Computational Fluid Dynamics

Prof. Dr. Suman Chakraborty

Department of Mechanical Engineering

Indian Institute of Technology, Kharagpur

Lecture No. #18

Discretization of Hyperbolic Equations: Stability Analysis

In couple of our previous lectures, we were discussing about the stability requirements or stability criteria for parabolic, and elliptic equations. Today, we will discuss on the same for hyperbolic equations. So, we will first try to develop a very simple prototype of a hyperbolic equation, and then try to see or try to examine the issues of stability.

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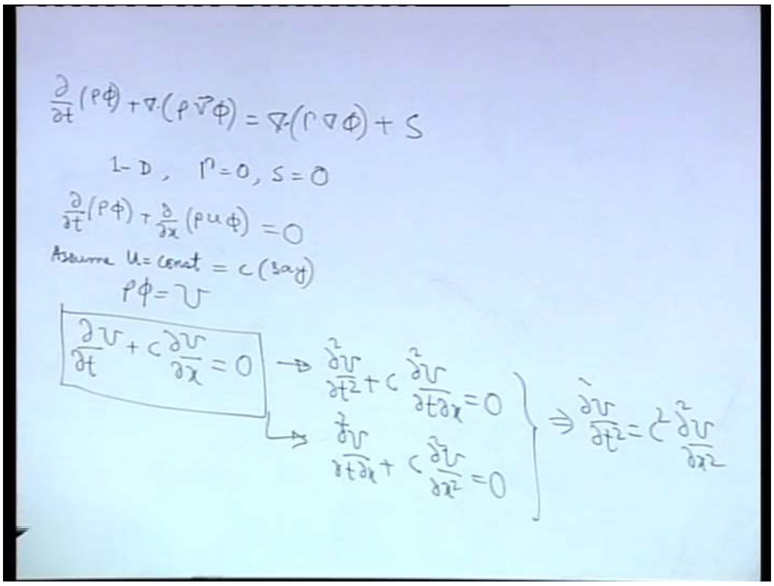

Let us say, that we have a general transport equation. Let us consider the following case, one-dimensional constant or not constant may be 0 diffusion coefficient, and 0 source term. So, if we have such a situation where there is no diffusion, then it is determined by the transient and the advection term.

Let us say that, u is equal to a constant which is equal to C say, and we give rho into phi say a name capital U. So, we come up with this following equation, this one, we can very easily assess the nature of this equation by comparing it with a standard prototype that we have already learnt while classifying the partial differential equations. So, to do that, we can differentiate these with respect to time and also differentiate it with respect to x. So, if you combine these two what follows is, so, you can multiply the second equation by C and then subtract these two. So, this is the prototype hyperbolic equation that we have earlier seen in the context of classification of second order partial differential equations. So, what we will do is, we will not consider the second order form to begin with, we will consider the first order form which is also hyperbolic and start assessing the stability of the corresponding numerical discretization schemes.

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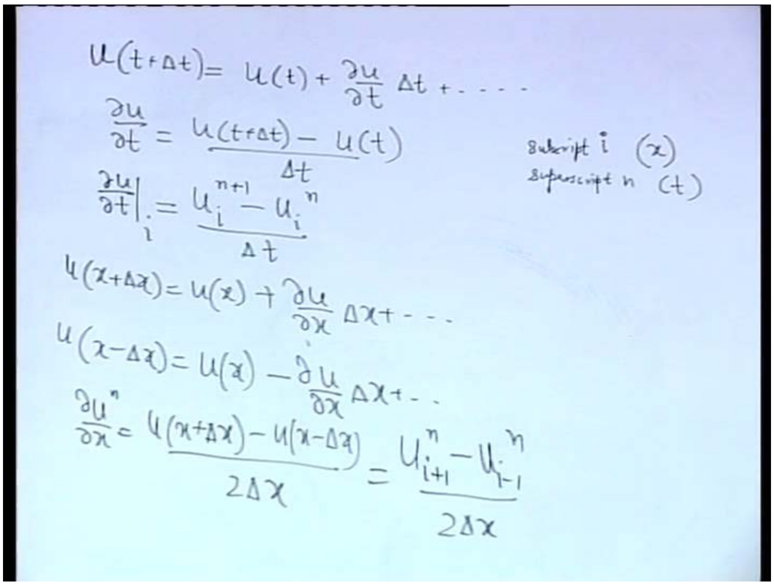

