Computational Fluid Dynamics

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Lecture No. # 14

Finite Volume Method: Boundary Condition Implementation and Discretization of Unsteady State Problems

In the previous lectures, we were discussing about the finite volume discretization of one-dimensional steady state diffusion type of problems. We discussed about some important aspects of discretization, some basic rules that one may possibly follow, but we did not discuss anything specific about the implementation of boundary conditions in using the finite volume method, so that we will do now.

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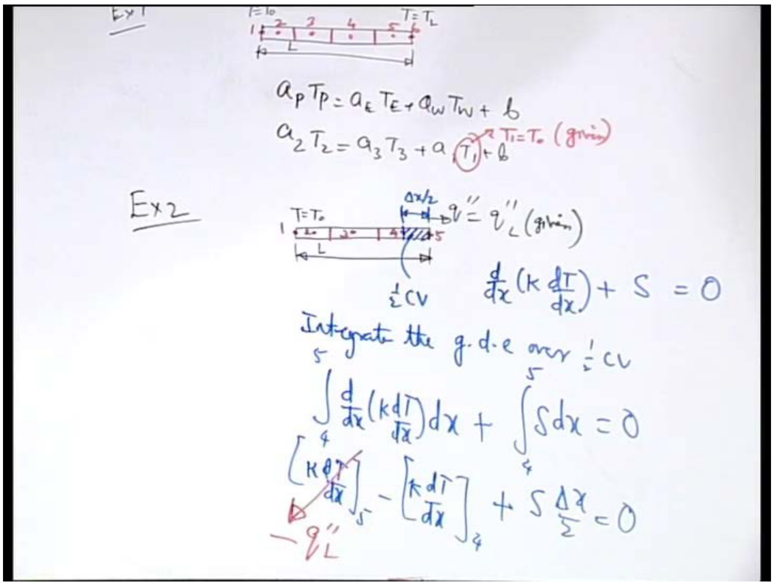

To illustrate that let us take an example, let us say that you have a rod like this of length L and let us consider that T is equal to T 0 at x equal to 0, and T equal to T L at x equal to L. So, we are discussing about implementation of boundary conditions, so this is the first example that we consider. Remember that while implementing the boundary condition, we must keep in mind that what is the discretized system of equations. So, let us say that we had divided the domain into a number of control volumes like this with grid points at the boundary.

So, let us give this grid point some number 1, 2, 3, 4, 5, 6 like that. Assume the heat transfer to be one-dimensional, so for any interior grid point, we have derived the discretization equation. So, that derive discretization equation is of the form a p T p is equal to a E T E plus a w T w plus b. So, for the grid point two, it is like a 2 T 2 is equal to a 3 T 3 plus a 1 T 1 plus b. And we know that given at the point 1, T 1 is equal to T 0; that is given.

So, that is the implementation of boundary condition when you consider the grid point 2 for the discretization equation. Similarly, for the given temperature boundary condition at 6, you can write the discretized equation for the grid point 5 and at T equal to T 6.You substitute T L, that is very straight forward. If you are not willing to make this substitution specifically, but still you want to impose a given temperature boundary you can use ultimately the penalty approach that we have discussed in the finite elements method context.

So, essentially after you get your system of algebraic equations, you will get similar system of algebraic equations as that of the finite element method and then, what you can do is you can for a specified temperature boundary condition you can use the same trick that we were discussing in the context of the finite elements method.

So, we will not discuss this case in more details, because we have learnt by this time that imposing a constant temperature boundary condition is very, very trivial; it is either you enforce its value specifically at the locations where it is appearing in the discretization equation or you write the discretization equation as it is, but you enforce its specific value by using the penalty approach. So, either of these two ways you can follow.

More interesting will be the other types of boundary conditions that we will consider with the subsequent examples. Let us say that you have a T equal to T 0 at left boundary, but at the right boundary you are given the heat flux at x equal to L is given, what will be the first step again divide the domain into a number of control volumes .Let us say we divide it in to 3 control volumes and identify the grid points.

