Computational Fluid Dynamics

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Lecture No. # 19

Part 1: Stability of Second Order Hyperbolic Equations

Part 2: Mid-Semester Assessment Review (Questions and Answers)

We will divide today's lecture into two parts. In the first part, we will continue with the consideration of stability of hyperbolic equations, and in the second part we will discuss about some of the questions and answers which we had as a part of our mid-semester assessment.

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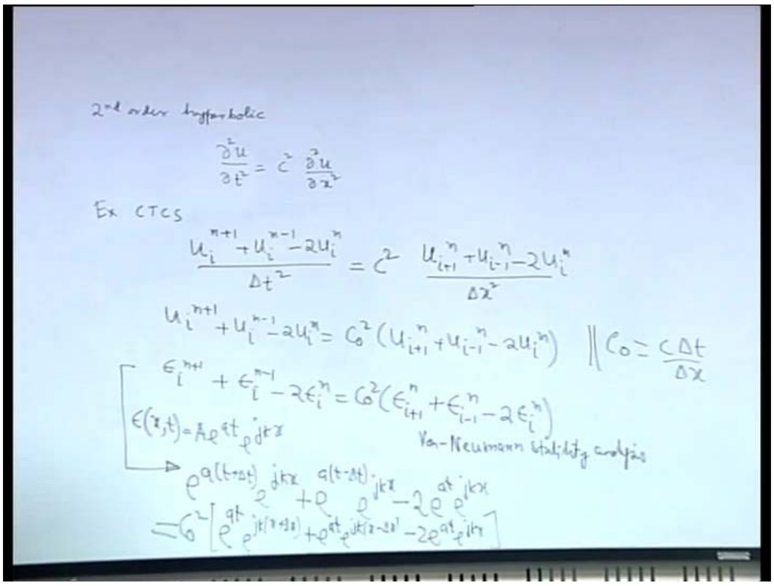

So, to consider the first part, we have till now considered a first order hyperbolic equation. Now, let us consider the second order hyperbolic equation, as we may recall the wave equation belongs to that category. So, you have… So, let us say that this is the prototype equation, and we want to study the corresponding discretization and its stability. So, this discretization can be done in many ways, we will consider one example; that we discretize it by CTCS scheme. Because we have examined through the first order equation that CTCS scheme gives a good stability behavior. So, we want to use it for the second order equation also. If we discretize it, that means it time will be central difference space also will be central difference. So, what will be the left hand side, u i n plus 1 i plus u i n minus 1 i minus 2 u i n divided by delta t square. Then this one u i plus 1 n plus u i minus 1 n minus 2 U i n by delta x square.

So, we can write U i n plus 1 plus U i n minus 1 minus 2 U i n is equal to courant number square U i plus 1 n plus U i minus 1n minus 2 U i n, where courant number is equal to C delta t by delta x. Let us write the expression of the corresponding error, so epsilon i n plus 1 plus epsilon i n minus 1 minus 2 epsilon i n is equal to courant number square epsilon i plus 1 n plus epsilon i minus 1 n minus 2 epsilon i n.

Now, as you recall we substitute epsilon (x,t) is equal to A e to the power a t into e to the power j k x, where j is square root of minus 1. It needs to be mentioned, perhaps I did not mentioned it earlier that the stability analysis that we are doing is due to the contributions of von Neumann, so this is known as von Neumann stability analysis. And it is essentially applied for linear problems. Now if we substitute this epsilon in the expression for the error, what we get out of these; e to the power a t plus delta t into e to the power j k x plus e to the power a t minus delta t into e to the power j k x minus 2 e to the power a t e to the power j k x that is the left hand side, that is equal to the right hand side courant number square into e to the power a t e to the power j k x plus delta x plus e to the power a t e to the power j k x minus delta x minus 2 e to the power a t e to the power j k x. We can now find out the amplification factor, by dividing both sides by e to the power a t into e to the power j k x.

