Computational Fluid Dynamics

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Lecture No. # 02

Conservation of Mass and Momentum: Continuity and Navier Stokes Equation

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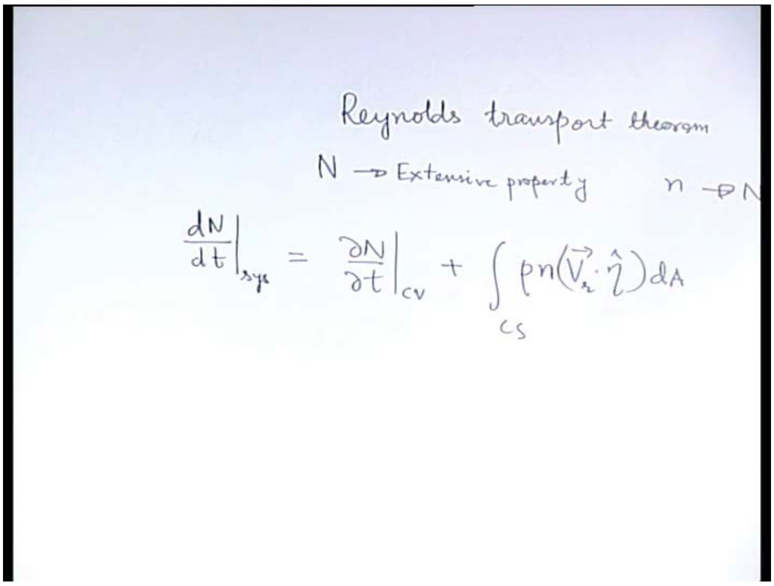

We continue with our discussions on the fundamental conservation equations; and we will start again with Reynolds transport theorem, which basically relates the rate of change of a quantity with respect to a system, with the same, with respect to a control volume. So, if N is an extensive property, then we can write…

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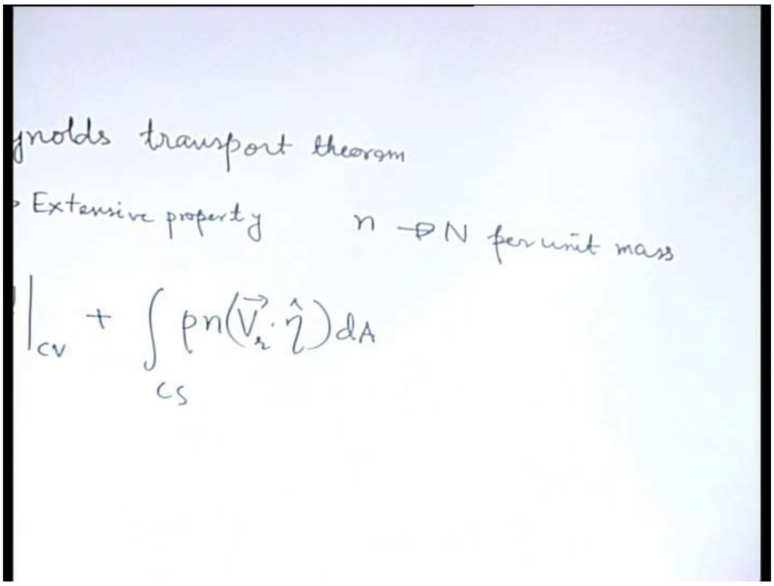

And n is the N per unit mass. So, let us just recapitulate carefully that, what are the meanings of the different terms.

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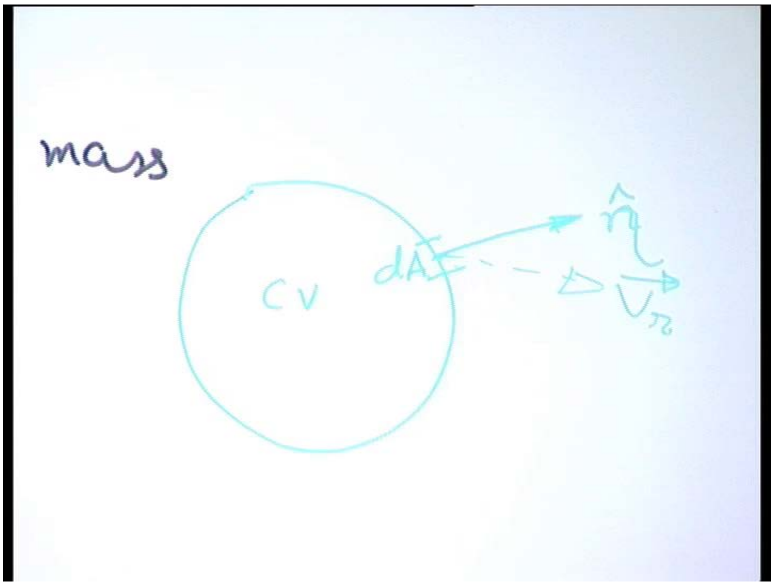

So if we have a control volume like this, and if you take a small elementary surface dA on that, then unit vector in the outward normal direction to the surface is eta; and Vr is the velocity of the fluid, relative to the control volume across that area. So, what these terms essentially mean? This is the total rate of change with respect to a system of something, of fixed mass and identity.

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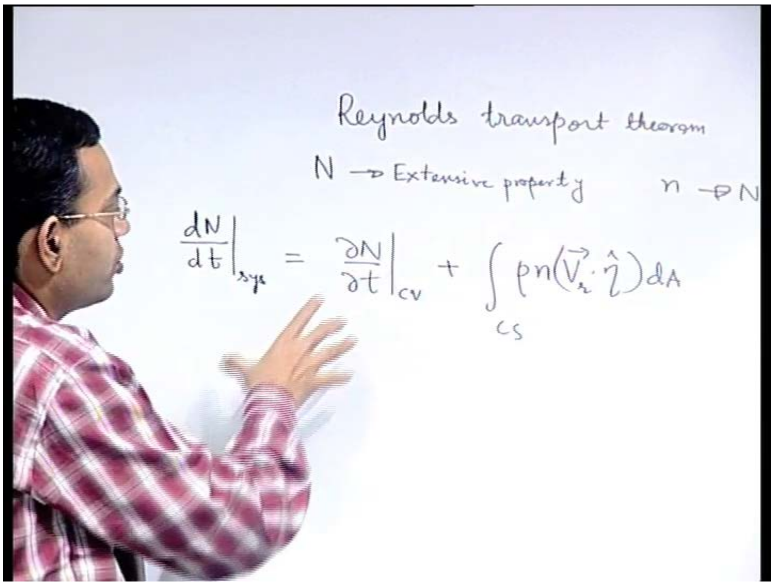

This is the change relative to the control volume and they are not the same; and they are adjusted by a term which represents the balance of outflow and inflow, across a system boundary, or across a control surface, to be more specific. Now, if you want to use this particular principle for various conservation laws, let us see that what is the outcome.

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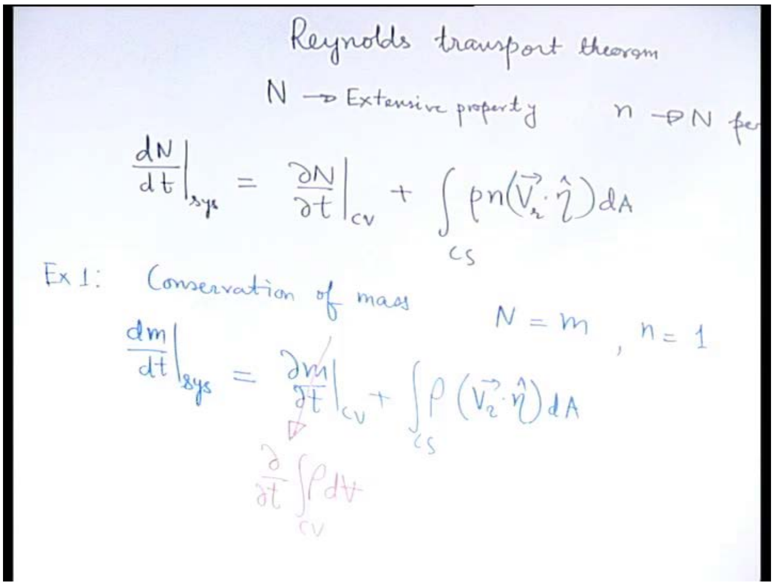

So we consider first example, conservation of mass. So, where N is equal to total mass of the system. So, you can write dm dt of the system; what will be n? n is 1, that is mass per unit mass, that is 1. So, integral of rho. Now, let us try to simplify, or sort of, write this term in a different way; this we can write partial derivative of the mass, inside the control volume. What is the mass inside the control volume? the rho times dV.