So, 1, 2, 3, 4, 5 now to implement the boundary condition that is the condition which is given in terms of the heat flux here we have to consider a control volume at the boundary and sometimes to do that one may take half of a control volume, it one may take also a full control volume, but if you take half of a control volume the boundary condition is more accurately implemented because, if you take a greater extent of a control volume.

Your gradient approximation becomes weaker and weaker if you take a smaller length you can capture the gradient more sharply. So, this is the control volume now what is your governing equation d dx of k d T dx plus S equal to 0. So, what we will do is we identify this as a half control volume. we will implement the boundary condition by using the governing equation and integrating the governing equation over the half control volume, if you do that then this heat flux will naturally appear in a boundary term the reason is that this is just like a natural outcome of the variational formulation, because in the variational formulation you have the natural boundary condition which is the flux boundary condition comes out automatically as a result of the variational formulation itself. Here, we have seen the finite volume method may be interpreted as a special type of weight weighted residual method where the weight is equal to 1 and therefore, similar sort of confluence may be expected.

Let us see that whether that can happen or not. So, what we do we integrate the governing differential equation over the half control volume. So, integral of d dx of k d T dx th e limit of integration is from grid point 4 to grid point 5. So, if the total length of the control volume is delta x then this is delta x by 2 that we have to remember. So, 4 to 5 the distance is delta x by 2. So, this becomes k d T dx at 5 minus k d T dx at 4, plus S delta x by 2 , equal to 0 what is k d T dx at 5, that is equal to minus of q L. So, in place of this one we will write minus of q L k d T dx at 4 for that we have to make a profile assumption to make a representation of that 1.

So, we make we can make a profile assumption as piece wise linear temperature profile between the grid points that we have earlier seen that it is a legitimate profile assumption. So, then it will be k into in place of d T dx it will be T 5 minus T 4 by delta x by 2.

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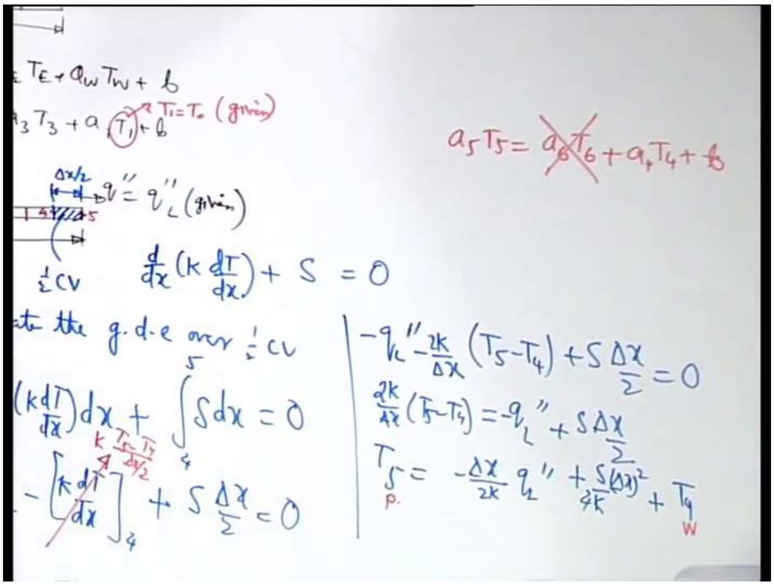

So, if you assemble it becomes minus q L minus 2 k by delta x T 5 minus T 4 plus S delta x by 2. So, 2 k by delta x T 5 minus T 4 is equal to minus q L plus S delta x by 2; that means, T 5 is equal to this one plus S by k. So, let us check the algebra T 5 equal to T 4 plus something what is that something that we will write T 4 plus this entire thing times delta x by 2 k. So, this one plus S delta x square by 4 k.

