Computational Fluid Dynamics
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Lecture No. # 15
Finite Volume Method: Discretization of Unsteady State Problems

We have been discussing on some examples of a the use of the finite volume method for diffusion type of problems. Let us look into one more example where unlike the previous cases; we try to discretize the governing differential equation in a cylindrical coordinate system.

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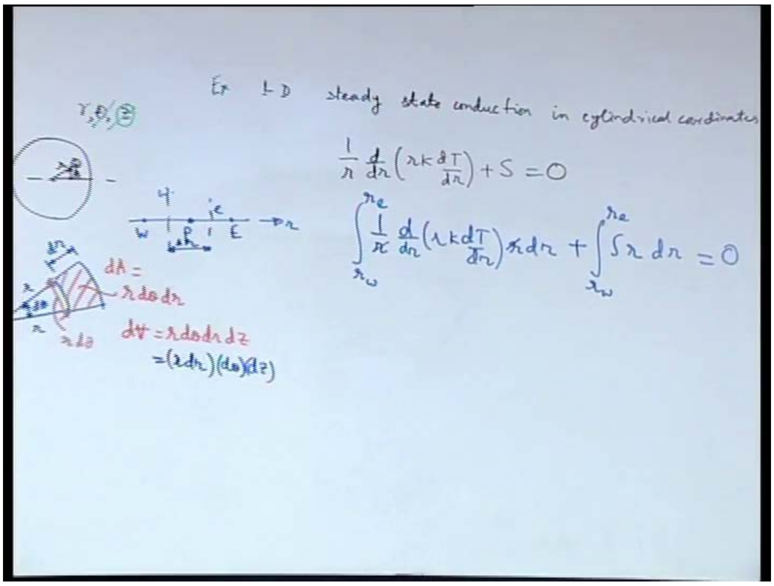

So, if we consider example of one-dimensional steady state heat conduction in cylindrical coordinates. So, in cylindrical polar coordinates you could have r, theta and z as the three coordinates. So, what are the r and theta coordinates? If you consider a reference line and you consider a point, the point can be represented by an angle theta and the radial distance r, which are basically the polar coordinates. And on the top of that if you have an axial coordinate then it becomes r, theta, z are the cylindrical polar coordinate system.

Now, here we are considering a case where it is a one-dimensional variation, first of all we consider the z direction to be large, so that there is no variation along z. And it is axially symmetric, so that there is no variation with respect to theta. So, only there is variation with respect to r. The governing differential equation becomes 1 by r d dr of r k dT dr plus S equal to 0. So, this is the equation that we are now interested to discretized.

Now, what is the first step in the finite volume method? We integrate the governing differential equation over an elemental control volume. So, what is an elemental control volume? Say if it was a two-dimensional system or a three-dimensional system, then if you consider that, there is a radial coordinate r, and you have an element of width dr, and let us say this angle is theta. So, what is the dimension of this element? So, one one of the lengths; so, we are talking about this element. One of the length is dr, the other length is r d theta. So, you have the total length or rather the total area of this shaded region as dA is equal to r d theta dr, because for small dr this is approximately like a rectangle.

If you consider an elemental length of dz along the other direction, then the elemental volume r d theta dr into dz. Now, in our case we do not have variations with respect to theta and z, but (( )) if you consider that it will remain in consequential. So, in our case what is that which varies? So, if you consider this dv it has three parts; one is r dr which is due to variation in r, then d theta, then dz. So, integrating the governing equation over the elemental control volume is as good as multiplying this equation by r dr. Because there is no theta or the variation or even if it is there it cancels form all the terms equally. There is no specific variation. So, what we do is, we multiply by r dr and integrate it with respect to r. Again we consider the radial direction say this one, whether grid points are P, E, W, phases of the control volume are small e and small w. So, we integrate it from r small w to r small e which is the control volume, and the length of the control volume we say is delta r.

