity of capital U, the bottom was plate is stationary and in between not at the midway, but roughly at one-third. So, exactly at one-third of the distance between the plates; that is if the total distance between the top and the bottom plate is H then at H by 3 from the bottom plate, you have another plate. And this plate is very much similar to the other plates of negligible thickness and this is also moving towards the right with a velocity capital U.

So, the description of the flow is as follows, you have steady, fully developed flow between the plates with inlet pressure same as the outlet pressure. Assume the fluid to be Newtonian and uniform of uniform viscosity. You have to determine the velocity profile that is small u which is the velocity along the x direction divided by capital U as a function of y by H. This you have to find out using finite difference, finite volume and finite elements method. You can take any arbitrary choice of the grid layout or the mesh layout as per your convenience.

So, let us try to first address the problem in terms of its fundamental governing equation. So, we start from the general conservation equation that is...

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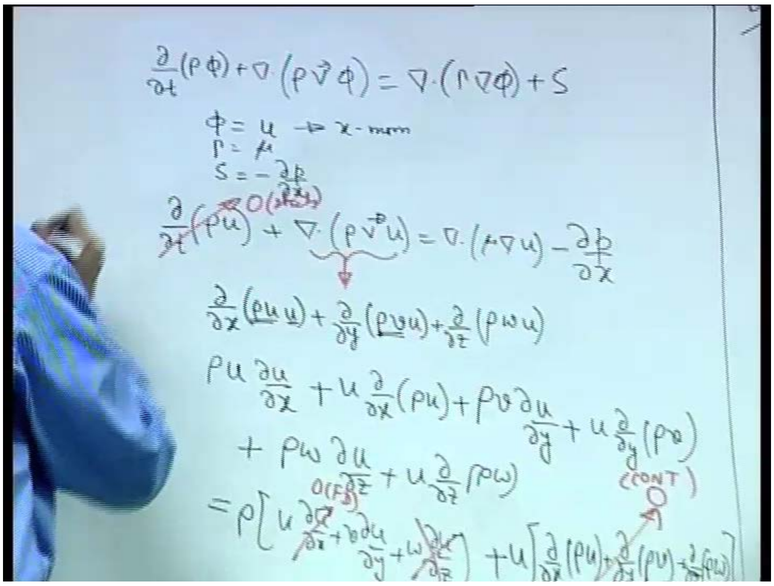

If you consider phi equal to u; it becomes the x momentum equation. Phi equal to u, gamma equal to mu, and S is negative of the pressure gradient along x .You have... There is no extra body force along x. This is the momentum equation now we can make certain simplifications. We can consider this term to be 0 because it is steady flow, we can also use the continuity equation along with this one to simplify this term we will not go through the details of that, but you can simplify this if you want we can also of course, go through the details. So, this will be...

So, you can club up rho into u as one term and u as another term. So, and use the product rule. So, it becomes rho u del u del x plus u del del x of rho u plus you can club up rho v as rho v del u del y plus u del y of rho v plus rho w del u del z plus u del del z of rho w and that is essentially rho u del u del x plus v del v del y plus w del u del z plus u del del x of rho u. Similarly, del del y of rho v and this term is zero ,because that is the continuity equation whatever is there in this square bracket that is 0. So, the left hand side is there rho into u del u del x plus v del u del y it is a two-dimensional flow first of all it is a flow in a plain. So, the third component is not important then fully developed flow means del u del x equal to 0. So, this will be 0 if you are considering an incompressible flow by continuity equation is even more simplified than this one it will become.

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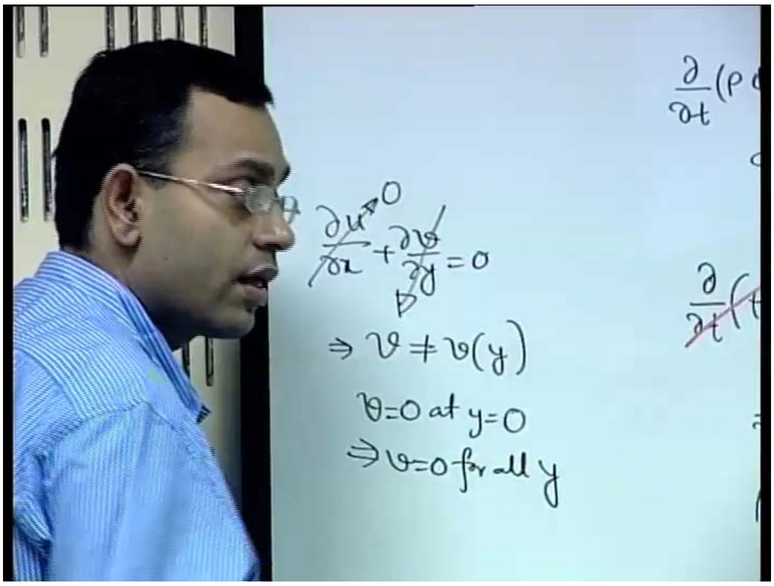

So, del u del x equal to 0 means, you have del v del y equal to 0; that means, v is not a function of y. We can say that v is equal to 0 at y equal to 0. By virtue of no penetration boundary condition at the valve say bottom valve as an example: which implies that v is equal to 0 for all y. It is also a consequence of the fully develop flow that v is identically 0 therefore, the left hand side becomes 0 right hand side if you simplify mu is a constant which is given.