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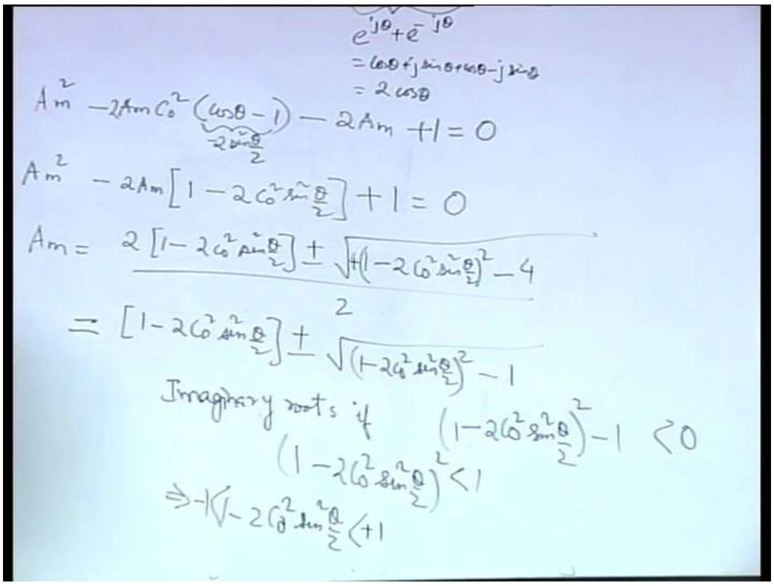

So we will do that now, the amplification factor is equal to e to the power a delta t. So we have, amplification factor plus 1 by amplification factor minus 2 is equal to C o square into e to the power j k delta x plus e to the power minus j k delta x minus 2. So this, from this equation we need to solve for the amplification factor. We consider k into delta x equal to theta. So, this is e to the power j theta plus e to the power minus j theta. So, cos theta plus j sin theta plus cos theta minus j sin theta. So, 2 cos theta is the corresponding expression. So, we have amplification factor square minus amplification factor C naught square into 2 into cos theta minus 1 minus two equal to 0.

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Minus two into (( )) So, 2 into A m square is there one, then there is one minus 2 A m plus 1 equal to 0. You can use cos theta minus 1, 1 minus cos theta is 2 sin square theta by 2, so minus 2 sin square theta by 2. So, A m square minus 2 A m into 1 minus 2 C 0 square sin square theta by 2 plus 1 equal to 0 . So, A m is minus of this one minus b. So, 2 into 1 minus 2 C naught square sin square theta by 2 plus minus root over of b square. So, we can cancel 2 from both numerator and denominator, this whole square. Now this clearly depending on the value of C naught, could assume real roots as well as imaginary roots. So, first of all if you consider the real roots you can see that, no matter whether the roots are real or imaginary. what is the product of the roots of this quadratic equation one. So if the two roots are real, if one of the roots is less than one then other will be greater than one.

So, that will give rise to instability. Therefore, the possibility of imaginary root of this one, needs to be assessed very carefully for stability. The real root that we have considered that, if one gives less than one amplification factor; the other will give greater than one amplification factor, but let us investigate the complex roots. That is the imaginary part of this solution more carefully. So imaginary root, if 1 minus 2 C naught square sin square theta by 2 whole square minus 1 less than 0. So, 1 minus 2 C naught square sin square theta by 2 less than 1, whole square less than 1. That means, 1 minus 2 C naught square sin square theta by 2, is magnitude should be such that it should lie between minus 1 and plus 1. So, that is square is less than 1.

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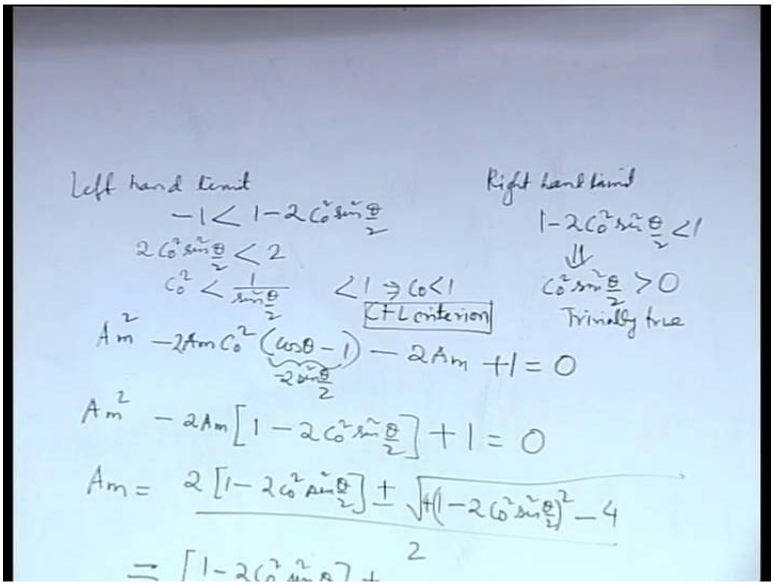