So, what we can see here is a very interesting thing that because of this multiplication by the r dr term, when we evaluate the integral this 1 by r singularity of the governing differential equation goes away. This is one of the intrinsic advantages of the discretization using the finite volume method. See in any other method you have to bother about what is 1 by r at r equal to 0; you have to treat it in a special way, because it at r equal to 0 it is singular. But here you do not have to care about what happens at r equal to 0. As if you are starting with r tends to 0 plus and in that condition no matter whatever the value of r is beyond 0 plus, you can cancel 1 by r and r, so that the effect of 1 by r does not remain. So, with that understanding you can integrate this term. So, let us integrate this term.

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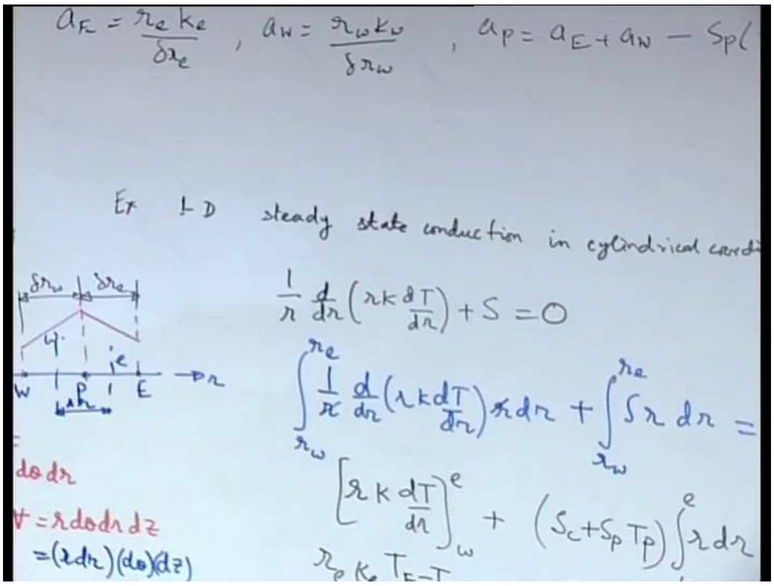

So, integral of that becomes r k dT dr from small w to small e plus... Let us assume that is is a constant or at least S is a function of temperature where the temperature dependent term is given by S c plus S p into T, S p into T is the temperature dependent term, and for that we assume a constant temperature over the control volume. So, we can write this as S c plus S p into T p integral of r dr. In the next step we have to make a profile assumption for dT dr. A profile assumption for T to evaluate dT dr. So, what can be an acceptable profile assumption for T? Piecewise linear between the grid points. So, let us make that choice. So, if you do that we can write this as r e k e into T p minus T E by delta r e, let us mark the dimensions, this is delta r w, this is delta r e. So that sorry this is T E minus T p by delta r e minus r w k w T p minus T w by delta r w plus integral r dr is r square by 2. So, r e square minus r w square by 2 equal to 0.

You can organise this equation in the following form, a p into T p is equal to a E into T E plus a w into T w plus b; where what is a E? r e k e by delta r e, a w r w k w by delta r w, a p is equal to a E plus a w minus S p into r e square minus r w square by 2 and b is S c into r e square minus r w square by 2. So, the purpose of casting the equation in this form is to show that eventually you get the same form of the discretizing equation. No matter which coordinate system you use, just the coefficient change depending on the coordinate system. If it was a spherical coordinating system it would have been 1 by r square d dr of r square k dT dr, and then you have to multiply it by r square dr instead of r dr, considering the elemental volume in a spherical coordinating system.

Now, let us move further ahead, we have till now discussed about steady state problems, but we are also interested about unsteady state problems and many interesting unsteady state problems occur in physical reality. So, let us try to discretize or see the discretization of the diffusion type of problems considering the unsteadiness of the governing equation.