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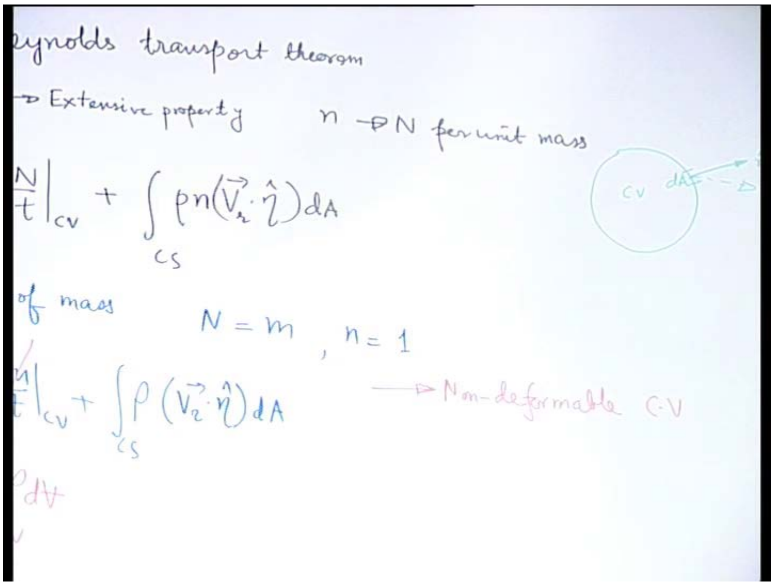

Now, with this, let us make certain simplifications. What are the simplifications? The first simplification is, we take a non-deformable control volume; when we say non-deformable control volume, by that what we mean? We mean that volume of the control volume is not a function of time. So, the volume of the control volume does not change with time.

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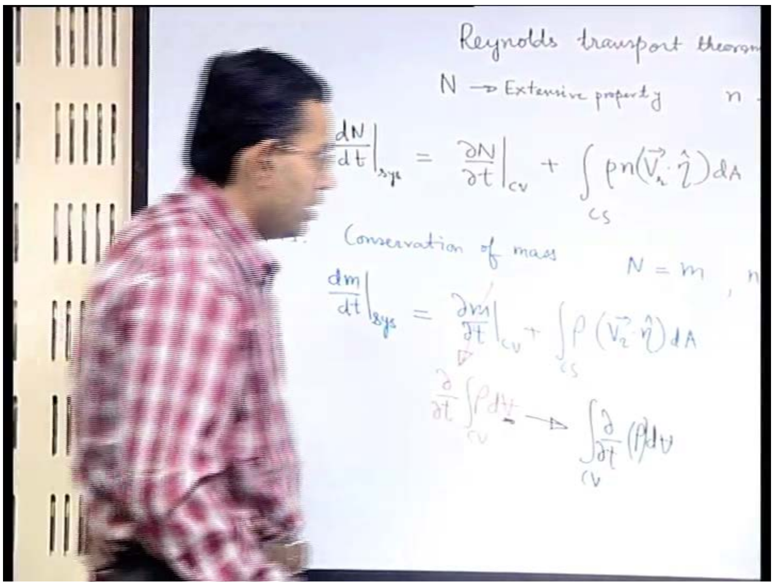

If it is so, then what we can make out from this is that, this variable V is not a function of the time. So, how can we make an adjustment to this one? How can we simplify this one? So, if this variable with respect to which we differentiate, is not a function of this one, then we can easily take this inside the integral; so, we can write this. So, if this was not the case, then what we should have done?

If this was not the case, then it would have been this term plus a couple of correction terms, which are because of the rule which converts a differential from outside integral to inside an integral, that is the Leibniz rule. Now, here we do not have to bother about that, because it is a non-deformable control volume. So, we can put this inside the integral, without requirement of any correction term.

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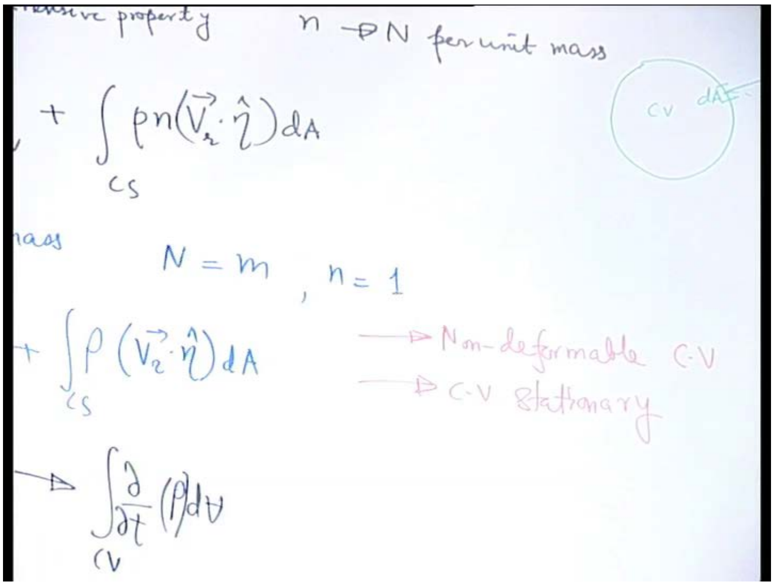

Then, next we consider the control volume to be stationary. When the control volume is stationary, what it means is that, the relative velocity of fluid is the same as absolute velocity.

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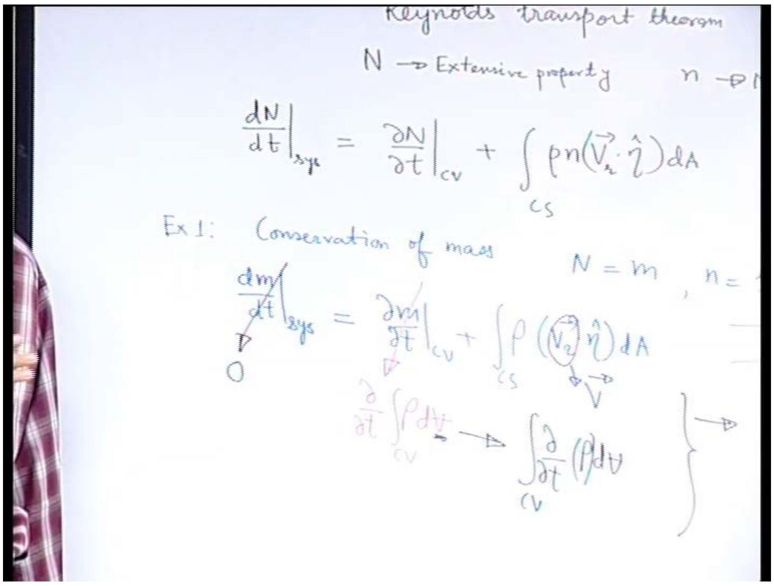

Because Vr is velocity relative to fluid to the control volume; since the control volume is stationary, it is the same as the absolute velocity.

So, keeping these two considerations, one can simplify this equation with a very important consideration that the left-hand side is 0; because by definition, system is something of fixed mass, so mass of a system doe s not change with time. Therefore, this is 0 by definition of a system.