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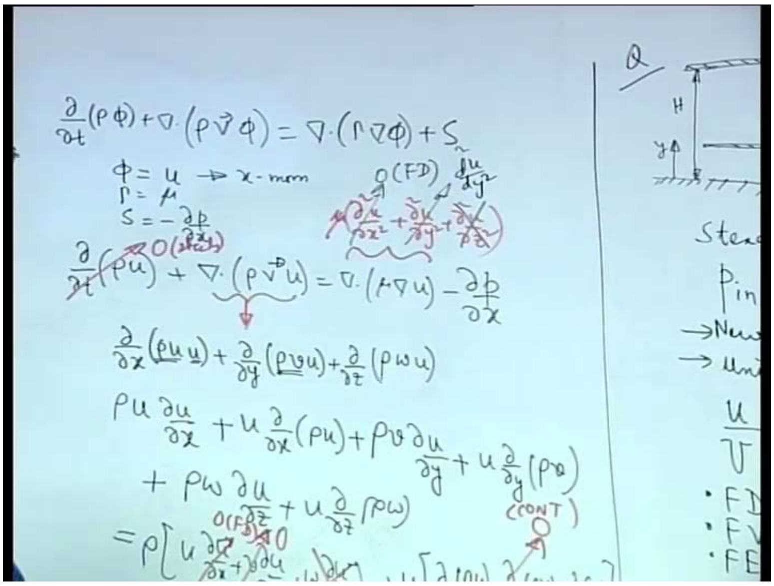

So, it will be mu because of two-dimensionality the last term is not there because it is fully developed flow del u del x equal to 0. So, its second order derivative is 0 and because it is fully developed u is the function of y only. This will become ordinary derivative.

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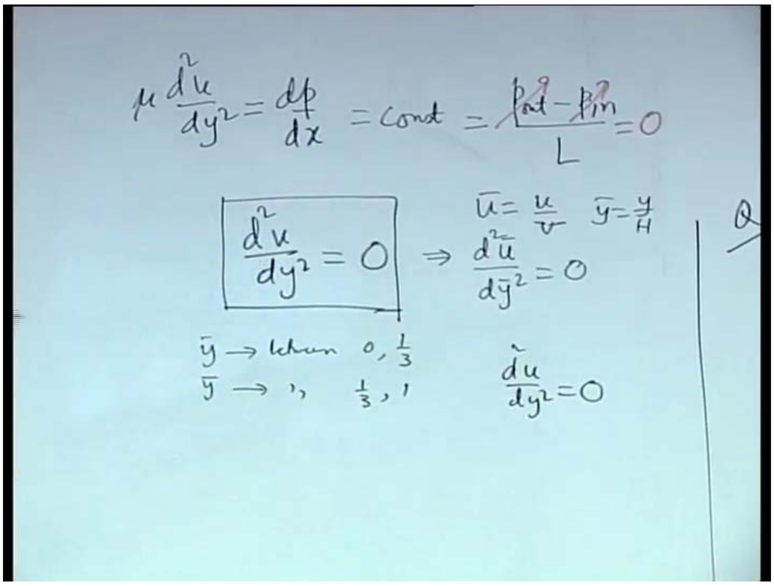

To summarize the simplification what we have at the end we have mu regarding the pressure gradient. We can use the y momentum equation to check that there is no pressure gradient along y. So, only pressure gradient that x along x. Therefore, this will become d p d x instead of the partial derivative it will become the ordinary derivative . This is equal to d p d x because the left hand side is a function of y only and right hand side is a function of x only they are equal only when it is a constant otherwise that is not possible . So, this is equal to a constant; that means, you can write this because d p d x is a constant you can write it as delta p by delta x because d p d x is a constant means p versus axis linear. This is equal to p out minus p in by the length of the channel, because it is given at the boundary condition p in equal to p out; that means, these are equal. This is equal to 0. Finally, we are left with the governing differential equation that d 2 u d y 2 equal to 0 .The first step of solving the problem is to come up with this simplified governing differential equation with appropriate logic based on which this equation evolves. Once this equation is there next we are interested to solve this equation by different methods.

Now, what you can see here is that you have the domain between 0 to H and in between a plate is inserted which is moving at the same velocity as that of the top plate . You have a sort of imposed condition in between the physical extremities of the domain and to handle that you can divide this entire problem into two parts. One is the bottom half which is H by 3 another is the top half which is 2 H by 3 and you can normalize this by writing a non-dimensional u. So, define u non-dimensional as small u by capital u and y non-dimensional as y by h. So, d 2 u d y non-dimensional square equal to 0. We have two sub domains one sub domain is y between 0 to 1-third and another sub domain y between one-third to 1. Just to help in writing, we will omit the bar at the top and we will write d 2 u d y 2 equal to 0 , but we will keep in mind that it is u bar and this is y bar just for the sake of writing we are doing that.

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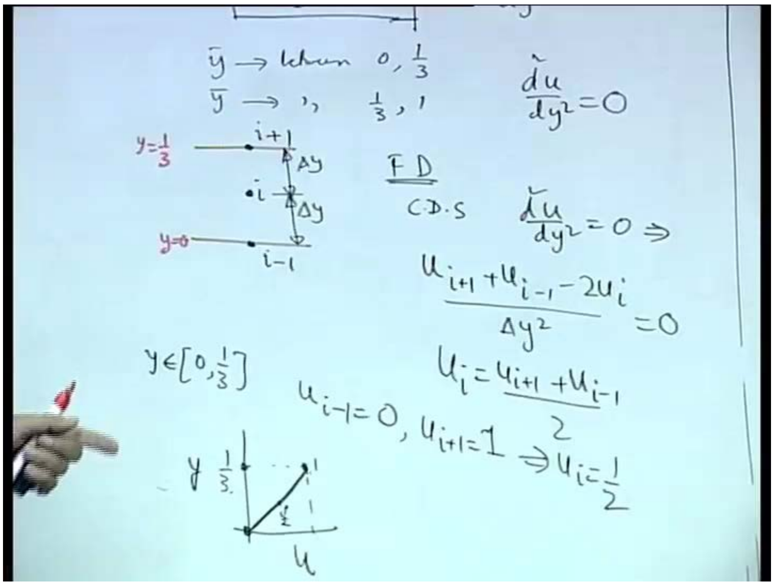