So, our objective is to look into this prototype equation and assess various discretization schemes for this equation. We will start with one scheme, with which we started for the heat equation also F T C S scheme, Forward Time Central Space. So, to do that, we have to keep in mind that the time derivative is to be represented by a forward difference. So, u at time t plus delta t is equal to u at time t and so on. So, partial derivative with respect to time for u is equal to u at t plus delta t minus u at t divided by delta t we are not writing that truncation error, just writing the difference formula. In terms of the symbols or notations if we use subscript, i for the grid point x and superscript n for time t, then how do we represent it? u this at i is equal to u at n plus one minus u n divided by delta t. Similarly, we consider the special derivative u at x plus delta x is equal to u at x. So, in terms of the subscripts and superscripts u at x plus delta x means i plus one minus u i divided by delta x.

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Oh we want central space, this has become forward space. So, to say to do central space what we have to do? u at x minus delta x also you have to find out. So, let us do that, u at x minus delta x. So, if you subtract these two. So, this becomes central space, the previous formula that we wrote was forward space. So, u i plus one minus u i minus one by 2 delta x, by looking into this formula you can appreciate one thing, that these formula are as if considering the rate of change by a linear approximation. So, d u d t is like u at t plus delta t minus u at t divided by delta t. So, as if the portion from t to t plus delta t is being considered, as a linear function. So, that it is the change in u by the change in t that is the slope of the curve. So, it is as good as linear profile assumption in a finite volume methodology.

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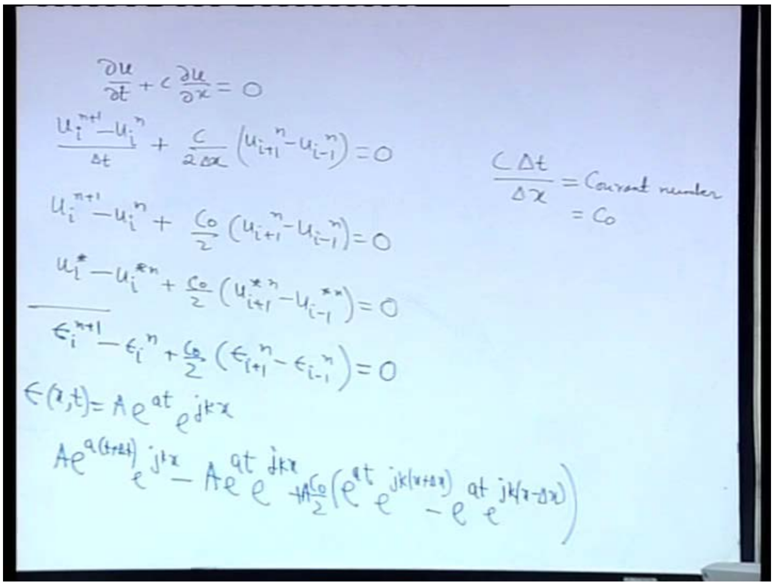

Now, let us. So, remember. So, this is del u del x at n. So, now, if you substitute the corresponding formula for the F T C S scheme, this is what we get, now we can multiply both the terms by delta t and represent this non dimensional number C into delta t by delta x as something called as courant number. We will come into the physical significance of this number, but if we just do it mathematically what we get is u i n plus

1, so, this is the difference equation the corresponding value of u, say which is the approximate solution if you solve the difference equation, remember that the actual solution will satisfy the difference equation as well as the differential equation. Now u star is an approximate solution which we have got from the difference equation. So, if you subtract this you will get the corresponding error. So, epsilon i n plus one minus epsilon i n plus courant number by 2 epsilon i plus 1 n minus epsilon I minus one n that is equal to 0.

Now, next is we will express epsilon as some A e to the power a t into e to the power j k x, in the same way as we did for the previous equations where j is square root of minus 1. So, if we substitute that, what we will get A e to the power of a t. So, e to the power of a n plus one means t plus delta t i subscript means x equal to x. So, e to the power j k x minus A e to the power a t into e to the power j k x plus C 0 by 2 e to the power a t into e to the power j k x plus delta x minus e to the power a t into e to the power j k x minus delta x.