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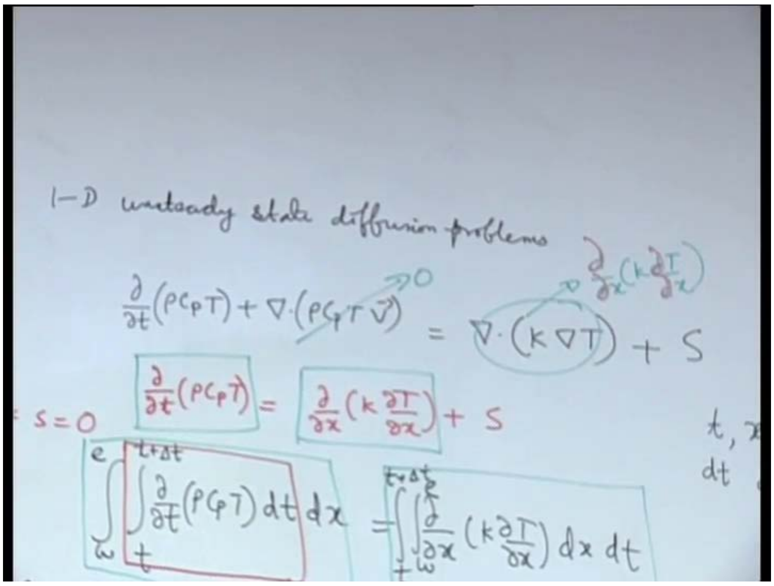

So, now we will move on to one-dimensional unsteady state diffusion problems And we are still considering the finite volume method for the discretization, so finite volume method for one-dimensional unsteady state diffusion problems. Let us start from the general governing differential equation.

So, this is the governing differential equation. Now, it is a diffusion type of problem. So, the fluid flow velocity is not there. That means this term is 0, for there because it is one-dimensional, this will become d dx of k dt dx, not d dx, but partial derivative. Now, because T is now a function of both x and time. So, we can write the simplified governing differential equation as

With consider an example with S is equal to 0; just for simplification, because our objective is not to show how to discretize the source term that we have already shown through the examples of steady state problems. Now, our objective is to specifically show that how to take care of this new term, which has appeared because of unsteadiness of the problem. So, the first step will be what? We have now two terms in the equation; one is an unsteady term, another is the conduction term. So, what will be the first step? Integrate the governing differential equation over the domain - elemental domain. Now, here the domain involves the specification of two coordinates - time and x. So, when you integrate it over an elemental volume in the domain space, the domain space will have elemental time also dt and dx.

So, we have to remember that it is not a physical volume that we are talking about. We are integrating the governing differential equation over the domain. If the domain is a physical volume then that is all right, but even if the domain is not a physical volume, it is the elemental space in the domain that we are considering. So, we have a time space, we have a we have a time coordinate and a space coordinate. So that elemental time into elemental space is the sort of element in the domain over which we are integrating the governing differential equation. That means we multiply it by dt dx and integrate; now, it is a double integral.

Since it is continuous, it does not matter whether you integrate with respect to x first or with respect to t first. So, according to the convenience of the terms, we either first integrate it with respect to x or integrate it with respect to t. So, what are the limits? The limits of time or from time t to time t plus delta t where delta t is the time step; so, just like we have divided the x domain into a number of sub domains. Similarly, we are dividing the time domain into a number of sub domains starting from time equal to 0 to then delta T, 2 delta T, 3 delta T like that. So, each small step is delta T where delta T itself may be a variable. So, one particular time steps start from time equal t and ends at time equal to t plus delta t. So, that is the elemental time domain.

What is the special domain? Let us keep in mind the discretization P, E, W, small e, small w. So, what is the x domain? From small w to small e; right hand side x domain small w to small e, time domain t to t plus delta t. Let us treat these two terms separately; so that we consider this as term 1 and this as term 2.

So, term 1, the term 1 has two integrals, and we can split that effort therefore, into two parts; first we calculate the integral in this red box and then calculate the remaining integral which is there in the green box. So, when we first calculate the integral in the red box, we consider the time to be a variable keeping the space coordinate unaltered; that is called as partial integration just like opposite of partial differentiation. So, if we do that then this comes out to be rho C p T then del of that one, so rho C p T at time t plus delta t minus rho C p T at time t, then integral of this one.