So, let us consider the part of the domain between 0 to one third. This is y equal to 0 this is y equal to 1 third if you want to capture a variation within this one what is the minimum number of grid points that you could take and we will take that . So, two must be there at the two boundaries because boundary condition has to be implemented and you have you at least should have one in between to see what is there inside the domain. You can call this as say I or may be this one as I this one as i minus 1 this 1 as i plus 1. So, for finite difference you can use different schemes say you use centre different scheme. So, that is also your choice if you use a centre difference scheme then d 2 u d y 2 equal to 0 this will imply what u i plus 1 plus u i minus 1 minus 2 u i by this is delta y this is 0. So, u i is u i plus 1 plus u i minus 1 by 2 u i minus 1 and u i plus 1 both are given by the boundary condition what is u i minus 1. So, for y between 0 to 1 third u i minus 1 is 0 u i plus 1 is not one ok non-dimensional yes. So, non-dimensional one. So, which means u i is equal to half. Your velocity profile in the bottom part at y equal to 0 it is 0 at y equal to 1 third it is 1 and at y equal to 1 sixth it is half . It is a straight line you may plot u along x and y along y what about the other part of the domain you can use the same discretization.

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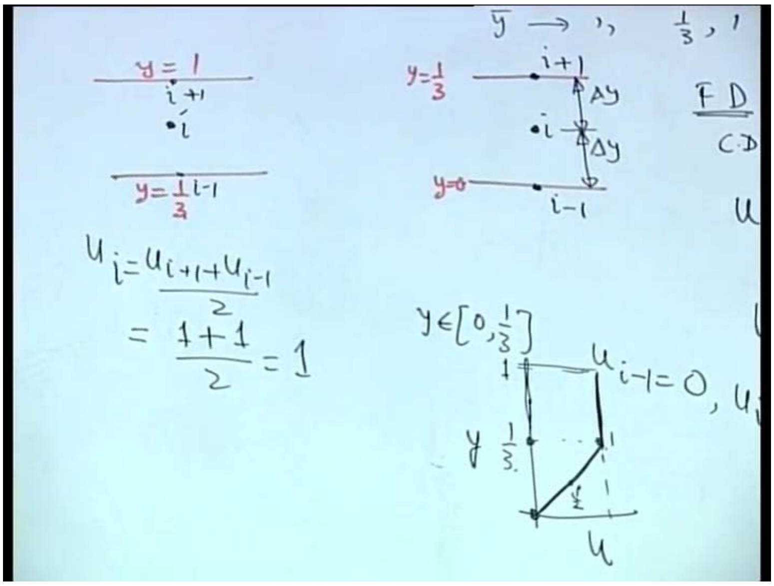

For the other part of the domain it is y between one third and y equal to 1. You can take another grid point in between. Again you can give this generic names i minus 1 i plus 1 and here also you will get governing equation is still the same. So, u i is equal to u i plus 1 plus u i minus 1 by 2 what is u i minus 1 capital U what is u i plus 1 non-dimensional of course,. It is 1 i minus 1 and i plus 1 both are 1. So, this is equal to 1. So, it is a uniform velocity profile in the gap between those plates between the middle plate and top plate this is one because the approximation of the Taylor series has been based on up to a linear term and the actual profile variations are linear these actual solutions are linear. You will get exact solution and the numerical solution the same because the exact solution also is a linear velocity profile. Then let us h look into the corresponding problem with the finite volume method.

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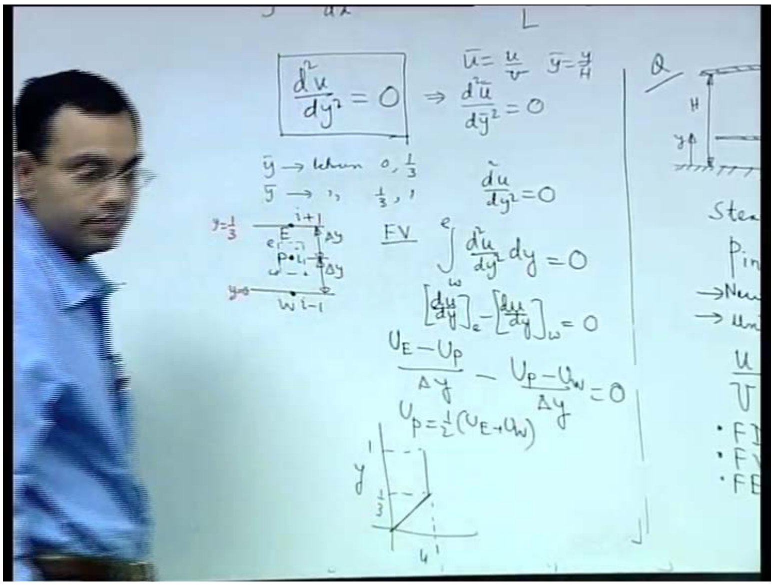

Finite volume method this is like the point I is like p i plus 1 is E and i minus 1 is W. So, integral the first step is integrate the governing differential equation with respect to the variable here it is y from. So, here control volume faces are this is small w and this is small e. So, integrate from small w to small e this is equal to 0; that means, you have d u d y at small e minus d u d y at small w is equal to 0 if you considered piecewise is linear profiles then this will become U E minus U P by delta y minus U P minus U W by delta y is equal to 0 .

So, U P becomes half of U E plus U W you can see that it is exactly the same as the finite difference discretization equation because the discretization equation is the same the solution is also the same. We need not go into the solution again you will find the same velocity profile that it is up to 1 third like this and then up to one it is a constant the agreement between the finite difference and finite volume method is not surprising because both in effect has considered a linear profile one through the Taylor series and another through a direct profile assumption. So, because both have considered implicitly or explicitly a linear profile the consequence the outcome is the same.