So, let us consider the two limits; left hand limit, minus 1 less than 1 minus 2 C naught square sin square theta by 2 and right hand limit, 1 minus 2 C naught square sin square theta by 2 less than 1. So, the right hand limit gives C naught square sin square theta by 2; greater than 0, which is trivially true, because C naught square is greater than 0, and sin square theta by two also greater than 0. So, the left hand limit needs to be examined more carefully. So, the left hand limit says 2 C naught square sin square theta by 2 less than 2, so C naught square less than 1 by sin square theta by 2. So, it should be less than 1, for a conservative estimate. That means, C naught should be less than 1, remember C naught is positive. So, what we can see here is that we get back the CFL criteria, Courant-Friedrichs-Lewy criteria for stability for the CTCS scheme for second order hyperbolic equation.

So, we will not illustrate this stability analysis any more, more and more we do it. It will become more and more monotonic, because it is basically the same exercise just a bit of algebra, different for different types of schemes. So, what message we have got from this analysis is that, we give a particular perturbation and see whether the perturbation amplifies or dies down. It depends on the behavior of the exponential term, from that we can get an amplification factor. And we essentially tend to adjust that, whether there are conditions corresponding to which the amplification factor has magnitude less than 1 and that condition ensures the condition for stability.

So, as the next agenda for today's lecture, what we will do is we will start looking into the questions of the mid-semester examination. First, I will read out the questions, describe the questions, and then we will try to look into the solutions of this questions. We will try to see whatever we can covered in today's lecture or and the remaining we will continue in the next lecture. So, the mid-semester questions essentially are from the portions that we have covered mainly with the variation formulation to begin with and then the fundamental considerations of the finite difference, finite elements and the finite volume method.

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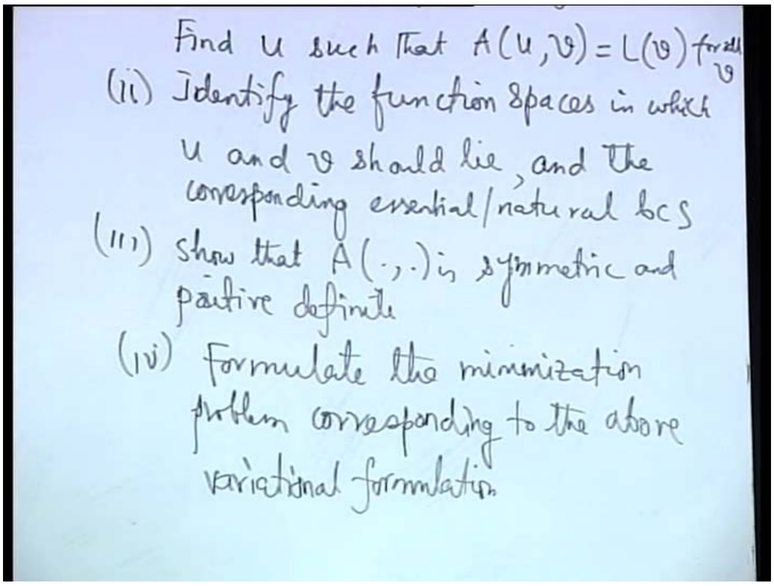

So, we will first consider one question from the variation formulation. So, the question number one for the mid-semester assessment. We consider the following differential equation. This is the differential equation, where a, b, c are known functions of x, so a, b, c are known functions of x, and a, b, c are greater than equal to 0 for all x. The domain of definition is x between x 1 to x 2. So, let us write down the questions in the right hand side, we will solve the corresponding questions in the left hand side. The first part of the question is, develop the variation formulation for the above differential equation; for this differential equation in the form find u such that A(u, v) is equal to L(v) for all v. That is number one, number two is identify the function spaces in which u and v should lie, and the corresponding, essential as well as natural boundary conditions. Number three show that A is symmetric and positive definite. Number four, formulate the minimization problem corresponding to the above variation formulation.