So, this is again an equation of the form, a 5 T 5 is equal to a 6 T 6 plus a 4 T 4 plus some B where there is nothing called a 6, because there is no grid point 6. The whole purpose of this is to show that this is also of the form a p T p equal to a E T E plus a w T w plus B where p is this one, w is this one and the remaining is B, a 5 and a 4 are one in this case and again see we are writing organizing the equation in such a way that it is a governing equation for temperature at the grid point 5. So, that comes in the left hand side; that means, boundary as a function of the interior that is how we are writing. It now it may be interesting to compare this implementation of boundary condition with the common finite difference based implementation of boundary conditions for the same problem. So, in the finite based difference condition often what we do?

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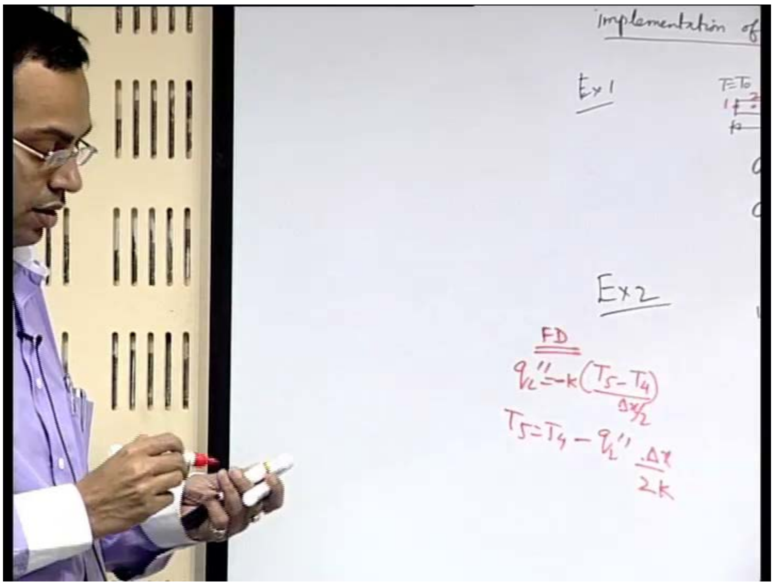

So, finite difference we write the heat flux is equal to k into T 5 minus T 4 by delta x by 2, that is what we write. So, then it becomes T 5 is equal to T 4 may be somewhere we had made a mistake here in the plus minus sign that we need to check. So, here sorry here Minus k d T dx . So, that was here you have a minus k d T dx. So, T 5 is equal to T 4 minus q L double prime k delta x by 2 k. So, with this one line exercise you can see that the difference between this finite difference on the finite volume way of writing the boundary term is the existence of the source term. It seems to be more logical to have the source term in the implementation of the boundary condition here in the finite volume method, because what it tries to do it tries to represent the boundary condition through the conservation of energy over the small control volume and there the heat source definitely plays a role.

So, you cannot write the conservation of energy by considering that the heat flux. Whatever is the heat flux entering, same is the heat flux leaving I mean that is of course, the case had there be no heat source, but now there is a heat source that is present many times, this is a short fall of the implementation of the boundary condition in this way one can get rid of the problem in most of the cases because the control volume being small this extra term is not of great importance, but technically this is a more accurate way of representing the boundary condition then in this standard way of doing it. Now we can consider a third example, where we implement a boundary condition which is of mixed type.

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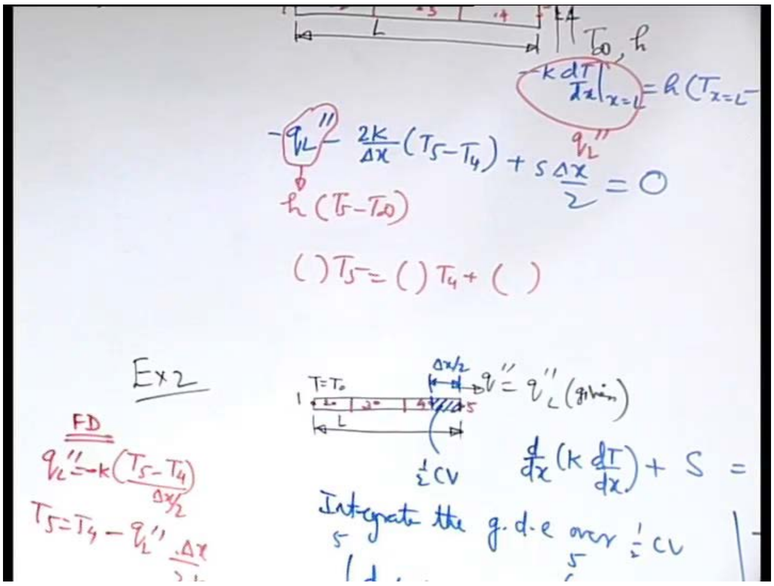