Now, here appears terms of the form integral of T dx that is temperature dx. To evaluate that we must have a profile assumption that how temperature varies with x. What can be the simplest profile assumption? Constant - piecewise constant within each control volume, because you do not have to evaluate any derivative. So, you can make such piecewise constant profile assumptions. So, this will imply. So, when T is a constant you can call it as T at p for each control volume. So, rho C p, let us consider rho into C p as a constant, it could also be a function of position and you you just have to integrate it by considering some profile. But for simplicity let us assume that rho into C p is a constant, if that is so that times. So, this entire thing becomes a constant not a function of x times integral of dx from small w to small e. So, that time delta x.

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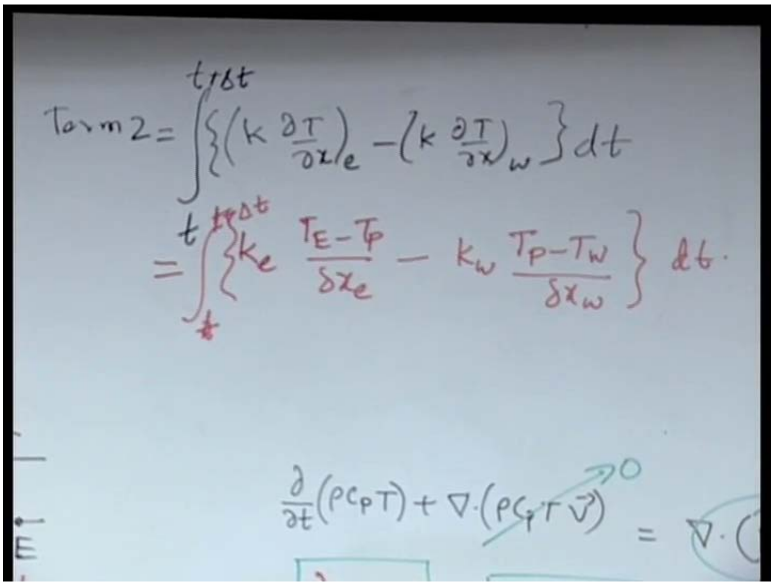

Next let us consider the term 2. To calculate term 2 again we have to calculate two types of integrals; first in this red box where we calculate the integral with respect to x, and then the remaining in the green box. So, when we calculate it with respect to x what does it become? So, the term two becomes k into del T del x at e minus k into del T del x at w times dt integral of that from t to t plus delta t. So, we require a profile assumption for T to evaluate the terms k dt dx type of terms. What profile assumption? We can take piecewise linear between the grid points. So, we can see that for different terms we can easily take different types of profile assumptions. We have earlier discussed that why we can do it, because at the end this history of profile assumption is loss once you evaluate the integrals. So, the profile assumption is surely for evaluation of the integrals, you do not require this for any further interpolation or any other calculation. And therefore, you have the flexibility of using different profile assumptions for different terms in the governing equation.

Now, you can write this as k e T E minus T p by delta x e minus k w into T p minus T w by delta x w where delta x e and delta x w are shown in this figure. Now, integral of this with respect to t. Now, here comes the important question. Here you have to evaluate terms of the form of the form capital T dt integral of that. So, there are four such terms; first two from this one T E minus T p and then second two from T p minus T w. So, basically you require evaluation of terms of the form integral T E dt integral T p dt and integral T w dt. So, these types of things are required.