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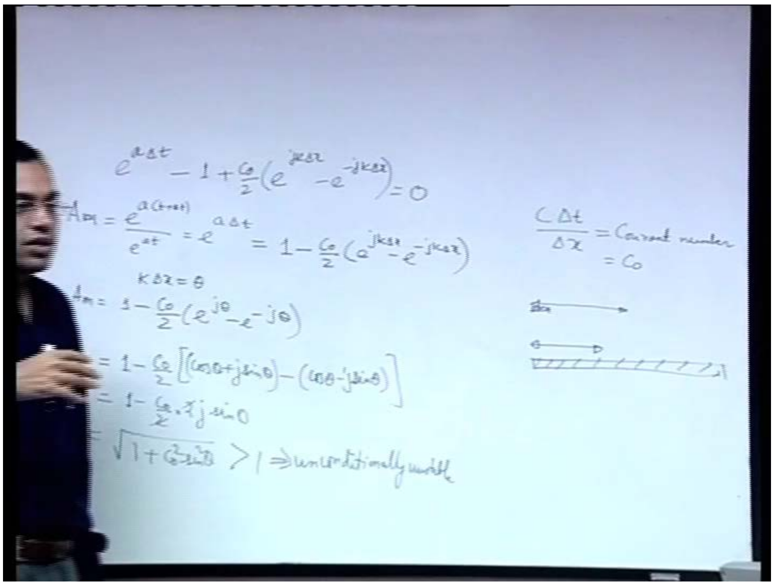

So, let us simplify this equation now. So, we have e to the power a delta t minus 1 plus courant number by 2 e to the power j k delta x minus e to the power minus j k delta x is equal to 0.

Let us recall that the amplification factor say we give it a name a m which is the amplification factor, which is e to the power a t plus delta t by e to the power a t, so e to the power a delta t. So, this is equal to 1 minus C 0 by 2 e to the power j k delta x minus e to the power minus j k delta x. Now for simplification one can consider say k is into delta x equal to theta. So, the amplification factor becomes 1 minus courant number by 2 e to the power j theta minus e to the power minus j theta. So, this is cos theta plus j sin theta minus cos theta minus j sin theta into 2 j sin theta. So, what is the amplitude of this one, square root of one plus C 0 square sin square theta. So, what can we talk about its stability, it is unconditionally unstable, because this is always greater than 1 the courant number is C delta t by delta x that is square of that is positive sin square theta is always positive therefore, this part is positive and it makes the total greater than 1. So, this scheme is unconditionally unstable.

So, we have to recognize one very important thing from this example that, whether the scheme is stable or unstable, it depends on the nature of the partial differential equation in addition to the nature of the scheme, the same F T C S scheme, when it was considered for heat equation, it was not unconditionally unstable. So, but here for hyperbolic equation it is coming out to be unconditionally unstable. So, it the nature of the equation also which plays a big role, not just the scheme itself.

Now, what about the significance of the courant number, the if you see the courant number the numerator C is a characteristic speed, just like you can considered it to be a sonic speed. So, it the speed at which a disturbance propagates through the medium. So, if you consider C into delta t that is the distance by which a disturbance propagates through the medium, and delta x is your grid spacing, what you generally expect is that your disturbances should not propagate faster than your grid spacing. So, your grid spacing should be able to capture the propagation of the disturbances. So, you should not have grid spacing so large or so small. So, if you see that there there there are there could be some limits. So, if you are having your grid spacing very very large then what happens, if you are having your grid spacing very very large, then your disturbance propagation has a characteristic length scale, it has a characteristic velocity into time it has a characteristic length scale and your grid spacing, if it is very large then at least one important thing at it might ensure that you are having a grid spacing a resolution which can definitely capture the propagation of the characteristic disturbances within the system.