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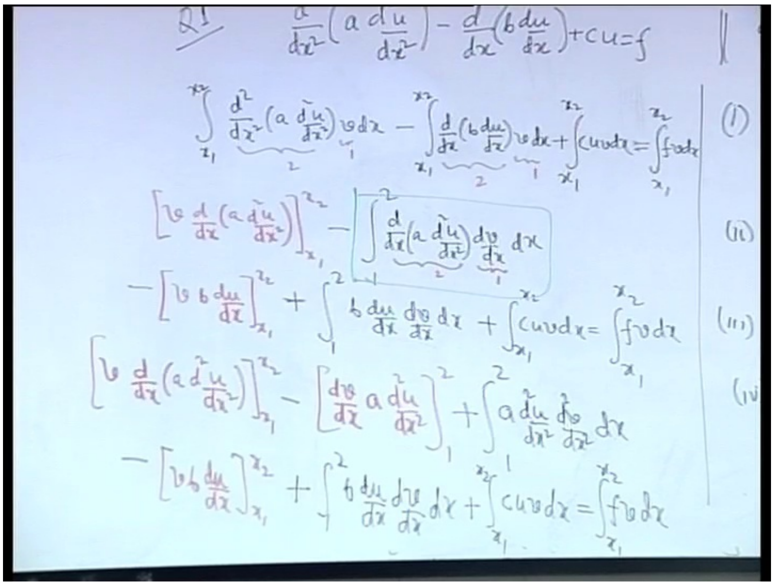

So, let us try to solve this part of these questions one by one. So, first to develop a variation formulation, what we need to do?

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We multiply this by V and then integrate with respect to x, so integral of (( )). So, that is the first step, next we will try to reduce the order of derivatives by integration by parts. So that, reduce the order of derivatives of u and that will increase the order of derivative of V as such it is like the total order of derivative as such is conserved. So, if you reduce the order of derivative of u that should be compensated by increase of derivative of the order of v, order of derivative for v. So, let us do that. So, v is the first function and this is the second function for all the terms. So, first function into integral of the second. I am writing the boundary term separately, because we will need this for specification of the boundary conditions, minus integral of derivative of first that is d v dx into integral of the second. So d dx of A d 2 u dx 2 into d v d x, that is for the first term. For the second term again first function into integral of the second minus integral of; so minus will become plus, integral of derivative of first into integral of the second. So, b d u dx into d v dx d x 1 to 2. We have to execute the integration by parts once more keeping this term in view. So, let us do that.

Let us first write the boundary terms v, this one minus now this will be the first function and this will be the second function. So, first function into integral of the second minus integral of the derivative of first into integral of the second, then there is a boundary term and the remaining terms as they were earlier. So, this we can write in the form that we desire, A (u, v) equal to L(v).

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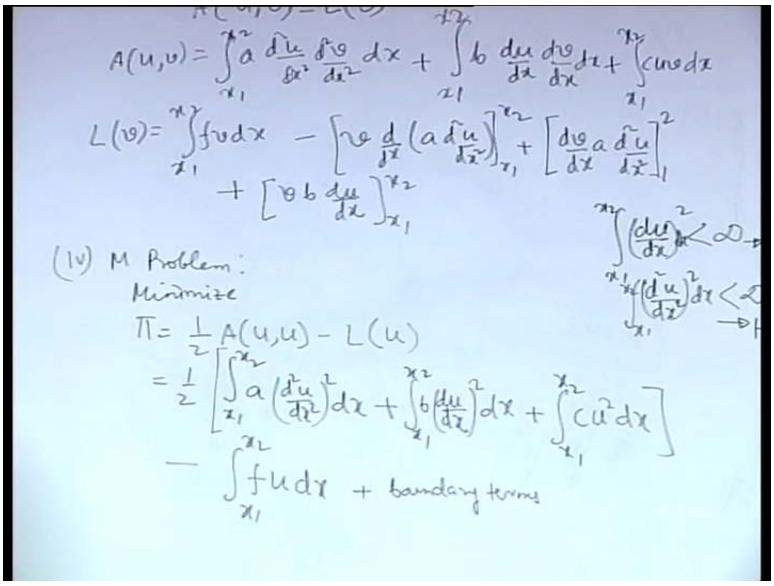