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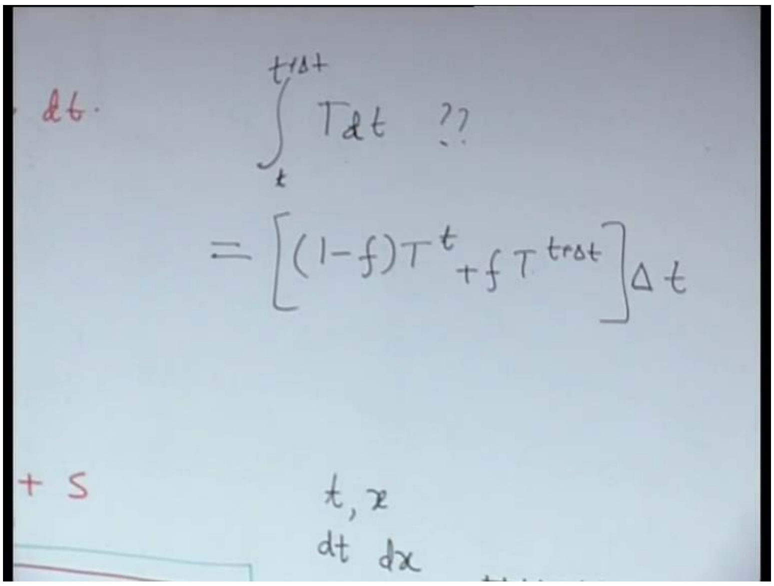

So, essentially your question is that you want to evaluate terms of the form integral of T dt from t to t plus delta t. So, how to do it? It requires again a profile assumption for temperature over time. Just like you require a profile assumption for temperature over space to evaluate integral T dx; similarly, you require profile assumption of temperature over time to evaluate integral of T dt. So, standard profile assumption towards this is written in terms of a parameter f which is a fraction as follows.

So, it is f is a weight and it is considered to be some weighted combination of temperature at t and temperature at t plus delta t, if this is p. So, in general if forget about the subscript p, it can be temporarily at any point. So, we do not write p, it depends on which point, if it is p we call it p, we if it is e we will call it e or w like that. So, this is an intuitive profile assumption where we are dumping some weight on the temperature at t, some weight on temperature at t plus delta t. Remember these are the two discrete points at which we know the temperature temperatures or rather we have to determine the temperatures. So, the entire temperature variation within the time step from t to t plus delta t is considered to be a weighted combination of something that has to do with the temperature at t, and some other things that has something to do with the temperature at t plus delta t, some weighted combination where f is a weight. So, this depending on the value of this weight we can have different schemes. So, let us try to write the general form of the scheme, and then we will assign different values of f, and see physically what it interprets.

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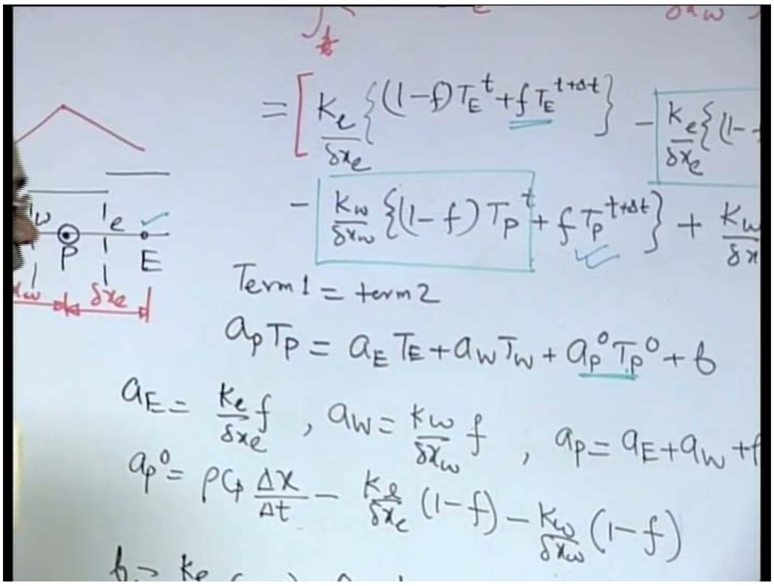

So, we have k e into T E minus T p by delta x e. So, in place of T E we will write 1 minus f T E t plus f T E t plus delta t, we will multiply all the terms by delta t at the end.

This is p right.

This entire term multiplied by delta t.

Now, you have term 1 equal to term 2, and for notational convenience we write T at time t as T 0 and T at time t plus delta t as T 1 which we call as just T, it is a notation; 0 superscript for the time at the beginning of the time step, 1 superscript for time at the end of the time step. And we usually drop the superscript 1 for just writing convenience. So, if you do not use any superscript we we mean that it is the end of the time step that we are talking about. So, if you have term 1 equal to term 2, then we have a p in to T p is equal to a E into T E plus a w into T w plus a p 0 into T p 0 plus b.