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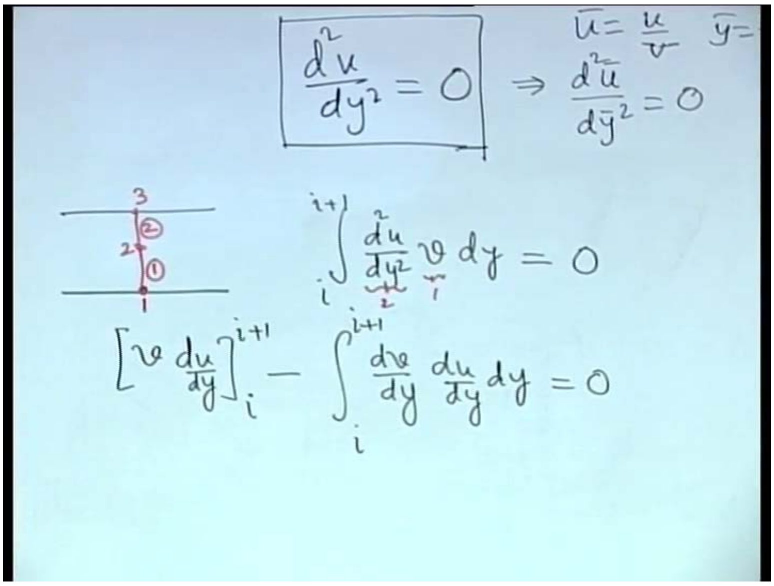

Then let us look into the finite elements method let us consider any two of the plates either the bottom two pairs or the top two pairs any of these two pairs and let us take some elements. How many elements minimum? You require two because you should at least have 1 node in between of course, the answer is not correct that how many elements you minimum require is 2 you minimum require number of element 2 provided each element has 2 nodes what happens if 1 element has 3 nodes then you could have taken just 1 element, but then you should have required a higher order interpolation function higher order shape function, but if you want to use linear interpolation function then your unknowns per element will be 2. So, that means; you will have two nodded element then two such elements at least will be required. The minimum number of elements required will depend on the order of the polynomial that you are using for interpolating the function if you use a higher order polynomial then you can use the single element, but may be with some interior node points within the element to accommodate the higher order interpolation function, but here we will use linear element. So, one element say let us give some names numbers 1 2 and 3. So, 1 element number 1 is 1 2 element number 2 is 2 3.

So, for each element let us formulate the finite elements equation. So, what is the first step you integrate it with respect to y after multiplying with a weighting function let us say v is the weighting function we are using the Galerkin's method and then we integrate it by parts if it is possible because that will reduce the requirement of the continuity in the order of the highest order derivative because we have used a linear element at the most first order derivative we can accommodate. Let us consider that one. We integrate it by parts by considering v as the first function and d 2 u d y 2 as the second function. So, v in to d u d y

Let us consider any generic element. So, from i to i plus 1 say some generic element which has 2 nodes 1 node number I another 1 number i plus 1 minus integral of derivative of first into integral of the second next we have to develop the shape functions n 1 and n 2.

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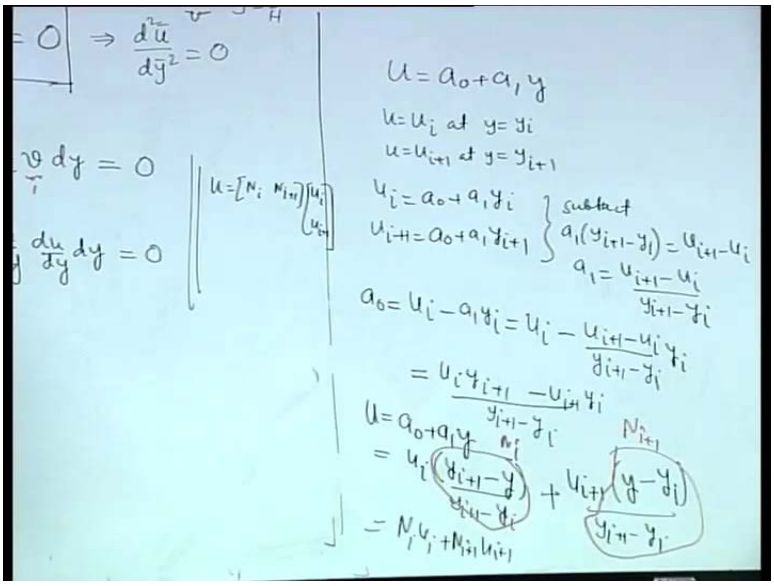

So, what we do we described u as a 0 plus a 1 y and we know that u equal to u 1 or u i at y equal to y i and u equal to u i plus 1 at y equal y i plus 1 . So, u i is equal to a 0 plus a 1 y i u i plus 1 is equal to a 0 plus a 1 y i plus 1 . So, from these two we can find out a 0 and a 1. So, to find out a 1 we just subtract. So, a 1 into y i plus 1 minus y i is equal to u i plus 1 minus u i. So, a 1 is u i plus 1 minus u i by y i plus 1 minus y i we can find out a naught as u 1 minus a 1 y i . So, u i minus u i plus 1 minus u i by y i plus 1 minus y i into y i . So, it is u i y i plus 1 minus u i plus 1 y i by y i plus 1 minus y i. So, u equal to a 0 plus a 1 y. So, u i let us first collect the terms with u i. So, u i into a 0 plus a 1 y. So, u i into y i plus 1 minus y by y i plus 1 minus y i this is with u i and u i plus 1 plus u i plus 1 2 i minus y i by y i plus 1 minus y i . So, this if we call as N i u i plus N i plus 1 u i plus 1 then; obviously, this is N i and this is N i plus 1.Whether the derivation has been find or not we can check that N i should be equal to 1 at y equal to y i and 0 at y equal to y i plus 1. similarly, N i plus 1 should be 1 at i plus 1 and 0 at I .