Now, if you this delta x very very small then what happens. So, you let us say that, this is the characteristic length scale over which your disturbance will be propagating and if you have this as the grid spacing. So, over the time interval of delta t your disturbance will not be able to even cross one grid point, there may be many such grid distances that it will be covering in in one particular short. So, in one particular short your disturbances travelling, many many grid distances. So, if it is travelling many many grid distances, then what is happening? Then each grid is not able to resolve properly that propagation of the disturbance, because the disturbance propagates so much. The grid is only up to this much, if the grid was large enough then; that means, that you could have 4, 5, 6, 10 grids or 20 grids or 100 grids say within these one, say you have the disturbance propagating in this way and you have each grid like this large, then your grid is able to resolve the propagation of the disturbance within a time step, there are several such disturbance propagation lengths within each grid. On the other hand if you have such a small grid, than what happens is that your disturbance will be propagating a large distance and that distance is not covered by a single grid. So, that single grid only covers smaller part of the disturbance propagation which is not resolved, remember delta t is the smallest time that is resolved, you cannot resolve a time less than delta t in a discredited system.

So, C into delta t is the smallest distance of propagation of disturbance that is resolved and if your grid is less than that one then; that means, your grid is not resolving a full propagation length whereas, if your grid is large it will resolve a few propagation lengths. So, we will later on see that out of these two, which one could be a preferred one, I am not committing anything I am just giving you the two limits, what is the consequence of small delta x and what is the consequence of large delta x, here we have seen that for this particular scheme it does not matter whether delta x is small or large or whatever this scheme is unconditionally unstable, but there are certain schemes which will have some conditions for stability and from those schemes we will try to conclude that what kind of delta x is preferred in compressing to delta t.

So, what we can see here is that displacement of the disturbance or propagation of the disturbance, is related to the grid spacing and that is how the time and space coordinates are connected to each other which is one of the important hall marks of hyperbolic equation, where you have the time coordinate and space coordinate they are strongly coupled with each other. So, here you have a time coordinate t a space coordinate x and you cannot choose their delta t is I am corresponding delta x is independent of each other, because you have to have a proper resolution of the disturbance propagation as against your grid resolution. So, these two, must be the matching of these two must be accounted for.

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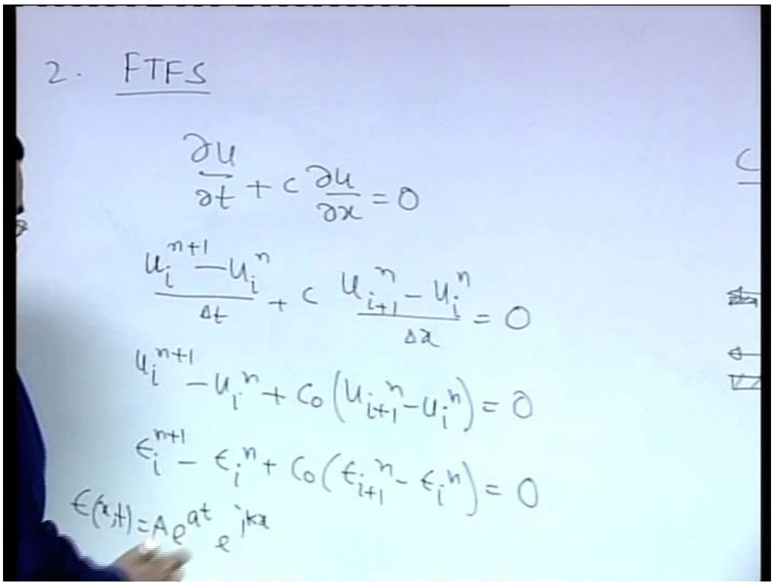

Next we will consider F T F S scheme forward time forward space. So, let us write the corresponding difference formula u i n plus 1 minus u i n by delta t plus C u i plus 1 n minus u i n by delta x equal to 0. So, u i n plus one minus u i n plus courant number into u i plus 1 n minus u i n equal to 0, the corresponding error equation is this one.