Let us completes the evaluation of these coefficients and then we will see what are the implications of this new coefficient a p 0 which has appeared. So, what is a E? a E is k e by delta x e into 1 minus f sorry not 1 minus f, but f, k e by delta x e into f, what we will do is, we will divide both the sides by delta t. So that we will write delta x by delta t here and that delta t is taken care of in this way. So, k e by delta x e into f that has come from which term - that has come from this term. Then any other coefficient for a E, no; then a w similarly, k w by delta x w into f that is from here.

No, it is not minus, it is plus, it is coming from this term. Then a p, a E plus a w then plus something we will write that plus something, then let us first write what is a p 0. a p 0 is the coefficient of T p 0; where is T p 0? T p 0 you have in both in term 1 as well as term 2. So, you have 1 T p 0 here, another T p 0 here and the third one here. And we have to remember that a p 0 and T p 0 they are in the same side as a E T E and a w T w. So, these two will be... This will be minus effect, this will be a minus effect and this will be a plus effect, because this will go in the other side. So, it will become rho C p delta x by delta t minus k e by delta x e into 1 minus f minus k w by delta x w into 1 minus f. And then what is b? The remaining terms, so k e by delta x e into 1 minus f T E 0 plus k w by delta x w into 1 minus f T w 0, then... And a p, a p as a E plus a w. we What are the terms which involve a p? Let us identify. This is one term, this is another term and the third term. So, this term is a E, this term is a w and then rho C p delta x by delta t. So, these two terms when they go to the other side they become plus and this is already plus.

So, even for the unsteady state problem, we are able to write or discretize the equation in a form a p T p equal to a E T E plus a w T w plus a p 0 T p 0 plus b, where now a new neighbour has appeared. Earlier the neighbour was the neighbours were e and w, now there is a time neighbour.

So, there are two types of neighbours now; one is a special neighbour, so for the point p you have the special neighbours as point e and point w, but for the same point p at time t plus delta t there is also a time neighbour which is that at time t. So that is given by this one, this term. So, it is a effect of the time neighbour, time neighbour at time t. And only one time neighbour we are considering, because whatever is at t plus delta t that is going to be influenced by what has happened at time t, but not what will happen at time t plus 2 delta t right. That is why you can see that there are two space neighbours, but only one time neighbour. Once you have this one time neighbour, you have to find out its implication that we will find out, and the implication of the time neighbour it depends on what is a p 0. Before doing that let us consider some special cases for different values of f.

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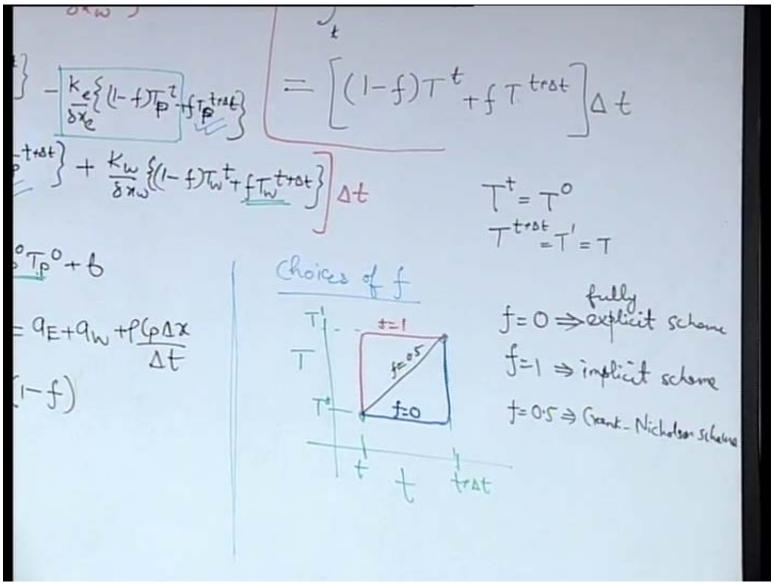