Let us say that, we have a boundary condition at this end, where there is a heat exchange with the ambient, the ambient temperature is T infinity and heat transfer coefficient is h. So, that one can write minus k d T dx At x equal to L is equal to h into T At x equal to L minus T infinity, remember that that minus k d T dx At x equal to L, this is nothing but q at x equal to L the heat flux at x equal to l.

So, if we consider the same discretization as that in the previous example. So, 1, 2, 3, 4, 5 then we will get back essentially the same thing that is minus q L minus 2 k by delta x into T 5 minus T 4 plus S delta x by 2 equal to 0, that is from this particular step. Now the heat flux at x equal to L you can write h into T at x equal to L minus T infinity. So, that is the only the extra effort that you have to put in place of this one we will write h into T at x equal to L is T 5 minus T infinity. So, the remaining work is the same. So, you can write it in this form something into T 5 is equal to something into T 4 plus some constant just by algebraic arrangement of the equation.

So, it is basically the same implementation of the mixed type of boundary condition is very much analogous to the implementation of the normal type of boundary condition. There only thing is you substitute the heat flux in terms of the heat transfer coefficient. Now, implementation of boundary condition has lots of issues there are certain interesting issues and tricks. That, we may discuss which in while somebody is coding it in a computer can use usually whenever there are heat flux boundary conditions. There are two different ways in which one can implement it one is that you implement it as a heat flux boundary condition and other is you implement it as an artificial source term. So, if you consider this control volume what does the heat flux do for the control volume. So, there is. So, let us take an example to illustrate that let us consider different example.

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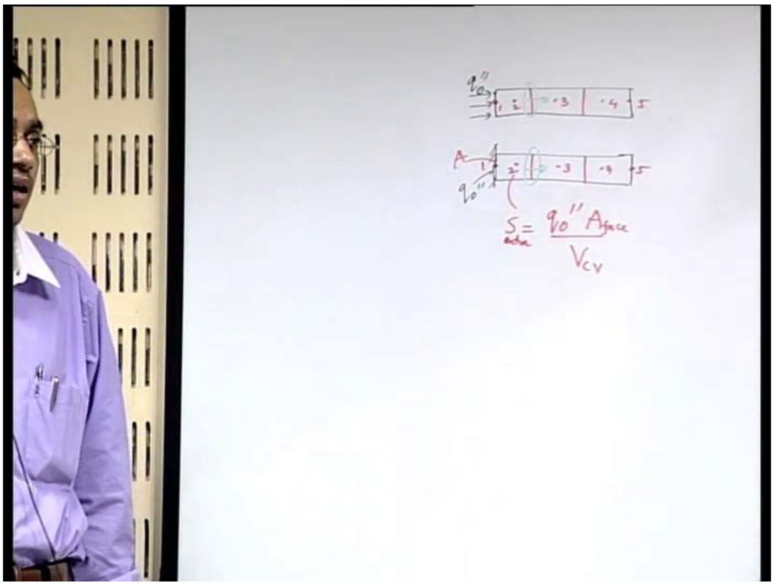

Let us say that you have a domain like this there is some heat flux which is into the domain at x equal to 0. So, what does this heat flux do if you consider a control volume like this. This heat flux essentially make sure that the same amount of heat flux leaves the other face of the control volume. Now compare it with a hypothetical case that this boundary is insulated that is the heat flux is 0.