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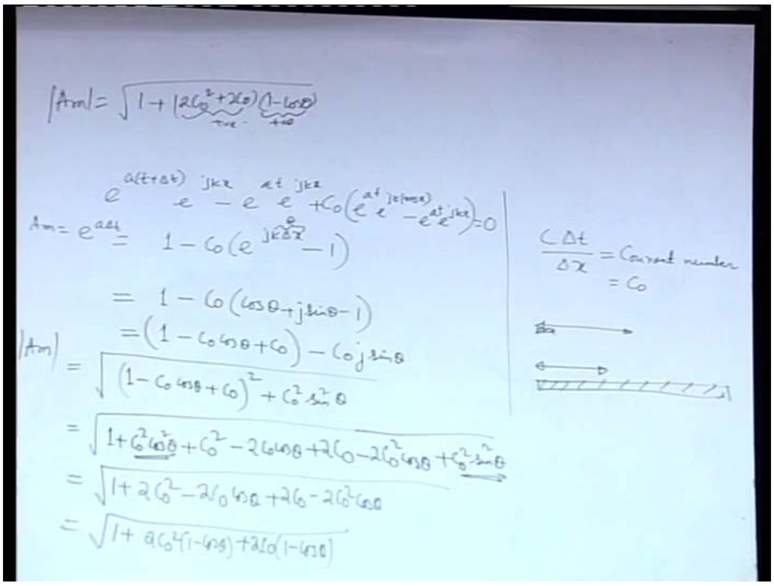

Again if we write epsilon at x t is equal to A e to the power a t into e to the power j k x, then what we get e to the power a t plus delta t into e to the power j k x minus e to the power a t e to the power j k x plus courant number e to the power a t e to the power j k x plus delta x minus e to the power a t e to the power j k x equal to 0. So, the amplification factor is e to the power a delta t which is equal to so, you divide all the terms by e to the power a t into e to the power j k x it is equal to 1 minus courant number into e to the power j k delta x minus 1k into delta x, we put theta.

So, cos theta plus j sin theta minus 1 so, 1 minus courant number cos theta plus courant number minus courant number j sin theta. So, mode of this amplification factor, we have just expanded the expression. So, C O square cos square theta and sin square theta together they will become 1. So, one courant number square plus another so, 2 courant number square, so, it will become 1 plus 2 courant number square minus 2 courant number cos theta plus 2 courant number minus 2 courant number square cos theta. So, one plus 2 courant number square into one minus cos theta plus 2 courant number into 1 minus cos theta. So, it will become 1 plus 2 courant number square plus 2 courant numbers into one minus cos theta.

Remember in the C delta t, C is the characteristics speed of propagation of the disturbance. So, it is considered to be a positive number and so, C is positive, delta t is positive and delta x is positive. So, by definition it is a positive number. So, 2 C C O square plus 2 C O this is positive, 1 minus cos theta this is also positive. So, this is also greater than 1, that means this is unconditionally unstable.

So, far we have considered two schemes and we have failed to get a particular scheme for hyperbolic equation, because both of these are unconditionally unstable. So, as a third possibility. So, we have considered forward time we have considered central space forward space. So, one option that is left to us is backward space, let us see that what do we get out of the forward time backward space scheme.

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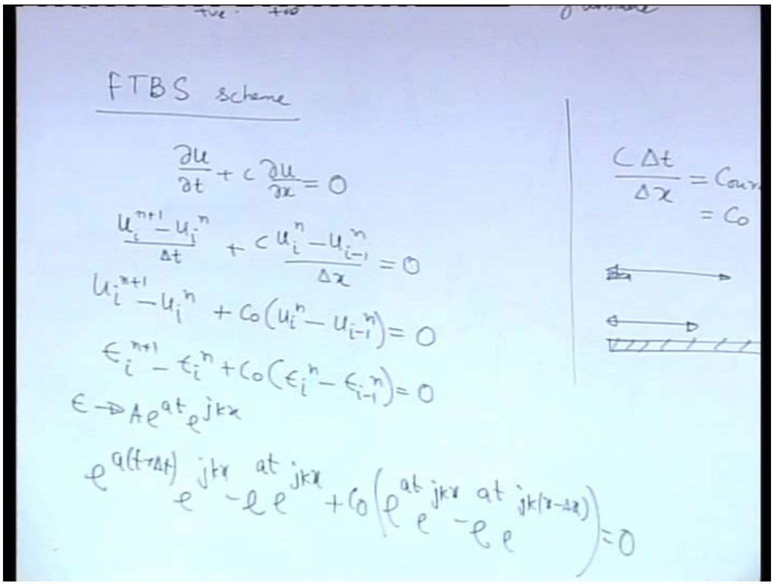