So, we next discuss the choices of f. Let us try to make a sketch of the variation of temperature with time between time t to time t plus delta t depending on what value of f we choose. Let us say that the values of the temperature at the instants of these two times; at T 0 at time t and T 1 at time t plus delta t. In between, how it varies? So, now look into this formula. First let us consider f equal to 0, remember here f lies between 0 to 1, so f equal to 0 and f equal to 1 are some special cases, f is a weight. So, any weight that can be assigned to either of these temperatures is either a 0 weight or a 1 weight or something in between, but nothing less than 0 or nothing greater than 1.

So, the limiting case, let us consider what is the consequence when f equal to 0. So, when f equal to 0, you see that there is no contribution of T 1; there is only contribution of T 0. So, then integral T dt becomes T 0 times delta t. So, what is the implication? Physical implication is we have considered a profile as if the temperature has been throughout T 0 over the time interval from t to t plus delta t. So, it is throughout T 0, but at the end it has to jump and match with T 1, because that has to be satisfied. So, this is f equal to 0. It is clear? f equal to 0 means the throughout the time interval the temperature is T 0, but of course it has to assume a new value of T 1 at the end. So, it suddenly jumps from T 0 to T 1 at the end. That is the profile. f equal to 1: f equal to 1 is the other thing that it means that throughout the interval, the temperature becomes T 1. So, what it does is? Suddenly it jumps from T 0 to T 1 and then remains T 1 throughout. So, this is f equal to 1. If you have a f equal to half as an intermediate example, then it is 50-50 combination of T at 0 and T at 1, and in between it can be interpolated linearly, so that this is f equal to 0.5.

These schemes... So, you can see that accordingly you can design different time integration schemes depending on what value of f you choose. What value of f you choose is your method is your method of designing the discretization problem. It is it is your headache, but we have to see that how to fix it up or what are the plus and minus points of choosing different values of f. But once you have chosen it you have to use that particular f for your discretization, and there are different limiting cases that we have identified, and accordingly there are different names of the schemes. If f equal to 0 this time discretization scheme is known as fully explicit scheme or in simple words just explicit scheme. If f equal to 1 it is called as implicit scheme, and if f equal to half it is known as Crank-Nicolson scheme. Let us try to figure out first the coefficients corresponding to different values of f, and see that what is the consequence of such coefficients.

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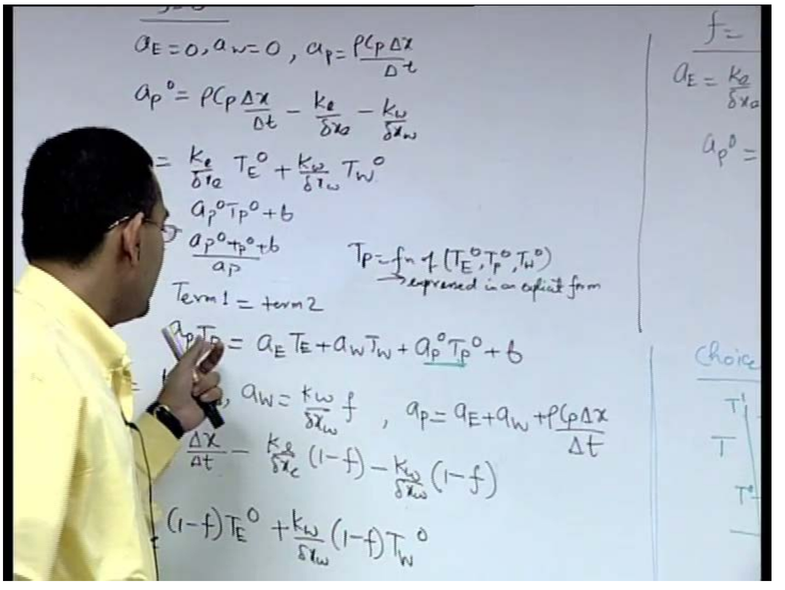