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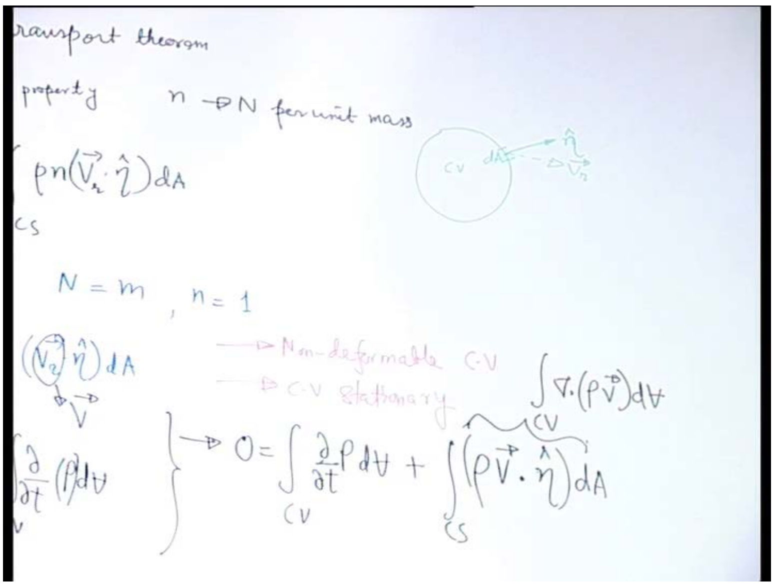

So, you have 0. Now, we can see that this equation has two terms, two integrals; one is the volume integral, another is the surface integral, and we can inter-convert, we can convert the surface integral into a volume integral by using the divergence theorem. So, we can write this term as divergence of row V, it is like, if f is a vector function, then f dot eta dA is divergence of f dV, where f equal to row into V.

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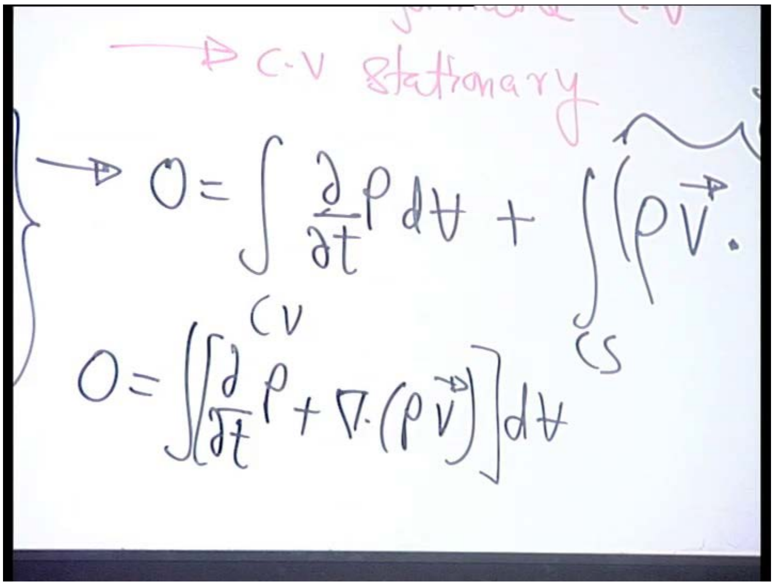

So, with these considerations, we come up with the equation, 0 equal to this one. Now, we have to keep in mind that the choice of the elementary volume dV is absolutely arbitrary.

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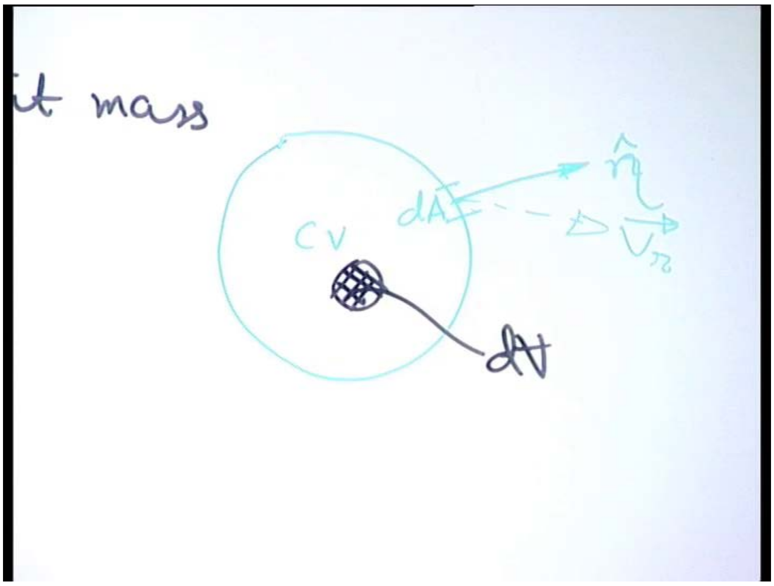

That means within the control volume, you choose any elemental volume and you call it dV; because the choice of dV is arbitrary, that means sort of the limits of integration of this are arbitrary.

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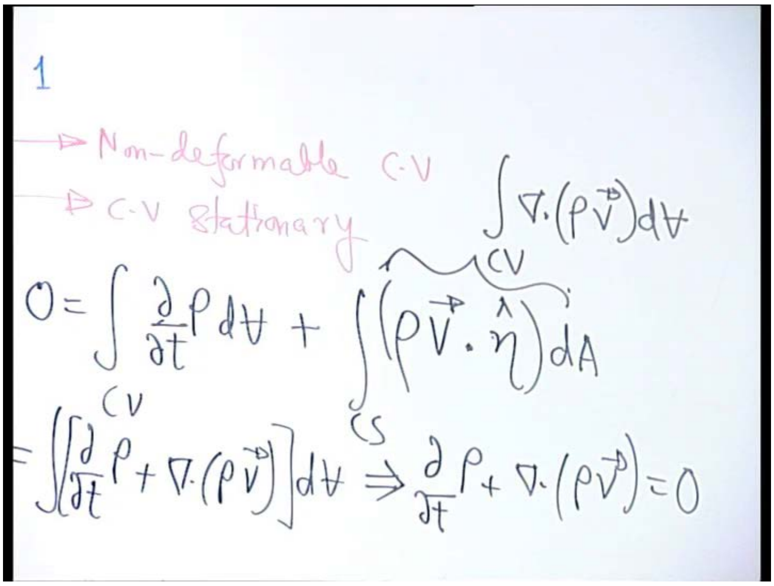

And this has to be satisfied irrespective of the choice of dV, that is where do you choose your elementary volume; and that implies, that the integrant itself is equal to 0.

So, we have to keep in mind that if the choice of this elementary volume was not arbitrary, then it is not always possible to conclude from this, that the function itself is 0, if the integral is 0. For example, you can have the function as a sin x function, which over its period, its integral is 0; but that does not mean that sin x itself is 0 at all points within the period, but we have to keep in mind that they are the choice of dx integral sin x dx, that dx is not arbitrary, it is confined within a particular period; whereas, here the choice of dV is absolutely arbitrary.

Now, this is an example of conservation of mass, and what this example has taught us is, how to convert an integral form into a differential form; we will see in our exercises of C F D that we will many times be requiring integral forms, many times we will be using the differential forms for deriving discrete sets of conservation equations from the continuum sets of equations. But it will be a very important consideration that how to inter-convert the integral forms and the differential forms. So, we start with the integral form here and then convert it into a differential form. All of you can appreciate that this is nothing but the well-known continuity equation.