But our objective is that the same heat flux. Whatever entered through this face the same will leave through this face, but this is insulated what should be adjusted here. So, that that is possible there must be a heat source inside right. So, if you compensate for this consideration of insulation with a heat source inside. Can you write the heat source as function of the heat flux see heat flux is the rate of heat transfer per unit area.

So, heat flux times the area of this face is the total rate of heat transfer and that divided by the volume of the control volume is the volumetric heat generation because the unit of the source term is rate of heat generation per unit volume. So, these two conditions may physically refer to a similar situation.

So, far as the heat flux through this face is concerned, but mind it physically they are not identical it is just an equivalence many times this equivalence is considered. So, for implementing a constant heat flux boundary condition sometimes what one can do is one can consider an insulated boundary condition and whatever was the heat flux there that one can dump as a source term in the corresponding adjacent control volume.

This source term remember this S extra. So, whatever was if there is some other source term this is not that one it is whatever was other source term plus this extra source term. To compensate for the non consideration of the heat flux at the boundary where it is actually there at the boundary question is why people try to play this trick. The reason is that what is the solution of a physical problem? See most of the problems will have very similar governing equations like you have the energy equation, the Navier strokes equation, these do not vary from one problem to another problem, but where the solution varies from one problem to another problem. Why does it vary? Of course, because different properties of the fluid, but most importantly because of different boundary conditions.

So, it is the manner in which you allow the boundary conditions to propagate inside the domain of the solution is the mechanism by which you get the solution. So, the solution mechanism physically is that you have a boundary condition by mathematics you are try allowing the boundary condition to propagate inside the solution of the domain. And that is how your domain knows that there is a boundary and the boundary condition has been imposed and so on more rapidly you can do it more rapidly, you can converge to the actual solution.

So, here this flux is at the boundary, but it is not penetrative inside the control volume in one shot here you have made the effect penetrative inside the control volume in one shot by making it as a source term and that makes it to converge faster in many cases as compared to the direct implementation of the boundary term it is of course, in many cases these the difference is marginal, but in some very extensive simulations these difference may be of some importance.

So, this is one trick in implementation of the heat flux boundary condition, when we are describing this boundary conditions and governing equations. Remember, we are giving one-dimensional steady state heat conduction as a example, but this is for any general flux type of boundary condition that we are talking about. So, it does not just specifically refer to a heat transfer function problem because it is more convenient to take as something as an illustration and give it a short by on the basis of that illustration that is what we are doing, but we should not keep this prejudice in mind that this is the only for a heat transfer problem that we are talking about. Now the other issue that we have discussed earlier that any condition at the boundary is not boundary condition right. So, it has to fulfill certain requirements. Now, let us ask ourselves a converge question, if there is some lack of condition at the boundary; that means, some at a boundary no condition is given still can you have a boundary condition.

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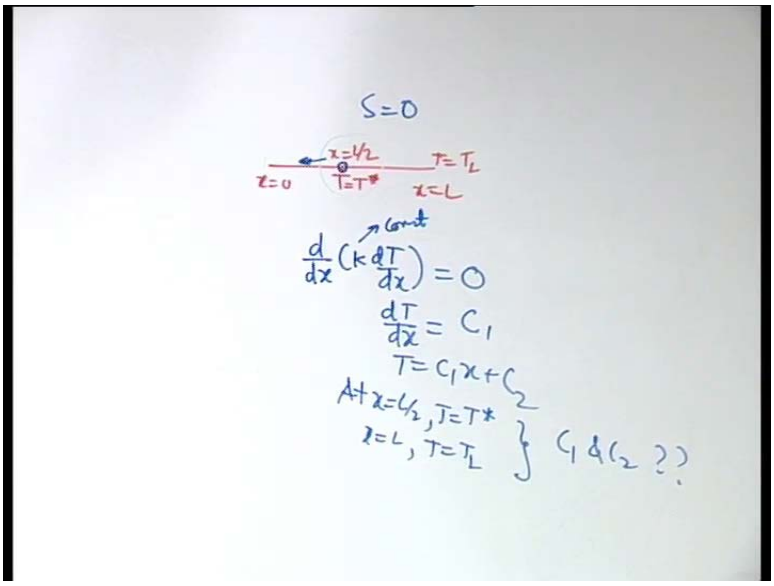

That means, let us take an example. So, let us say that you have a domain x equal to 0 to x equal to L. We are giving a boundary condition that at x equal to L by 2 T equal to say T star and at x equal to L T equal to T L, actual physical domain boundary x equal to 0 we have not given a boundary condition here.