So, f equal to 0, if f equal to 0 you can readily see that a E equal to 0, a w equal to 0, a p is rho C p delta x by delta T that is the... p 0 is rho C p delta x by delta T minus k e by delta x e minus k w by delta x w, b is equal to k e by delta x e T E 0 plus k w by delta x w T w 0. So, you can write for that case a p T p is equal to a p 0 T p 0 plus b. So, T p you can explicitly express in terms of T at the previous time step. So, you can see that here in this T p 0 you have temperature at the point p at the beginning of the time step. That is at the end of the previous time step. In b you have T E 0 and T w 0. So, what what what do you infer from here? That the temperature at the point p at the current instant of time that is at the end of the time step, you can express explicitly as a function of the temperature at the same point at the beginning of the time step plus at the neighbouring points at the beginning of the time step; so that T p is a function of T E 0, T p 0, T w 0 expressed in an explicit form.

This essentially implies that if you know what is the temperature at a particular time, you can just match with time and find temperature at any other point at the subsequent time by explicitly expressing the equation as the value at the current time, as an explicit function of the value at the neighbouring points at the same point at the end of the previous time. And in this way you can just match ahead with time. So, you need not solve a system of linear algebraic equation. You can just consider one algebraic equation from that you can you can calculate temperature at that grid point, then the next algebraic equation the corresponding temperature at the corresponding grid points. So, you need not consider them to be coupled as a system. So, you can treat individually equations, and from individual equations you can obtain the corresponding values of the temperatures. Now, that is one of the very important advantages.

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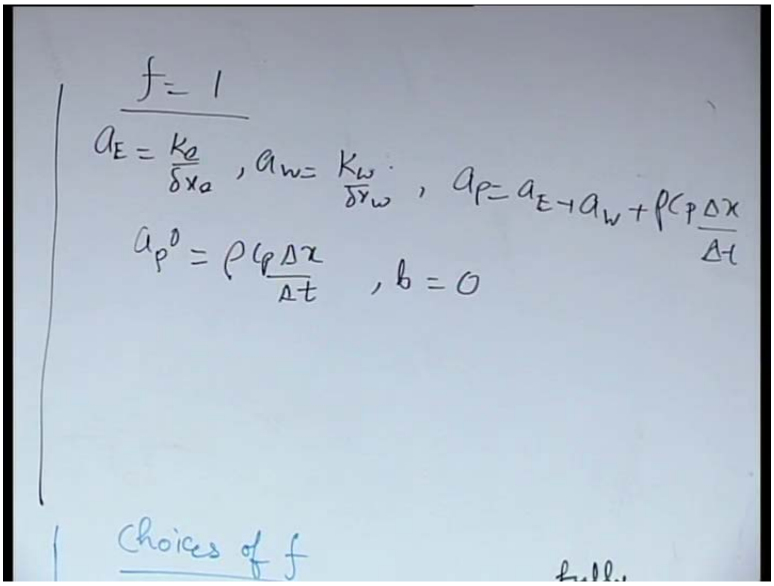

Now, let us consider f equal to 1, then a E is equal to k e by delta x e, a w is equal to k w by delta x w, f e is equal to whatever a E plus a w plus rho C p delta x by delta T, a p 0 is equal to rho C p delta x by delta T and b is 0. So, here what you can see is that the temperature at current time, you can write as a function of temperature at the current time at the other neighbouring points. So, T p now is a function of T E now, T w now and so on.

So that you can not explicitly obtain T p now from this equation, you have to simultaneously solve for all the grid points the coupled equations to get T p now from the system of equations. And you have to start of course, with an initial condition that what is the value at time equal to 0. So, that means in this equation in this particular form T p now is expressed as an implicit function, not as an explicit function of that temperature at the previous time step and that is why it is called as an implicit scheme. The Crank-Nicolson scheme behaves in between, and in our next class we will see that how satisfactorily these schemes do perform based on your constraints over time step spacing and grid size spacing. That we will take up in the next class. Thank you.