So, this is of the form A(u, v) equal to L(v), where what is A(u, v) is equal to a d 2 u dx 2 d 2 v dx 2 dx plus b d u dx d v dx d x plus c u v d x, and what is L(v) integral f(v) d x, and then the boundary terms taken in the other side. So, this minus, this is the variation formulation where we have written it in the form A(u,v) equal to L(v); for all v. So, that is the solution to the first part of the question.

What is the second part of the question? Identify the functions spaces in which u and v should lie, and the corresponding essential and natural boundary conditions. So you can see here, that in this formulation you require the first order derivative and the second order derivative to be the important order derivatives. These two are the two important order derivatives. So, you have d u dx square integral from x 1 to x 2 that should be less than infinity as well as the second order derivative less than infinity. This is true for both u and v. So, the first order derivative of u, as well as the first order derivative of v should be square integral. The second order derivative of u as well as the second order derivative of v should be square integral, because these are the two derivatives which appear in the corresponding variation formulation. So, this implies that it will belong to H 1 space and this will imply that it will belong to H 2 space. So, these are the function spaces in which, you expect these to lie.

Then, what are the corresponding essential and natural boundary conditions. So, for that let us look into the boundary terms. So, possible boundary conditions. Let us make a table; essential and the corresponding natural. There are three boundary terms. So, the boundary condition should come from three possibilities. First we consider this term. So, from this term, what is the corresponding essential boundary condition? See, v is the variation in u, so v is delta u. So, v is to be taken care of by specifying u. So, if u is specified then variation of u is 0 or delta u is o. So, essential is u specified. So, remember essential variation essential boundary condition is the specified value on the variable, the corresponding variation of which appears in the boundary term. So, delta u is equal to v therefore, u specified is the corresponding essential boundary condition and what is the natural boundary condition.

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The remaining what is there in the boundary term. So, d dx of a d 2 u dx 2 specified. Remember that either of this may be specified, but not both. But when we say what are the possible boundary conditions, we have to identify all possibilities. So, this one pair next let us considers this particular pair. So, in this particular pair we have d v dx and v is delta u. So, d u dx specified and what is the corresponding natural boundary condition? a d 2 dx 2 specified. Then we consider the third boundary term, in the third boundary term you again have v, v is delta u; that means, u is specified and the corresponding natural boundary condition b d u dx specified. So, this could be the possible specifications of the boundary terms.

Then, the third part of the question which is pretty straightforward, show that A is symmetric and positive definite. So, we can see that if we interchange u and v in the expression for capital A it remains the same. So, A u comma v is equal to a v comma u; that means, it is symmetric. And how to test the positive definiteness, what is A(V,V). So, this is the expression for A(v, v), and if you look into the given condition small a small b small c are all greater than equal to 0 for all x. So, a small a small b small C are greater than equal to 0 and there multipliers are also greater than equal to 0. So, we can see that A(v, v) is greater than equal to zero. So, it shows that it is positive definite.

Then the final part, part four formulate the minimization problem that is m problem corresponding to this variation formulation. So, the m problem will be what, minimize pi is equal to half A(u, u) minus L(u). So, half integral of x 1 to x 2 a d 2 u dx 2 whole square dx plus b d u dx whole square dx plus c u square dx minus L(u). So, minus integral of f (u) dx plus some boundary terms. I am not writing it again anymore, just it should be L(u), in place of v you replace it with u. So, minimization of this problem, because it is symmetric as well as positive definite you can formulate and m problem from the v problem. So, that completes the discussion of the answer to the first question.