So, no condition at the boundary can it be a boundary condition till now we have answered the question that whether any condition at the boundary condition or not now we are trying to answer a question that there is a problem where we are not giving a condition at the boundary, but still can it be a well posed boundary value problem.

So, this like I mean this the origin of this discrepancy or dilemma comes from the fact that we have been trained to learn boundary condition as the condition at the boundary.

So, boundary of the physical domain is very, very important and definitely this point does not belong to the boundary of the physical domain, but let us see whether this is appropriate.

So, let us try to solve this problem analytically. So, you have d dx of k d T dx equal to 0 let us say that the source term is 0 just to simplify the calculation our objective is not to highlight that and let us say that k is constant throughout the domain. So, d T dx is equal to C 1, T is equal to C 1 x plus C 2. So, if you use that at x equal to L by 2, T is equal to T star and at x equal to L, T equal to T L then from that you can find out C 1 and C 2 right you will get 2 independent algebraic equations and two unknowns from that we will get it there is absolutely no discrepancy this is very, very simple.

So, what lesson we can learn from it this is a problem where apparently from our common sense prospective we are not given a boundary condition, but still we are able to come up with a solution. We have given a boundary condition, but that is not physically at the domain boundary now what we can make sure is that no matter whether we have given the boundary condition here its effect will be also propagated towards this side the same cannot be talked about for a initial value problem, because we have already discussed that time is a one way coordinate system.

So, in place of space coordinate if it was time coordinate then that would have not been possible because whatever is happening now cannot influence what has happened 10hours back it can only influence what is going to happen in the future, but space wise whatever is happening here can easily influence what is going to happen here and that in an fashion for an elliptic boundary value problem.

So, it all depends on when we say boundary condition we mean not just a boundary condition, but also initial condition if the time dependence is there. So, all these critical things we have to keep in mind while designing a problem. Now, that we have learnt the finite volume discretization. Let us take one or two concrete examples to illustrate a bit more of implementation of that and for that for the heat conduction case.

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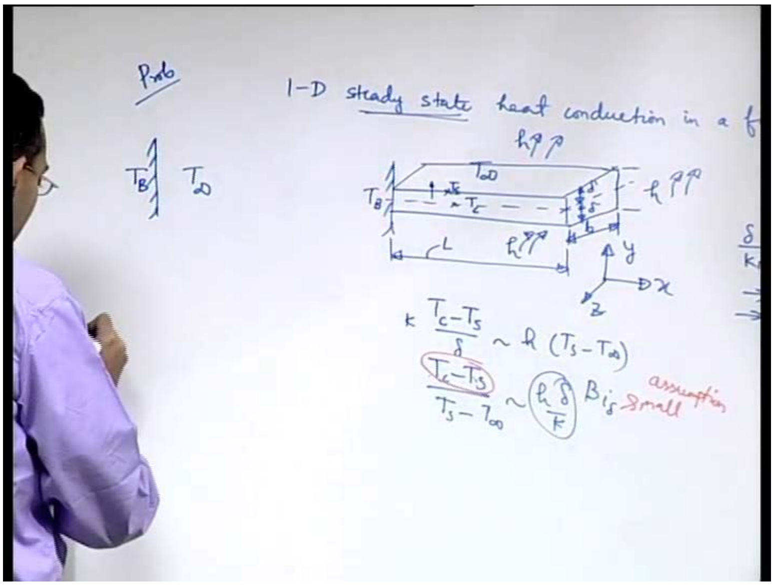

We will take the example of this is you can assume an illustrative problem one-dimensional steady state heat conduction in a fin. So, what we will do is I will partially work out this problem towards the discretization, and leave the remaining part for you as an exercise to complete it through the use of a computer program.