So, u for this one, u i n plus 1 minus u i n by delta t that is forward time, then backward space u i n minus u i minus 1 n by delta x is equal to 0. So, you have u i n plus one minus u i n plus courant number u i n minus u i minus 1 n is equal to 0. The corresponding error equation is, epsilon i n plus 1 minus epsilon i n plus courant number epsilon i n minus epsilon i minus 1 n equal to 0. We substitute epsilon A e to the power a t into e to the power j k x. So, e to the power a t plus delta t into e to the power j k x minus e to the power a t into e to the power j k x plus courant number e to the power a t e to the power j k x minus e to the power a t e to the power j k x minus delta x that is equal to 0.

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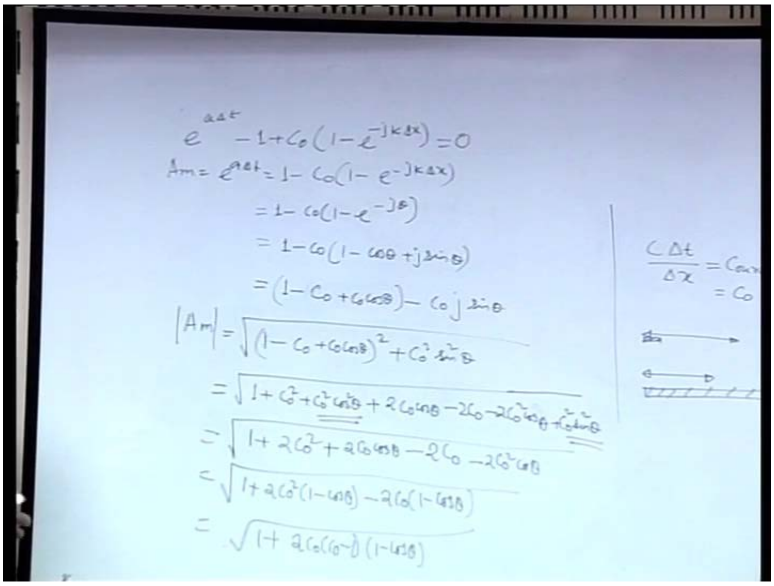

So, e to the power a delta t minus 1 plus courant number into 1 minus e to the power minus j k delta x equal to 0. So, the amplification factor is, e to the power a delta t 1 minus courant number into 1 minus e to the power minus j k delta x. So, 1 minus courant number into 1 minus e to the power minus j theta. So, e to the power minus j theta is cos theta minus j sin theta. So, it will become minus cos theta plus j sin theta. So, this is 1 minus courant number plus courant number cos theta minus C naught j sin theta. So, what is the magnitude of the amplification factor?

Let us work out that, so, we can take the cos square theta plus sin square theta term. So, it will become 1 plus 2 C naught square plus 2 C naught cos theta minus 2 C naught minus 2 C naught square cos theta. So, 1 plus 2 C naught square into 1 minus cos theta minus 2 C naught into 1 minus cos theta 1 plus 2 C naught into C naught minus 1 into 1 minus cos theta.

Now, let us look into this it is this expression, C naught is positive 1 minus cos theta is positive. Now C naught minus 1 can be positive or negative depending on whether it is greater than 1 or less than 1, if it is less than 1 then what happens, if it is less than 1 then it is negative, the other case is more trivial if it is greater than 1 then 1 plus a positive number will make this greater than 1. So, that will be unconditionally unstable. So, for stability we cannot allow it to be greater than 1, if it is less than 1 so, this is negative if C naught less than 1. So, if it is negative than 1 minus a positive number will make the amplification factor less than 1.