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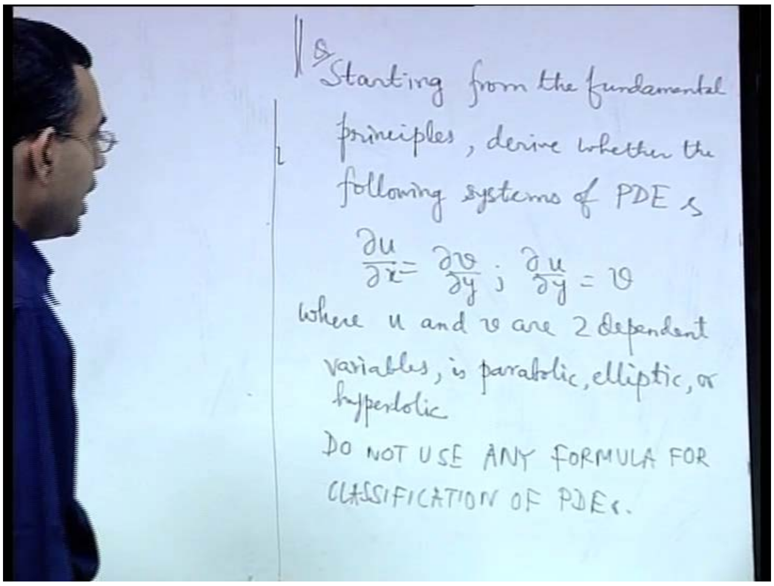

Now, let us consider the second part of the first question or a different question. Again let us write the question in the right hand side. Starting from the fundamental principles, derive whether the following systems of PDE s; that is partial differential equations del u del x is equal to del v del y and del u del y is equal to v, where u and v are two dependent variables is parabolic, elliptic or hyperbolic. Most important is do not use any formula on classification of PDE s. So, you have derive it from the first principles. So let us try to answer this question. So, there are two variables u and v of course, you can deal with two variables simultaneously. But it becomes much easier if you eliminate one of the variables.

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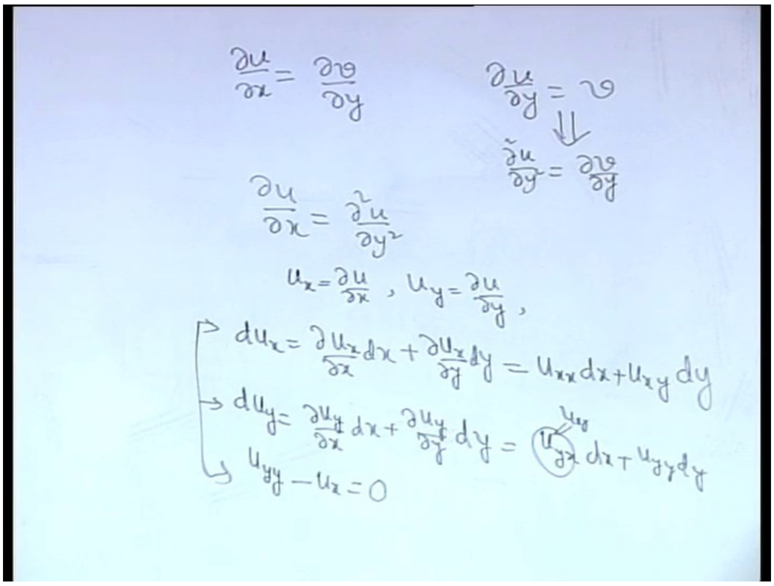

So, let us say you eliminate v. So you have, this one and the second equation del u del y equal to v. So, if you differentiate this with respect to y. Now if you equate the first equation with the final one that we obtain, you will get del u del x is equal to del 2 u del y 2. So, it is the nature of this particular equation that we have to determine. To do that, what we can do is we can use short hand symbols like u x equal to del u del x u y is equal to del u del y and so on. Similarly second order derivatives like mixed partial derivatives and so on. So, we can write what is d u x, because these are functions of x and y. So, by shorthand notation, this is u x x dx plus u x y dy. Similarly what is d u y? So, u y x dx plus u y y d y; for continuity in the second order partial derivative u y x is same as u x y, and what is the given equation. Given equation is u y y minus u x equal to 0. So, these are the three equations that we get, and we can arrange this in a matrix form.