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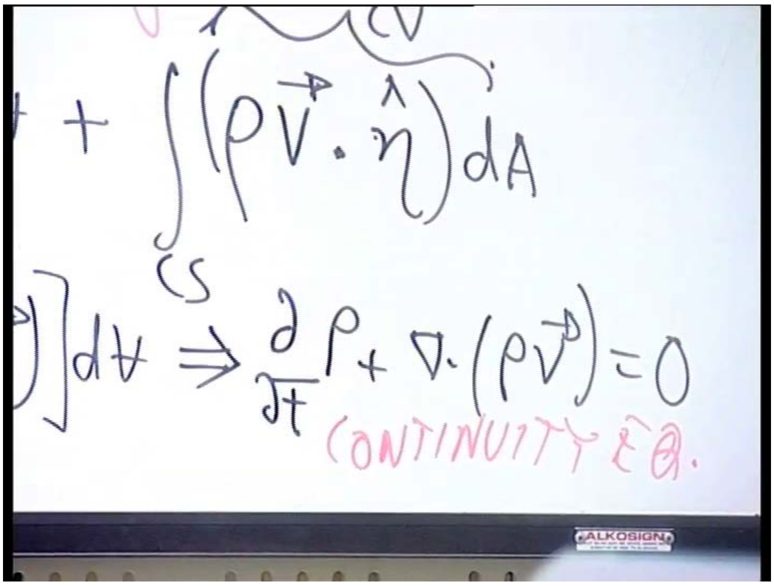

With this consideration, let us now move on to a second example. As the second example, we consider the conservation of linear momentum.

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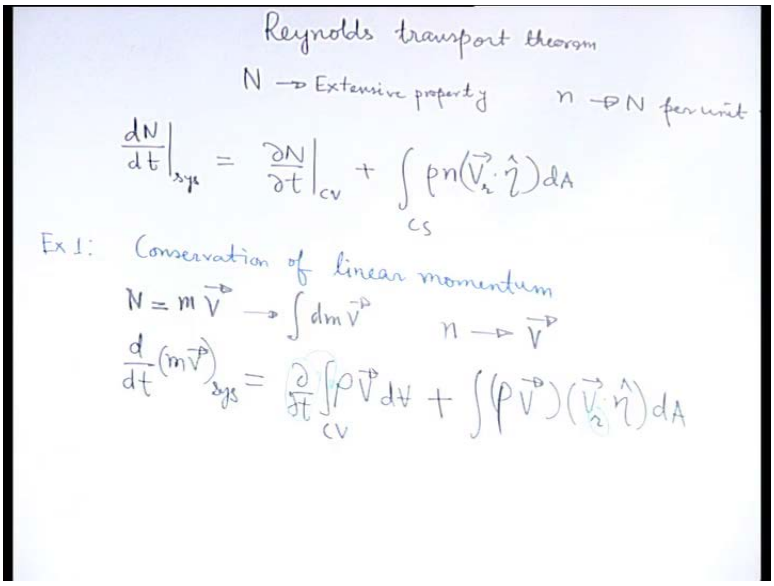

So, when we consider conservation of linear momentum, we will be again keeping the two assumptions, same as the previous case, that is the control volume is non-deformable and the control volume is stationary.

Now, with the conservation of linear momentum as a consideration, what will be N? N is the linear momentum of the system, that is m into v. Now, it is, for a fluid, you may have or even for any non-deformable body, you may have different v at different locations. So, you can also write it as integral of dmv or whatever. But just for simplicity, we write it as mv, where you can sort of represent this v as, v of the center of mass or something like that; but you can also write it as integral of dmv.

Now, what is the small n? This per unit mass; so, that is equal to v. Now, let us apply the Reynolds transport theorem for this particular case. So, we write d dt of mv of the system, or basically integral of dmv, but just to just for simplicity, we write it d dt of mv, is equal to… so, what will be here? Integral dm into V. So, what is dm into V? dm is rho Vdv, dm is rho into d v of the control volume plus… Now, let us make the simplifications. What simplifications we make?

First of all, a non-deformable control volume, so that we can take this inside the integral, that is the first thing; and second is, stationary control volume, so that Vr is equal to V. So, with these considerations and noting that, see this is the utility of writing it in terms of a system. Now, if you write it for a control volume, this is basically the rate of change of linear momentum for a control volume. But that you cannot write it as the total force, but the rate of change of linear momentum of the system, you can write that as the total force, because Newton's laws of motion are applied for a system of particles, but not for a control volume directly. So, we are basically trying to apply the Newton's law, because we know the Newton's law for a system, we use the inter-conversion between the system and the control volume, so that we will be effectively writing Newton's law for a control volume, that is nothing but the linear momentum conservation for a control volume.

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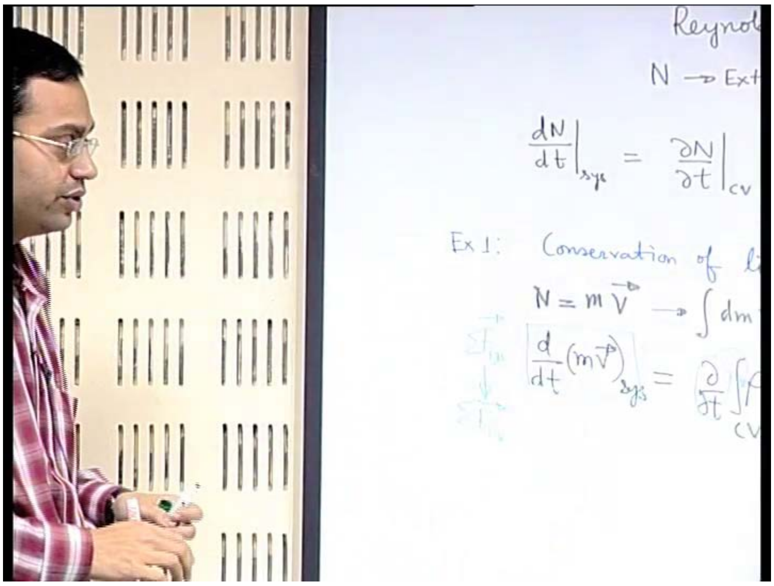

So, this will be resultant force acting on the system. And we have derived the Reynolds transport theorem for the limiting case, when the time interval delta t tends to 0, so that the system and the control volume almost converge on each other; so, that means, this is the resultant force acting on the control volume in the limit, as delta t tends to 0.

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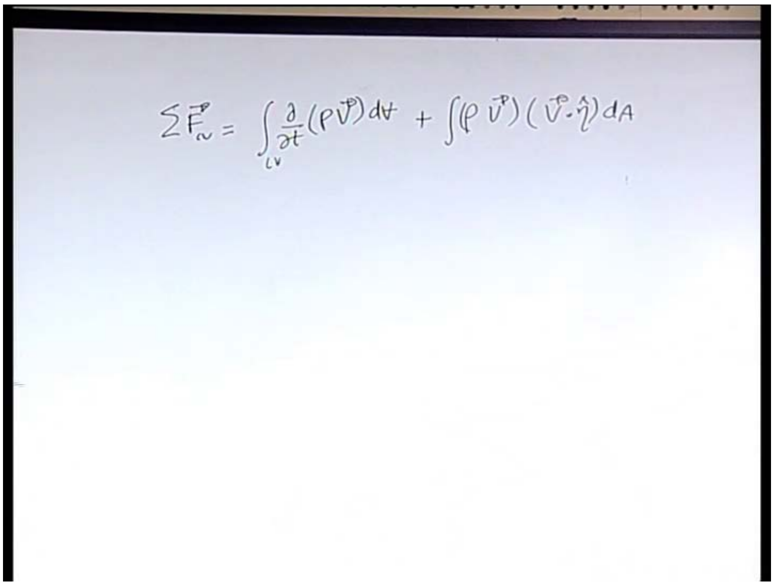

So, we can write, what we can write that resultant force acting on the control volume is equal… So, this is the basic integral form of the equation of conservation of linear momentum. Now, we have to simplify it further, for our special cases; to do that, we will move step by step and first see that what are the forces, which are acting on the system. Because, here we have just generically represented the force on the control volume, but what are these forces?