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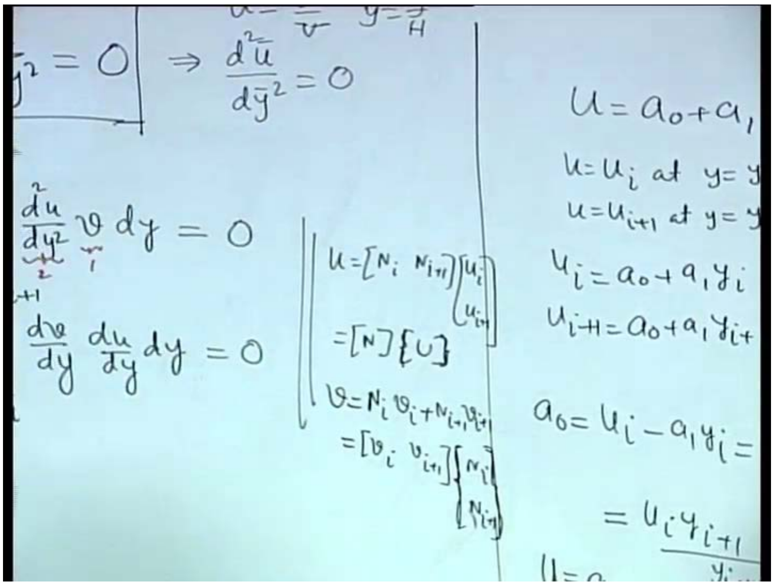

So, you are what you are doing essentially is you are writing that u is equal to N i N i plus 1 u i u i plus 1 this is what you are writing. So, we call it N into u matrix into u vector in the Galerkin’s method the trial function and the weight function are the same. So, v is also N i v i plus N i plus 1 v i plus 1. So, you can write this as v i v i plus 1 into N i N i plus 1 just in the transpose form of the previous one because that will help us in matrix manipulation.

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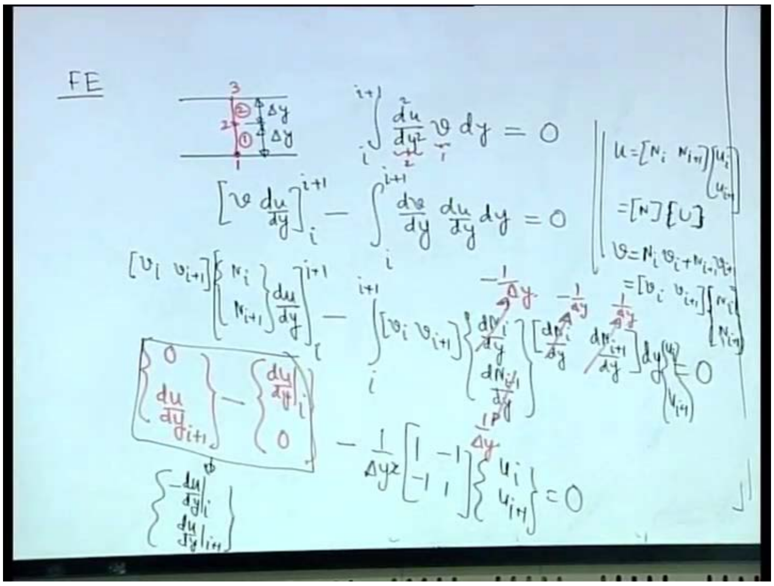

Let us substitute those here. So, in place of v we write v i v i plus 1 into N i N i plus 1 multiplied by d u d y at i i plus 1 minus integral of i 2 i plus 1 d v d y. So, again we can put v i v i plus 1 and this will become d N i d y d N i plus 1 d y because N's are function of y, but v’s are not a function of y similarly in place of d u d y you can write d N i d y d N i plus 1 d y dy. So, into u u i u i plus 1 which are your unknowns that you need to solve now considering that each element has length d y sorry delta y that is y i plus 1 minus y i equal to delta y. So, what is d N i d y minus 1 by delta y. So, this is minus 1 by delta y this is plus 1 by delta y this is minus 1 by delta y this is plus 1 by delta y

The next step what we do we consider that this variations are arbitrary. So, essentially we can forget about this and write the terms as if we have cancelled the variations from the or both the terms arbitrary most importantly, but not zero if they were 0 then it would have been division by 0 now N i into d u d y what is N i at i plus 1 0 N i plus 1 at i plus 1 is 1.This will become the upper limit will become 0 1 into d u d y at i plus 1 right we are writing the upper limit minus the lower limit. So, this is like this evaluated at i plus 1 minus the same thing evaluated at i. So, this evaluated at i plus 1 minus the same thing evaluated at i. So, at evaluated at I this will be one times d u d y at I and this is 0. So, this is the boundary term minus 1 by delta y square then 1 minus 1 minus 1 1 there is 1 by delta y square then there will be one integral of d y. So, that will be delta y. One delta y will be remain right this into u i u i plus 1 equal to 0. So, the first two terms the boundary terms we can write together as minus d u d y i and plus d u d y i plus 1.

Next what we will do. So, this is for each element now we will assemble that two elements 1 is element number 1 where i is 1 and i plus 1 is 2 then element number 2 where i is 2 and i plus 1 is 3.

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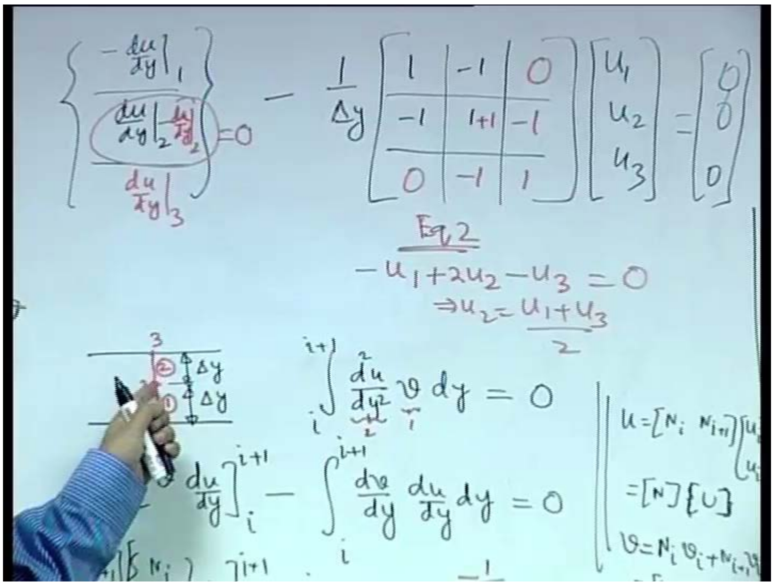