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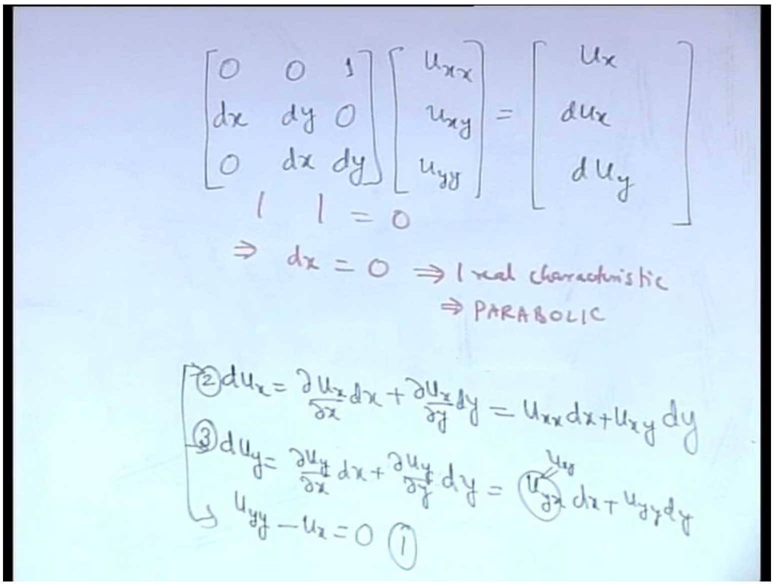

So, if we consider this as equation number 1 then this as 2 and this as 3, from equation one we have there is no u x x. So, u x x into 0 plus u x y into 0 plus u y y is equal to u x. For the second equation, u x x into dx plus u x y into dy then there is no u y y is equal to d u x. From the third equation, there is no u x x u x y into dx plus u y y into dy is equal to d u y. For getting the characteristics of the equation, we need to get the characteristics of the equation, because the characteristics of the equation will essentially decide or the number of real characteristics will essentially decide whether it is parabolic, elliptic or hyperbolic. This is a second order PDE of known standard form. So, we know that across the characteristics, there may be discontinuities in u x x u x y and u y y. To accommodate that, you must have determinant of the coefficient matrix, that must be equal to 0. So, determinant of this one is equal to 0 that essentially implies that dx equal to 0. So, x is some constant. So, that is one real characteristic, which implies that it is a parabolic partial differential equation.

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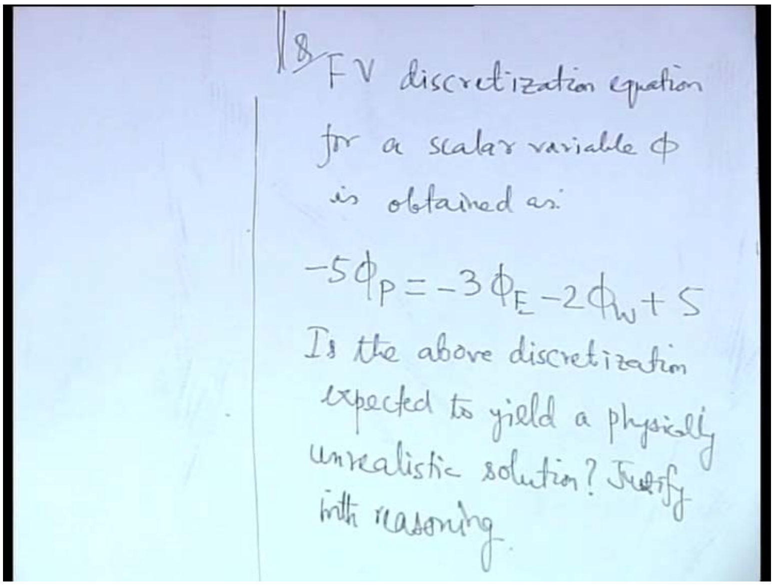

Then let us consider the next question, finite volume discretization equation for a scalar variable phi is obtained as minus 5 phi p is equal to minus 3 phi E minus 2 phi w plus 5. Question is, is the above discretization expected to yield a physically unrealistic solution? Justify with reasoning. So, a key towards understanding this particular question is that we have only limited information on the nature of the equation which is discretized, and we should assess it on that limited information only. See, it is not told whether this is discretized on the basis of one-dimensional steady state heat conduction with whatever. So, with constant property, variable property this that nothing about the physics of the background information based on which this equation is derived is told to us.