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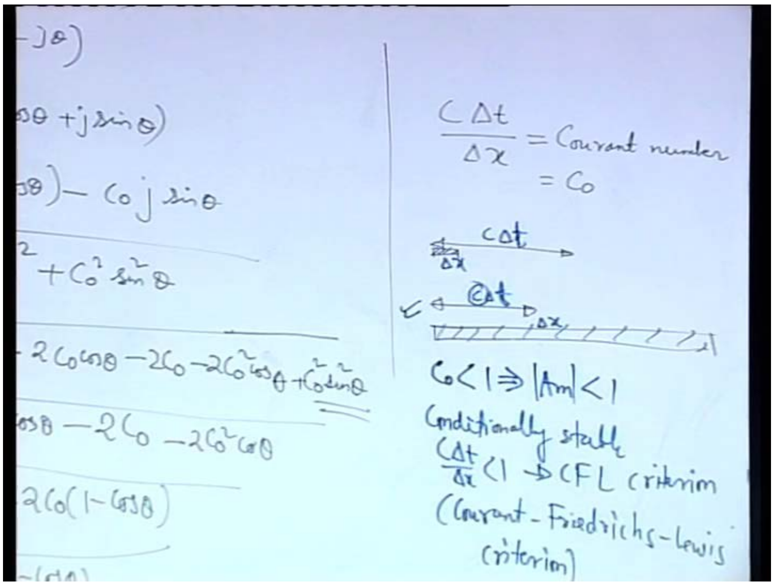

So, C naught less than 1 will imply that the amplification factor is less than 1. So, this scheme therefore, is conditionally stable and this particular criterion for stability that is C delta t by delta x less than 1 is called as C F L criteria Courant Friedrichs Lewis criterion. So, now, can see that let us let us compare this two cases now C into delta t is this much and this is delta x. So

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C into delta t right. So, what you expect is that delta x is greater than C into delta t for stability. So, you expect this case. So, what we expect is that, whatever is your grid it should large enough. So that, it can accommodate a few propagation of disturbances within a particular time step, a few propagation length of disturbances within a time step. So, you cannot indiscriminately put small grid spacing, knowing otherwise that small grid spacing may give you more accurate solution, but here you are putting a large grid spacing at least large enough. So, that C delta t by delta x less than 1. Of course, you have to keep in mind that these we are talking about these genetically, what it means is that it depends on C. So, if if you if you are really interested to compare these two it depends on the value of C. So, what we have considered is that C into delta t has some absolute value less than delta x, but we have not considered the value of C now if the value of C is very very small, think about that limiting case. The limiting case is well taken care of as well as the large C, because it does not depend on whether it is large C or small C what it depends on what is C into delta t as compared to delta x. So, it is not the absolute delta x that is important, it is not how large or how small C absolutely that is important, it is a comparison of C delta t with delta x that is important. So, if C is large you have to accommodate it with a proportionate large delta x, for a given delta t. So, if you increase delta t, you have to increase the corresponding delta x, if you decrease delta t you can decrease delta x commensurately, but the ratio should always be less than less than 1 for stability.

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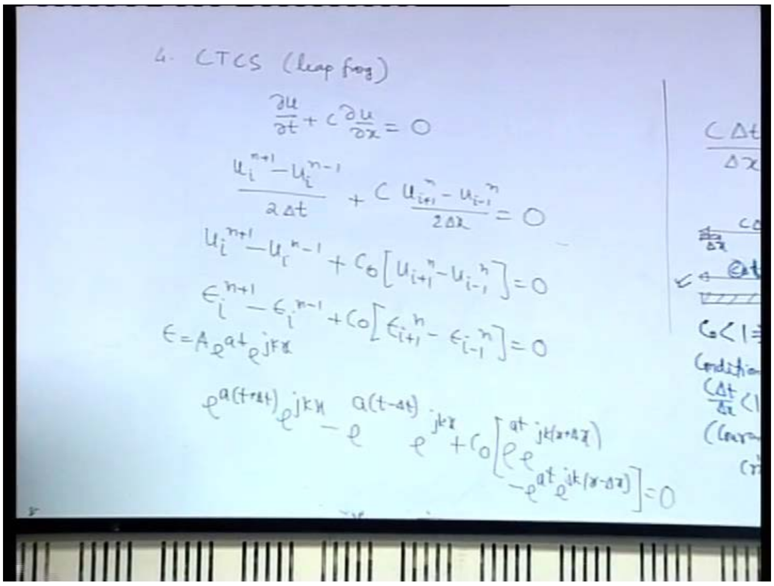