If you assemble this then we can write minus d u d y 1 next term is d u d y 2 we will add the f x of the second element subsequently, but first let us write the corresponding entries. how many entries are there? 3 entries here because there are three nodes and it the corresponding stiffness matrix will be 3 by 3. So, then for the let us write for the first element for the first element 1 and 2. So, 1 minus 1 minus 1 1 for the second element the points will be 2 and 3. So, it will be minus d u d y 2 this will be plus d u d y 3 minus i and plus i plus 1. So, i is 2 and i plus 1 is 3 here it will cover 2 and 3. So, 1 minus 1 minus 1, 1 our objective is to solve for u 2 because 1 and 3 are located at the boundaries no matter whether it is top half of the domain or bottom half of the domain. So, you can see that this become 0. So, for the second equation what you get equation number 2. So, here basically we will get 3 equations 1 for u 1 1 for u 2 1 for u 3 equation for u 1 and u 3 you do not require because there you have boundary condition values you have Dirichlet boundary condition that is specified values of u 1 and u 3, but you required the equation for the u 2. So, what is that minus 1 into u 1.

So, minus u 1 plus 2 u 2 minus u 3 that is equal to 0 that is the equation two from this system of equations that is u 2 is equal to u 1 plus u 3 by 2 again after. So, much of effort you can see that you get back the same equation as what you got for the finite volume or finite difference that is the midpoint is the average of the two points located in the boundary and this is also not unexpected because here also you have considered piecewise linear profile which is the shape function of the interpolation function in the language of finite elements.

So, because you have considered the linear profile you will get the same consequence as that you get using a linear profile from a finite difference or finite volume it is a linear problem. It does not have any scope of having any other variation. You can see that this is a simple example where through different approaches we merged to the same answer and with the same profile assumption either explicitly or implicitly we get back the same outcome from a very simple governing differential equation. This is about the discussion on the questions that we had for the mid-semester examination

Now, we have discussed about one-dimensional problems, but in reality we know that problems are two-dimensional or in the most general case three-dimensional therefore, we need to graduate slowly from one-dimensional to two-dimensional or may be possibly three-dimensional we have to keep in mind; however, that if we increase the dimensionality the basic principle of solution of the problem does not change and that is why one-dimensional problem is. So, important to get a grasp on the or grip on the method of solution because once you understand that it is a straightforward extension of that to multiple dimensions that one is looking for two-dimensional and three-dimensional problems.

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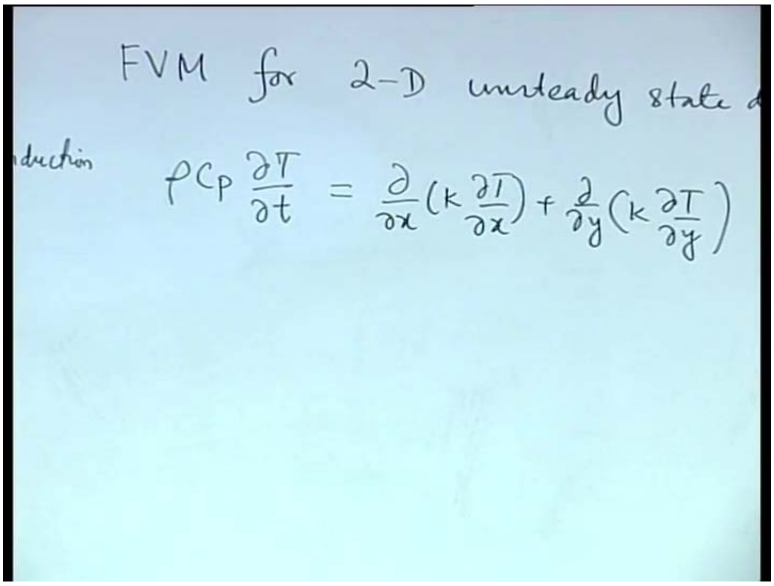

We will illustrate this for the case of finite volume method for two-dimensional unsteady state diffusion problem. So, what is our governing equation let us take the example of heat conduction rho C p del T del T is equal to this 1.

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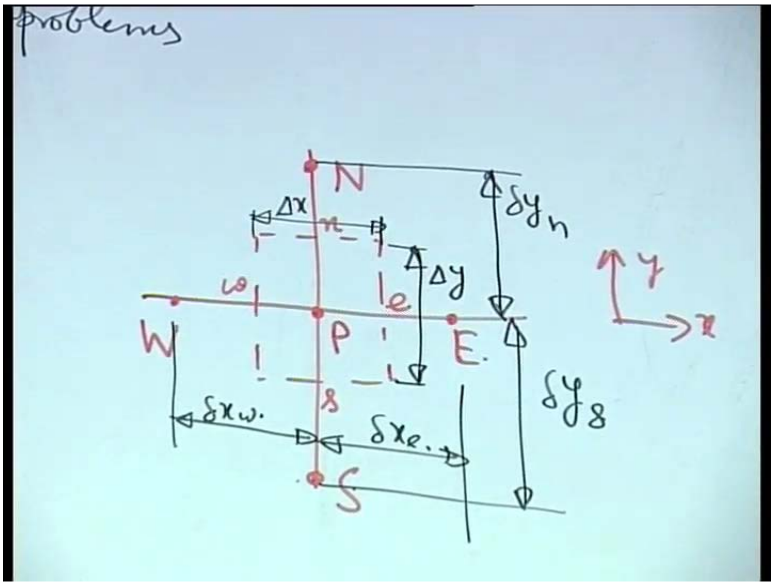

We need grid layout for the finite volume method now it is a two-dimensional problem. So, our control volume is like this with p as the central point e and w are east and west grid points now it is a two-dimensional problem. If this is x this is y along y direction we have grid points N and S north and south thus control volume faces are represented by small letters. So, small e small w small n and small s and various dimensions are given as follows delta x e delta x w delta y n then delta y s delta x and then delta y. What is a first step? Integrate the governing differential equation with respect to all the variables so, time and special variables.