So, but we can understand that at least it is a steady state problem, it is a one-dimensional problem. That much we can understand out of it, and there is some source or sink. Now to assess, whether it is expected to yield a physically correct solution or not, we have to consider that what are the basic rules that are pertinent to this discretization equation. Out of the basic rules that we consider, one of the most important basic rules that we can use regardless of the background physical information is that all coefficients must be of the same sign.

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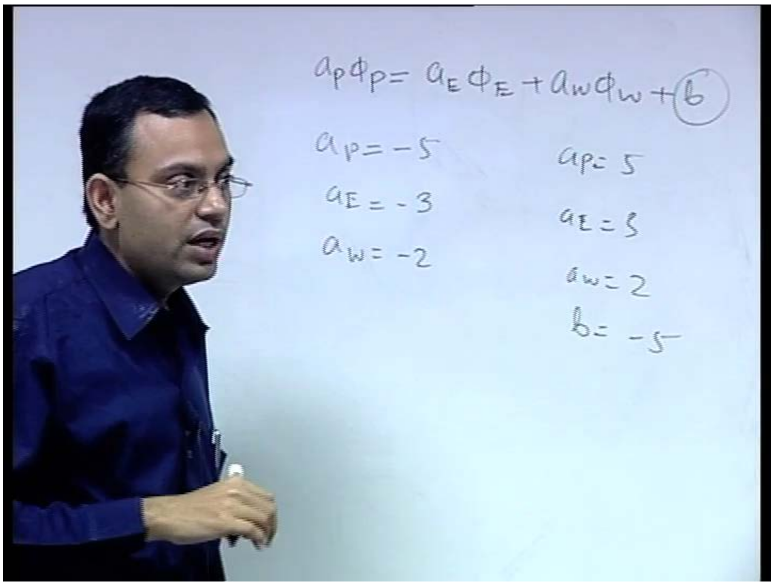

So, here the you can write this equation in the form a p phi p is equal to a E phi E plus a w phi w plus b, where a p is minus 5, a E is minus 3, a w is minus 2. See, do not try to assess this equation on the basis of whether a p is equal to a E plus a w, because that particular consideration is valid only for, a special case of a linear equation with no source term. So, do not try to assess it on the basis of that here coincidentally it happens fine, but it has nothing to do with the assessment. I am telling this, because that is one of the temptations that you might have to assess it on the basis of such requirements. So, you just assess it on the basis of the fact that these are the coefficients, these coefficients all have same sign. Remember b is not a coefficient. So, do not try to assess it on the basis of sign of b. I mean, these are common mistakes that students make. That is why I am iterating this. So, b is not a coefficient are the coefficient and all these all the coefficients are having the same sign since all the coefficients are of the same sign, it is a very simple question. Do not try to over assess the complication of the question. It is not a overly complicated question. It can be the simplest question possible, but you can of course attempt to complicate it by looking into the corresponding physics from which it would have perhaps evolved, and then several other consequences fluxes at control volume phases and so on.

So let us not go into that; let us always try to learn a very simple obvious thing. Take a simple thing in a simple manner. Do not try to complicate any simple thing. So here the most simple consideration is, you have all coefficients of the same sign, you could have perhaps written it as a p equal to 5, a e equal to 3, and a w equal to 2 then b would have been minus 5. So it is not, it does not matter whether it is all coefficients are positive or negative, all coefficients are of the same sign to get a physically consistent solution. Therefore, from the phase value it is expected to yield a physically realistic solution, not an unrealistic one; that is the answer to this question. So, we will stop here today, in the next lecture, we will continue with the remaining question of the mid-semester examination. And after that is over, we will take up the discretization of two-dimensional problems. Still now we have considered the discretization of one-dimensional problems, We will take up the two-dimensional problems in the next class. Thank you.