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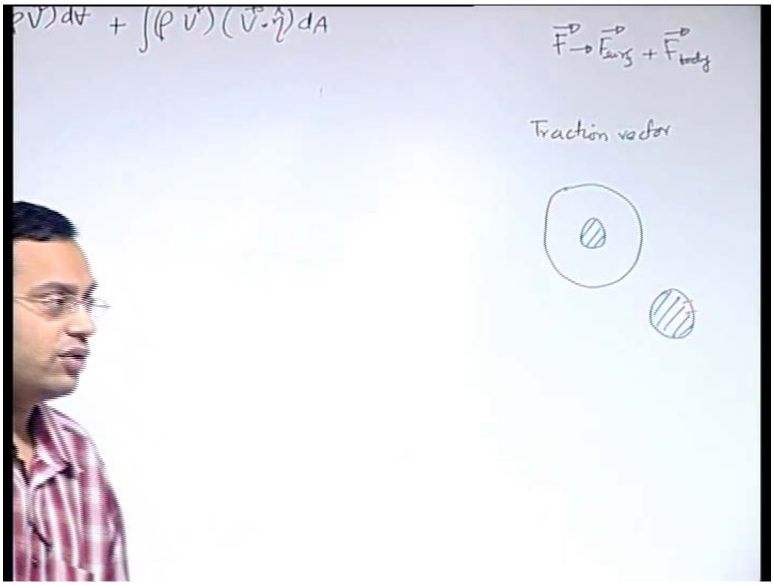

So, in continuum mechanics, we have two types of forces; these forces are the surface forces, and the body forces. So, the force is surface force plus body force. What is the surface force? A surface force is a force, that acts on the surface of a bounding volume and it can be expressed as force per unit area, in some way; and the body force is a force that acts throughout the volume of the body, like the gravity force. Now, with this consideration, let us first try to access, what is a surface force. To do that, what we will do is, we will introduce some concept of a traction vector, which is a vector, which designates the surface force in a formal way.

How does it do? Let us consider that you have a volume from which you take out a chunk, this chunk is bounded by a surface. So, once you take this chunk out, and you take a small element, let us say that you take a small element of area on the chunk, then there must be some force which is exerted by the other part of the material on this chunk. This is just like Newton's third law type of interaction that the other type of material will exert some force on the chunk and chunk will exert an equal and opposite force on the other part of the material. So, that interaction force now can be represented, because you have removed this chunk from the material. So, as if you were drawing a free body diagram of the chunk.

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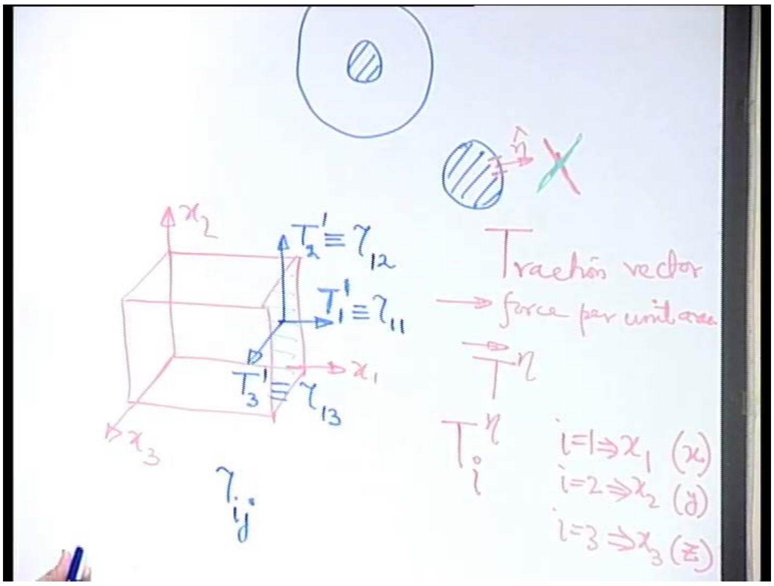

So, when you are representing the surface force, you can represent it in different ways. So, one of the representation is representing it through a force per unit area. So, when that is done, it is usually represented by a vector called as traction vector. This is force per unit area and usually, it has to be given certain additional features, just by mentioning the traction vector, it is not sufficient to prescribe it, why? Because, if you consider an area at a point, the force will not only depend on the choice of the magnitude of the area, but also the orientation of the area.

So if the area is chosen in this way, and at the same point, the area is chosen in a different way, say area is chosen in this way, then because of the change in orientation of the area, say from this orientation to a different orientation, both considering about the same point, you will see that the force will change. So, force at a point, because of this interaction, which we will call formally call as a stress, subsequently, will not only depend on the choice of the location of the area, but will also depend on the choice of the orientation of the area.

So, orientation of the area, as we have seen, is specified by a unit vector in the outward normal direction, we call it eta. So, T with superscript eta, is a formal way of designating the traction vector, where the superscript eta signifies the orientation of the normal to the surface on which this traction vector is calculated. Now, to proceed further, what we will do is, we will consider the components of this one. So, this is a vector, it has its own components, it has its x component, y component, z component like that.

So, if we consider this vector, it’s any component can be represented by an index i, where i equal to 1 implies x1 or x; i equal to 2 implies x2 or y; i equal to 3 implies x3 or z. So, this is known as Cartesian index notation. So, you are using an index for certain purposes, here when you are considering the act of a quantity like a vector, then a single index i is sufficient to describe its component. Now, this traction vector, this is used for any arbitrary surface, for any arbitrary orientation of the area.

It is important to see that when this arbitrary orientation coincides with either of this x1 x2 or x3, then what happens to it; because that, sort of, standardizes the traction vector, in terms of its direction of orientation of the surface, any surface can be arbitrarily oriented, but there are certain standard surfaces, which are oriented having their direction normals either along x1, or along x2, or along x3. So, those are special surfaces on which we have special effects on the traction vector.

What are those special effects? So, to consider that, let us draw a simple diagram. Let us say, this is x1 direction, this is x2 direction, this is x3 direction. This index notation is quite helpful in many ways; first of all, you can write a big expression in a very compact index notation; not only that, if you want to translate it into a computer program, you can use sort of loops, where the loops are according to the indices of different terms in an expression. Now, let us consider the traction vector on this surface.

So, how do you designate it? You have T. What will be the superscript? Superscript will be, we do not write x1, we write 1; it means of course x1; and now, it depends on the component, which we are considering. So, if we are considering the x1 component, so, let us draw these components. So, this we call as with subscript 1; similarly, this we call as with subscript 2; and this as subscript 3. For these special surfaces, we have equivalent notations using tau.

So, we call this as tau 1 2, we call this as tau 1 1, and this is tau 1 3. So, in general, if we call this as tau ij, where these two indices are effectively, the indices like eta and I, we are replacing eta by j, for all cases we can do that; but only, when eta coincides with either x1, or x2, or x3, we can use an index for it.

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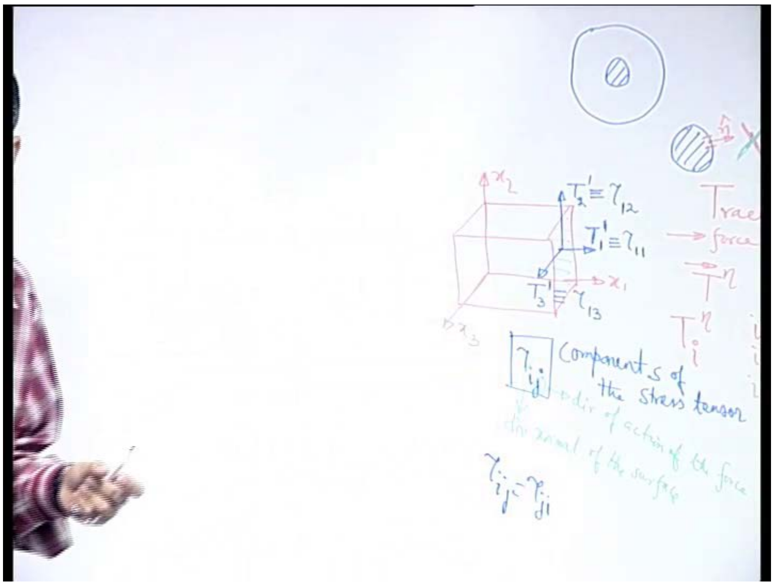

So, we write tau ij, where what is i? Direction normal of the surface, and what is j? Direction of action of the force itself. Remember, this is described as force per unit area, so, all these are expressed in terms of per unit area. So, with this understanding, we can clearly make out that, when we are writing this tau ij, these are formally called as components of the stress tensor. So, let us make a note of that. So, this is nothing but components of the stress tensor. So, you require two indices i and j, then what is the requirement for the two indices, because the stress at a point does not only depend on the force per unit area, but it also depends implicitly on the orientation of the area that is chosen, to specify the stress; and there comes the index i. So, it requires two indices for its specification and more formally, it is called as second order tensor; in that way, a vector is a first order tensor, because it requires only one index for its specification, scalar is a tensor of order zero, because it requires no index for its specification.