We will consider one more scheme, the central time central space scheme C T C S scheme or the. So, called leap frog scheme. So, u central time, so u i n plus 1 minus u i n minus 1 by 2 delta t and central space u i plus 1 n minus u i minus 1 n by 2 delta x. So, you have u i n plus 1 minus u i n minus 1 plus courant number into u i plus 1 n minus u i minus 1 n equal to 0 the corresponding error equation is epsilon i n plus 1 minus epsilon i n minus 1 plus courant number epsilon i plus 1 n minus epsilon i minus 1 n equal to 0 epsilon is A e to the power a t into e to the power j k x. So, it becomes e to the power a t plus delta t into e to the power j k x minus e to the power at t minus delta t into e to the power j k x plus courant number e to the power a t e to the power j k x plus delta x minus e to the power a t e to the power j k x minus delta x, that is equal to 0.

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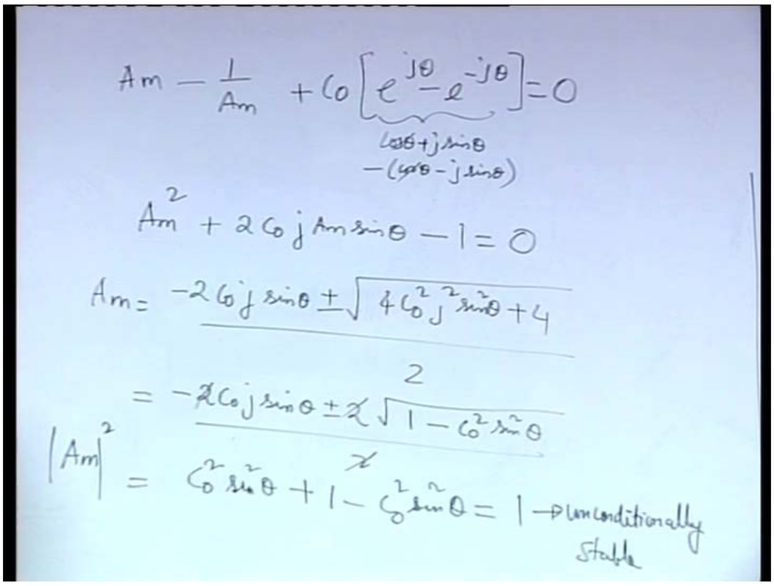

So, the amplification factor if you divide all the terms by e to the power a t into e to the power j k x, you have amplification minus 1 by amplification, because it is e to the power minus a t so, a delta t plus courant number e to the power j theta minus e to the power minus j theta equal to 0. So, this is cos theta plus j sin theta minus cos theta minus j sin theta. So, you have amplification factor square plus so, these what will get cancel cos theta will get canceled. So, plus 2 C naught j an m sin theta minus 1 equal to 0 right. So, A m is minus 2 C naught j sin theta plus minus square root of 4 C naught square j square sin square theta plus 4 by 2 j square is minus 1, because j is square root of minus 1. So, you can have 2 cancelled from all these terms. So, what is mode of amplification factor square? So, you have this one, C naught square sin square theta. So, this is the imaginary part, this is the real part. So, C naught square sin square theta plus one minus.

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Plus 1 minus C naught square sin square theta. So, that is 1. So, it is unconditionally stable, because even if it is 1 it is fine, if it is if amplification is greater than 1 then that is a problem. So, it is limiting, but still it is unconditionally stable. So, what we can see here is that out of the four schemes that we have seen for the first order hyperbolic equation we initially had two schemes which were unconditionally unstable then we are having one, we saw one particular scheme which is conditionally stable and this is unconditionally stable of course, very limiting, but it does not matter whether its limiting or not. So, long as it is unconditionally stable it is fine.

Now, of course, we have also to go through the second order hyperbolic equations, and the second order hyperbolic equations may be thought of as natural extensions of the first order hyperbolic equations, but we will consider the second order hyperbolic equations separately. And that we will take up in the next class, so in our next class our agenda will be first to consider the second order partial differential equation as a prototype form of hyperbolic equation, and look into its stability conditions. And the other main agenda will be we will look into mid semester review that is questions of the mid semester examination, and review of the solutions to those questions. So, that we will take up in the next lecture. Thank you