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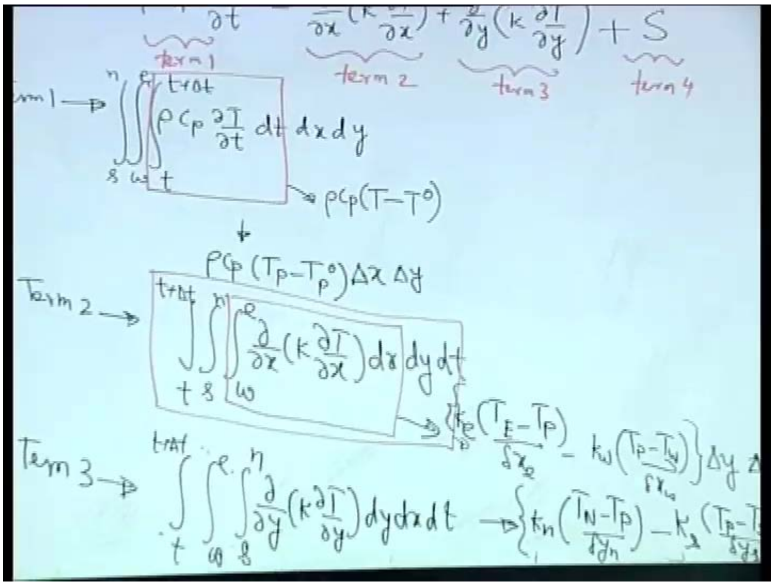

Let us call this as term 1 this as term 2 this as term 3 if you want to put a source term let us put a source term also term 4 to make it very general. So, for the term 1 what it will give integrate it with respect to time x and y. So, from time t to t plus delta t what is limit of x small w to small e what is a limit of y small s to small n.

So, let us do the integrals 1 by 1 first we considered the time integral. This will be rho C p into t at time t plus delta T minus t at time t. This will be rho C p T minus T 0 at time t plus delta t we use without any superscript as we have considered for the one-dimensional unsteady state problems same notation we are using minus T at the old. So, t at present minus T at the old time step.

So, this d x d y. When we integrate this with respect to x what is the profile assumption that we can take for this we can take piecewise constant because there is no derivative we need not take it as something different form of a constant. If it is a piecewise constant temperature profile within each control volume then this t becomes T P. So, with respect to x if you integrate the next step will be it will be rho C p T p minus T p naught is whole thing is constant. It will come out of the integral d x will become x e minus x w x small e minus x small w is what delta x and that integrated with respect to y is delta y. So, that is the term 1

Let us consider term 2. So, integral d x d y d t x from small w to small e y from small s to small n time from t to t plus delta t now. So, this of course, depends on what is the time integration scheme that we are using, but before that let us consider the special integration. So, first this integral. So, it will become k del T del x at small e minus k del t del x at small w right what kind of profile you can take piecewise linear. You can see that it is a basically locally one-dimensional calculation that you are doing and adding dimensionality over that this is a local one-dimensional calculation. It will become k d t d x at small e minus k d t d x and at small w. So, I am writing it in one step.

It is K e into T E minus T P by delta x E minus K w into T p minus T w by delta x w . The final term by considering the entire integration will be what this term integrated with respect to y into delta y and integrated with respect to time. So, integral of t d t it depends on what scheme you are using explicit implicit or Crank-Nicolson. Let us considered fully implicit scheme remember this discretization will be different if you considered any explicit scheme or Crank-Nicolson scheme like that.

So, you considered this as a fully implicit scheme. So, that the temperatures remain this one and only you are multiplying by delta t because in the fully implicit scheme the temperature remains temperature at the end of the time step that is taken as the volume of the temperature throughout the time step similarly what will be term 3 we can write now easily by analogy with the term 2.

Let us just complete it. So, d y d x d t. So, y is from s small s to small n x is from small w to small e and t is from t to t plus delta t. So, what will be the corresponding term in place of k small e it will be k small n into t n minus T p by delta y n minus k small s into T p minus t capital S by delta y s into delta x into delta t .

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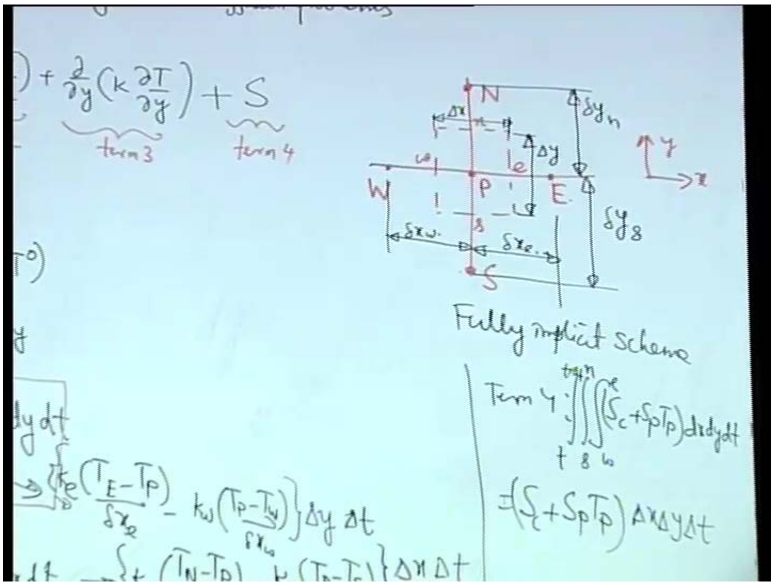

So, here also what you do you first integrate with respect to you first integrate with respect to y and then with respect to x and with respect to t because it is fully implicit t becomes t remains same as that at the end of the time step and for the term four you can consider S as term four is integral S you can consider as S c plus S p T p into d x d y d t x from small w to small e y from small s to small n t from t to t plus delta t . So, what it will become S c plus S p T p will be a constant with respect to all those integrals. So, remembered we are considering a linearize source term. So, that we can able to write it in this form this into delta x delta y delta t. So, we have got all the four terms the final job is to assemble these terms.