Of course, there are more interesting properties of a second order tensor, and while deriving the momentum equation, we will come across one such interesting property; but like this sort of, helps us to recapitulate and familiarize ourselves with the tau ij notation, which we will be following for subsequent derivations. Now, the big question is, that well if we know this tau ij, so, how many such components are there? So, i can be from 1 to 3 and j can be from 1 to 3. So, you could have 3 into 3, 9 components; but you have 6 independent components, because it can be shown from the conservation of angular momentum, that tau ij is equal to tau ji.

So, usually we do not have separate equation for conservation of angular momentum for fluids. So, what except for the cases, in which fluids have particles which are rotating, or fluid elements themselves have some sort of particulate nature, or the sometimes, some force, some very special fluid known as micro-polar fluids, and so on. But we are not going into such details; so, we are considering such cases, where the fluid elements do not have any body couples, so when the fluid elements do not have, just like you can have body forces, you can also have body couples. But at normal circumstances, if it does not have embedded particles like that it, will not be able to sustain any body couple; and if it cannot sustain any body couple, the angular momentum conservation will automatically give tau ij equal to tau ji. So, basically you have 6 independent components of tau ij; when you have 6 independent components of tau ij, that is so nice; but that is a bit restrictive, because you can use those, only for surfaces which are having normals oriented along x1, x2, or x3. But what happens for surfaces which are arbitrarily oriented, that is the normal does not coincide with either of x1, x2, x3. So, to do that, let us consider a simple exercise.

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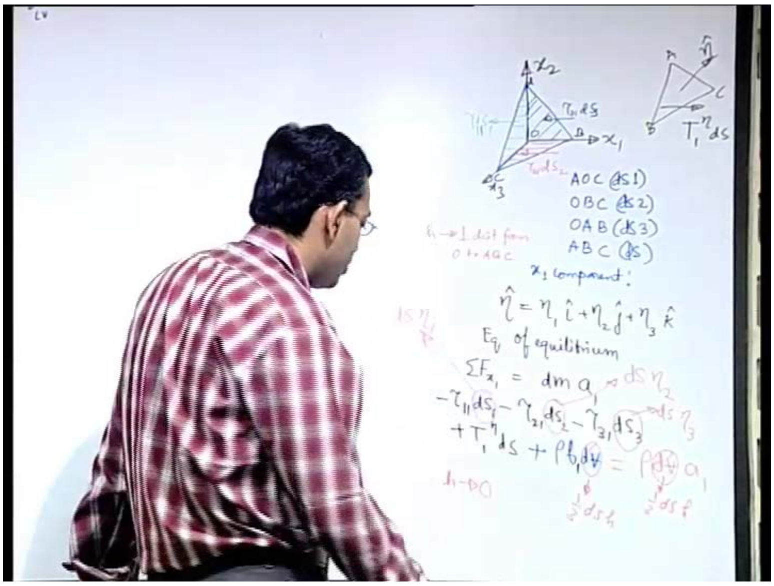

Let us try to construct, such a surface, by constructing a volume. So, what we do is that, we construct a volume with these four vertices A, B, C, O. So, how many surfaces are there? You have four surfaces, let us mark these surfaces, say AOC, we give it a name S1, surface number 1; OBC, we give it a name S2, surface number 2; OAB, we give it a name S3, that is surface number 3; and ABC, which we give it a name S. Now, why we have chosen such a surface?

Why we have chosen such a volume? This volume is bounded by four surfaces, out of which three surfaces have their normal directions either along x1, or along x2, or along x3. And only the fourth one, the surface S is arbitrarily oriented. So, if we write an equation of equilibrium for this particular volume, then the force on the surface, for which the orientation of the normal is arbitrary, can be expressed in terms of the forces on surfaces, for which normals have directions either along x1, x2, or x3.

So, we will be able to write an arbitrary traction vector, in terms of the tau ij components. How to do that? Let us just consider only one component, let us say, we are interested about x1 component. So, let us identify the forces, first let us identify the force on surface S1. So, we are interested in only x1 component of the force; we will follow a sign convention that the tau ij will be acting along positive j, if the normal is along positive i.

So, here if you consider this surface as an example. So, if we consider it as a tau ij notation, first of all, what is i? i is 1, plus or minus, we will account while showing the direction of the force, but not while writing it; and what is the j? We are interested about the x1 component, so tau 11; but because, the normal is along minus 1, we will show the force component itself along minus 1, that is the sign convention. So, tau 11, the second one is the direction of the force, we could have shown it either along plus 1 or along minus 1, because the normal is along minus 1, we show the positive component of the force, as per sign convention along minus 1.

So, tau 11 times S1, because remember this, tau ij or T these are forces per unit area. If we are interested to construct a volume which is differentially small, because eventually, we will write an expression which is valid at the point O; so, we will swing the volume to a differentially small volume around the point O. So, it is formally more logical, if we call this as dS1, dS2, dS3, and dS, just for notation; because we eventually, consider them to be differentially small surfaces, and the volume also differentially small.

So, x1 component, you write, on this surface tau 11, dS1; then let us write that for the bottom surface, what will be that? tau 21 dS2; similarly, for this surface tau 31 dS3; and what about ABC? We have to specify a normal direction for ABC; for ABC, we have to keep in mind, that we cannot use the tau ij notation; because tau ij notation, you can use only for surfaces, which are oriented along x1, x2, or x3; but ABC are arbitrarily oriented.

So, let us say that normal to ABC, let us just make an arbitrary sketch, that if this is ABC; let us say that normal to ABC is oriented along eta. So, you have to use the t notation, rather than tau notation. So, what will be the x component of force on ABC? t eta 1, this is the direction of action of the force times dS; and eta, let us write this as eta 1 i cap plus eta 2 j cap plus eta 3 k cap, where eta 1, eta 2, eta 3 are the direction cosines of the unit vector, eta. So, basically the components along x, y and z. Now, let us write the equation for equilibrium.

So, if we write equation for equilibrium, resultant force along x1, if the fluid element is under rest or uniform motion, then resultant force is zero. But if it is under dynamic equilibrium, and if it is accelerating, then it is same as the mass of the fluid element times acceleration, along one. When we consider the force, it should be some of the surface force and the body force. So, let us first write the surface force. So, minus tau 11 dS1 minus tau 21 dS2 minus tau 31 dS3. Then, plus T1 eta dS plus. So, this is total surface force.

Then, let us write the body force. So, the body force, let us say that b is body force per unit mass. So, what is the mass of an elemental volume? rho into dV. So, rho bdV is the body force, we want its component along one; so, rho b1, b1 is the body force per unit mass along x1 direction. So, we will be able to write this, in terms of several other quantities; but just let us write it in this way, first. It is equal to dm, that is rho dV times a1. Now, you can make certain geometrical simplifications. What are the geometrical simplifications?

From this figure, you can see that dS1 is nothing but the projection of dS on x2 x3 plane. If you see the area dS, that is ABC is projected on x2 x3 plane, then that becomes dS1. So, that means in terms of an area vector representation, dS1 as a area vector is the projection of the dS, as an area vector on the x2 x3 plane. So, that means we can writ,e dS1 is dS times eta 1. How do you get it? dS, in the vector notation has eta 1 i plus eta 2 j plus eta 3 k; dS1, in the vector notation has dS1 into i cap, with of course minus sign, but forget about that, because here we are writing the magnitude minus has already been taken care of.

So, you have eta 1 i plus eta 2 j plus eta 3 k dot with i; to get the component, you make the dot product of two vector. So, eta 1 i dot i becomes eta 1; so, it becomes dS eta 1 component of dS, along that direction. Similarly, this will be dS eta 2; and this will be dS eta 3. Now, you can also calculate the volume, if you know that, let us say h is the perpendicular distance from O to ABC; then what is the volume of this elemental, volume? It is one third into dS into h. And this also, one third dSh. Now, we are interested to express the traction vector, in terms of the stress tensor components at the point O. So, we shrink the total volume, so that it converges to the point O; that means, we take the limit as h tends to 0; as h tends to 0, you will see that the body force starts, when the acceleration term will vanish.

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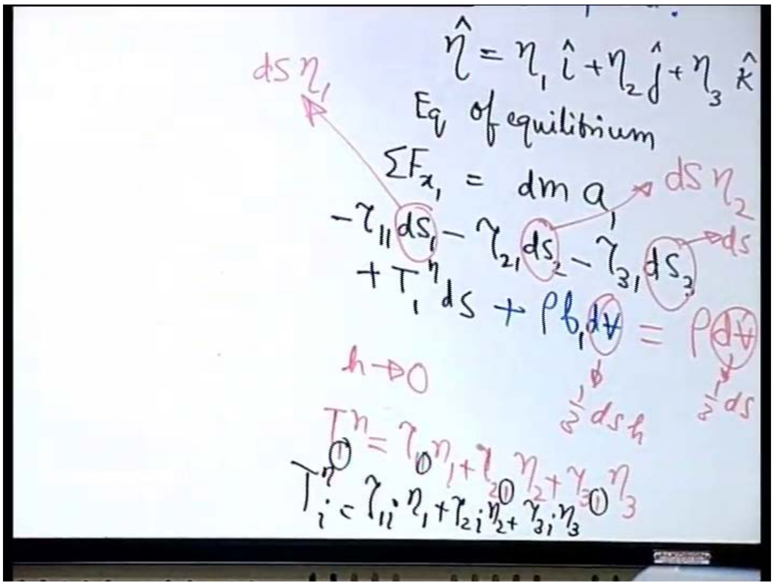

And you will be left with, t eta along the direction 1 is tau 11 eta 1 plus tau 21 eta 2 plus tau 31 eta 3. So, you can see that, this index 1 is same as what index 1 appears here. So, in general you can write replace this 1 with I, for any arbitrary direction; so this could be 1, 2 or 3 to generalize, we call it I; tau 1i eta 1 plus tau 2i eta 2 plus tau 3i eta 3.

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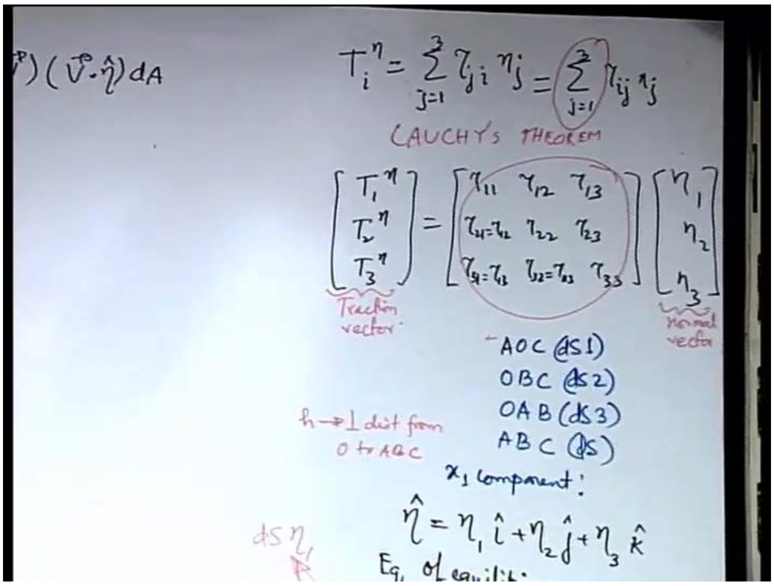

So, in a compact notation, we summarize our result, that we write, tau i eta is equal to summation of tau ji eta j, where j is equal to 1, 2, 3. Now, because tau ji is equal to tau ij, this is as good as. When we write this, we usually omit this summation, that is known as

Einstein's convention, that is when you write this, you keep in mind that, there is an invisible. How do you know that there is an invisible summation? If there is a repeated index, so, you have j as the repeated index, that means, the repeated index is summed up, and the repeated index is dummy because, instead of j you could use k, l, m, n, whatever, it makes no difference; only thing is that, index is summed up or this expression is summed up over that index; this expression is known as Cauchy's theorem. It is a very important theorem because, it relates the traction vector on an arbitrary plane with the stress tensor components, which are sort of referenced with the known planes.

It is also possible to write it in a matrix notation. So, you can write its components. You can see that we get two interesting vectors, this is the traction vector; this is the normal vector. Vector is of course, expressed in terms of its scalar components. And you can see, this stress tensor component collection, what it is doing is, it is mapping the normal vector on to the traction vector. So, this is an interesting property of a second order tensor, that it maps a vector onto a vector.

Here in this physical example, it is mapping a normal vector onto the traction vector. So, this helps us in writing the expression of the force or expression of the traction vector in a formal way. So, this is about the surface force, and writing the body force is more or less trivial. So, what we will do now, we will use this understanding that we have discussed for over a period of time.

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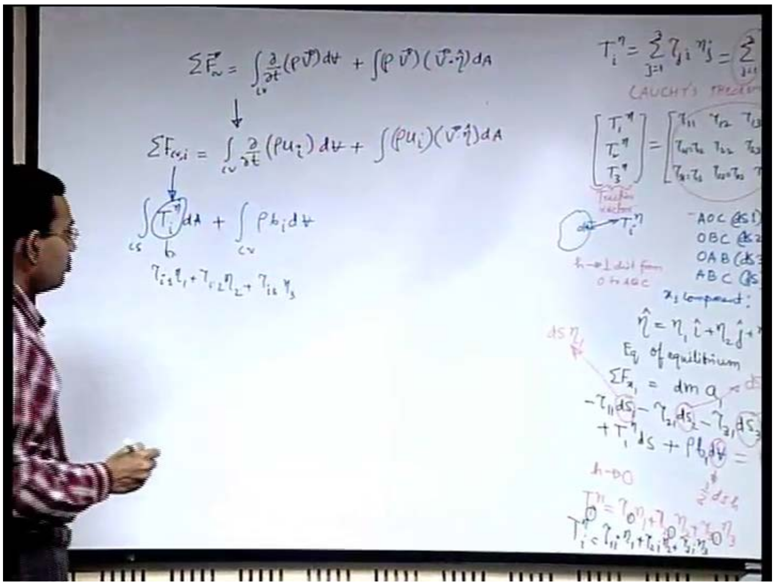

To write now, this expression for conservation of linear momentum; but for a specific direction, say direction i; so, resultant force on the control volume along the direction i, we will use the notation ui for velocity component along the ith direction; so, ui is the i th component of the velocity. Now, what we will do is, we will write the force on the control volume, as sum of the surface force and the body force.

So, what will be the surface force? If we consider a small elemental surface, let us say dA, then and let us say that eta is in its normal direction, so, its traction component along i is Ti with superscript eta, and that is per unit area, integral of this one over the control surface, plus the body force. Now, this one, we can, what we can do is, we can write this as, we have just seen that what is the expression of these, that is tau i1 eta 1 plus tau i2 eta 2 plus tau i3 eta 3. So, it is, we have written here, tau 1i, tau 2i, tau 3i, but because, tau ij is tau ji, we tau i1, tau i2, tau i3, like that.