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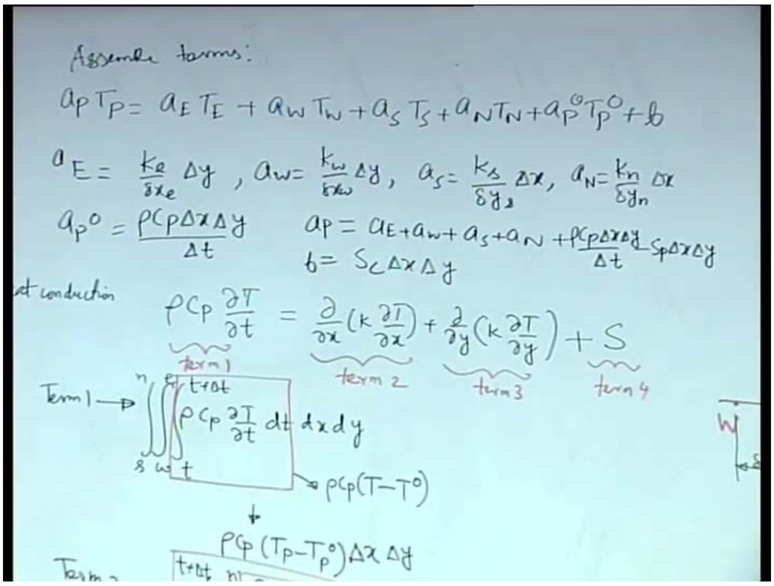

So, to assemble these terms what we do is we divide all the terms by delta t we divide all the terms by delta t if you do that we will get the equation of this form in this form. So, let us try to fill up the corresponding coefficients what is a E K e by delta x e into delta y what is a W? K w by delta x w into delta y a S K s by delta y s into delta x a N K n by delta y n into delta x a P 0 rho C p into delta x into delta y by delta t a P a E plus a W plus a S plus a N then plus rho C p delta x delta y by delta t then there is 1 S p term minus S p into delta x into delta y and b is S c into delta x into delta y.

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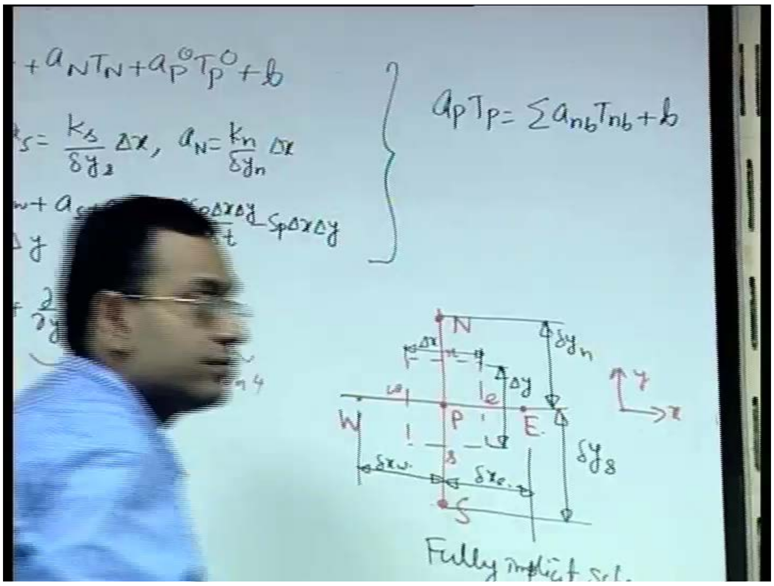

So, what we can see that this equation also can be cast in the same form as that of the generic form that we have developed for a one-dimensional problem what is that a P T P is equal to summation of a neighbour into T neighbour plus b where here there are 4 space neighbours and one time neighbour this cast the system into a system of linear algebraic equation, but now there are four neighbouring special neighbours and one temporal neighbour. So, those appear instead of a fewer number of neighbours for one-dimensional problem that is a first observation.

Second observation is that you can see the physical significance of various coefficients what is this a E for example, what is this delta y? delta y is. So, this length is delta y length perpendicular to the plain of the figure is one. So, unit width is considered. So, delta y into one is the face of this area. So, it is just like conductivity into area of the face by length k a by l what is that conductance inverse of that is resistance thermal resistance due to conduction l by k a this is k a by l. These coefficients physically represent thermal conductance of various faces and the other important thing you may observe here is that this is an unsteady problem, but you can convert it into a steady problem by setting delta t very large because if delta t tends to infinity then extra terms which come out of unsteadiness become 0.

This is some trick that many times we play say you have a program which is given to you or which you have developed for an unsteady problem, but you want to use the same program for a steady problem what trick you can play just use the same program set artificially delta t equal to very large. So, that the term which contains delta t will become 0.

To summarize we have now got an idea that how to develop the discretization equation for a two-dimensional problem. I give you a homework that you do the same thing for a fully explicit scheme. We have done it for a fully implicit scheme you repeat the same exercise for a fully explicit scheme and similar extension you can also do for a three-dimensional problem. Till now we have seen what is the basic principle of discretizating diffusion type of problems in one-dimensional, two-dimensional or even this can be extended to three-dimensional at the end. We get a system of algebraic equations our next objective will be to see that how to solve this system of linear algebraic equation that we get from this discretization that we will take up from the next lecture thank you