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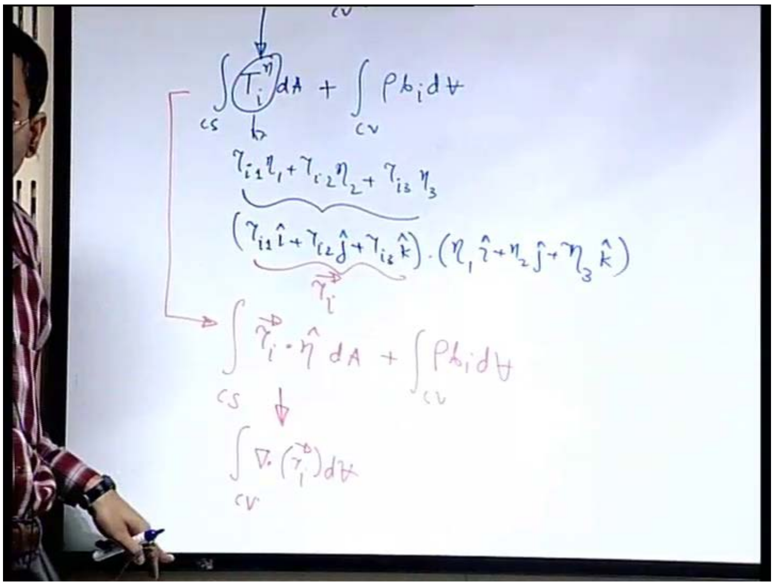

So, we can write this in terms of a dot product of two vectors, tau i1i plus tau i2j plus tau i3k dot with eta 1i plus eta 2j plus eta 3k. We can just make this vector by taking a dot product of these two. So, this the first vector, let us give it a name, vector tau I; this is just a name that we are giving which has components tau i1, tau i2 and tau i3. Keeping that in mind, we can express this as integral tau i dot eta dA.

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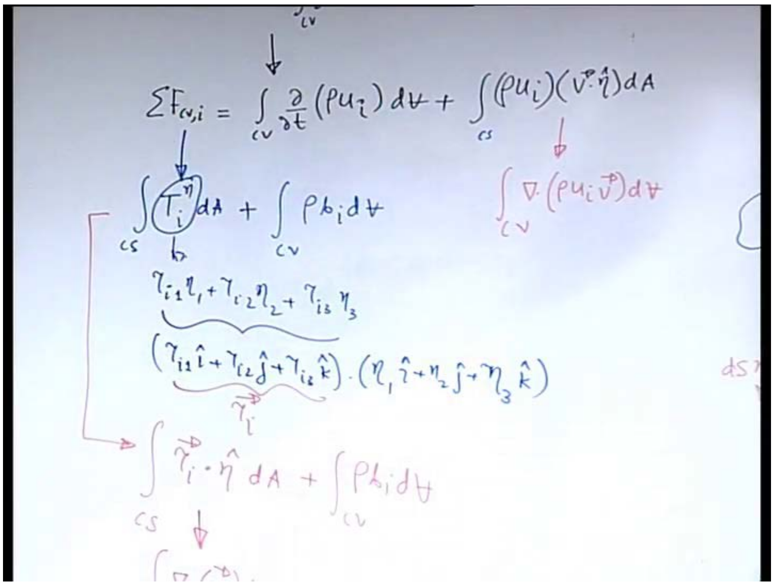

Using the divergence theorem, we can express this in terms of a volume integral. This is nothing but divergence of tau i. So, the left hand side both terms are expressed in terms of the volume integral; right hand side, the first term is already expressed in terms of the volume integral, the second term is an area integral; so, you can write this in terms of a volume integral.

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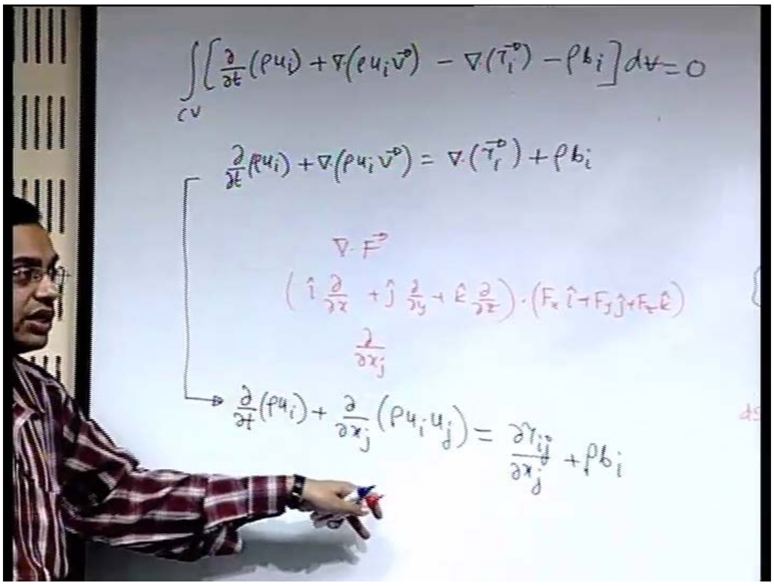

Now, the left hand side and right hand side, all terms are expressed in terms of volume integral, so, we can take all the terms together; if we take all the terms together, we have this equal to 0. Just similar to the case of conservation of mass, in this case also, you have the volume dv is arbitrary; because, the volume dv is arbitrary, the integrant is 0, that means you have. So, the integrant itself is 0. Now, you can write this entire thing in terms of an index notation, how you can do that?

If you write, you consider an example, let us say that we are considering divergence of some vector F. so it is, so, you can see that its x component is partially derived with respect to x, y component with y, and z component with z. so, that means, you can write it in terms of in general partial derivative with respect to xj, where j varies from 1 to 3, because there are three partial derivatives that you are essentially doing with the components.

Keeping that in mind, you can simplify this, in this way, where uj is the j th component of the velocity; so, you are differentiating the j th component with xj, just like you are differentiating the x component with x, y component with y, z component with z. So, this there is no difference between these two, we have just written this using the index notation and considering the divergence operator using the index notation.

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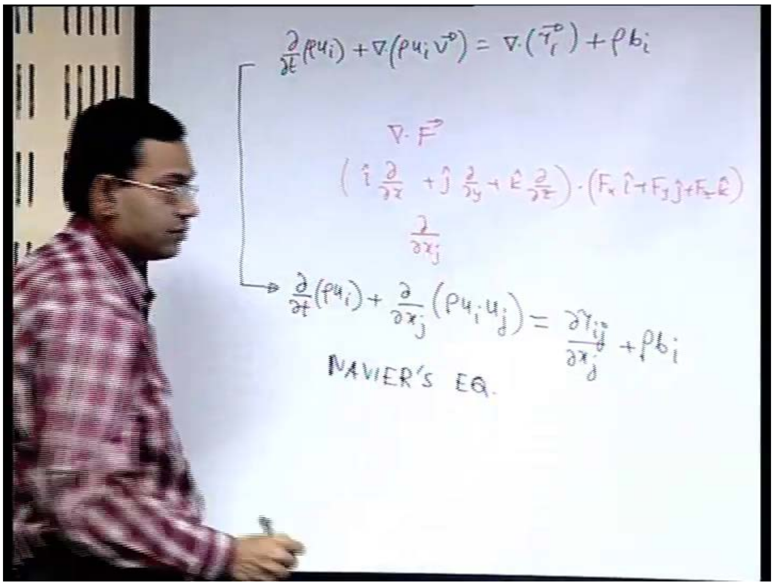

This equation is known as Navier's equation of equilibrium. This equation is not itself complete, because we have not specified what is tau ij, and that depends on how we can relate the behavior of the fluid in terms of its stress response to the rate of deformation, that we will see in the next class.