So, let us say that you have a fin like this. All of you know what is a fin, But just to discuss in terms of a common sense prospective say if you have a boundary from which you want to dissipate heat now if you want to dissipate heat and there is a high concentration of heat one of the ways. So, how to enhance the heat transfer? Remember the heat transfer from a particular location of a solid boundary to the ambient is proportional to several things. One is the heat transfer coefficient which depends on the ambient conditions the other is the temperature difference between that and the ambient and the third thing is the area over which the heat transfer is taking place. So, the fin tries to take care of the enhancement of the area.

So, what it tries to do it tries to provide with an extended surface and through the extended surface you can have a greater amount of heat to be dissipated to the surroundings because the essentially it has more available area, but in doing, so attempting to do. So, you are now playing with that temperature difference the temperature difference. So, if the temperature earlier was T boundary and the ambient was T infinity. Now the temperature inside the fin will be different from T boundary.

So, this is still T infinity this is T boundary because heat is conducting mainly in the axial direction then what will happen the temperature inside that fin will fall from T boundary because heat will flow from high temperature to low temperature. So, because the temperature inside the fin will fall that difference between the fin temperature and T infinity will decrease and therefore, the driving potential difference for heat transfer will decrease. So, on one hand you are adding surface area. So, that is augmenting heat transfer.

On the other hand you are reducing the temperature difference which is the driving potential for heat transfer. So, the combination the fin may be effective or ineffective depending on whether the combination is good as a design or the design is a bad one. We will not go into the details of that one, but this is just to give you a common prospective of whether a fin can be effective or not. Now, let us say that we are interested to write a control volume balance for a fin derive a governing differential equation and write a finite volume discretization based on that one. So, for illustration let us consider that the fin is a rectangular one. Of course, it is not necessary to do that, but just for illustration let us say that the half height of the fin is delta the heat transfer coefficient to the ambient from all sides is h, and width of the fin is B; the length is L; h is the heat transfer coefficient.

So, first of all let us see that what is the important mode of heat transfer then what is the directionality of heat transfer. So, first we considered a steady state heat conduction; that means, we consider that the fin is at a steady state that is temperature at a given location in the fin is not changing with time now if you consider the fin to be wide enough as compared to its thickness that we will see later on that what is its implication, but even before that if you consider up two- dimensional heat transfer to begin with.

Then let us look into the different thermal resistances along different directions. So, if you say that you are interested in analyzing a heat transfer let us say x is this direction, y is the perpendicular direction and z is the direction perpendicular to the plane of this board then which direction will be important for that temperature gradient analysis y or z. x definitely yes, but out of y and z.

So, in the y direction you have the thickness delta much, much less than the z direction thickness. So, the thermal resistance in one case is delta by k just like L by k. Of course, in terms of reduced units in another case it is b by k here the area is area of L by k A remember. So, area corresponding to the flow here or heat flow is say here a 1 and here it is a 2 the two directions are different areas, but physically without looking into these numbers you say that here you have a small length across which the heat is flowing here. You have a large length across which the heat is flowing and this length L is even larger.

So, you can consider two different cases - one is B much greater than delta; another is delta much greater than B this all these examples that you can consider. So, out of these cases, of course, the figure we have drawn may give an indication that B is much much greater than like here it is better to use two delta because half of this we have considered as delta or two delta much much greater than B. These two are different cases to cut things short on this discussion out of these two considerations any one of the considerations will make it a two-dimensional problem instead of a three-dimensional one. Whether the heat transfer has to be considered along z or y is a matter of issue, but the thing is that once you make such a consideration you can convert it into an equivalent two-dimensional problem.

Let us say that we consider that our objective is not to go into deep into the physics of a fin problem. So, we do not go into the details of the thermal resistances and So on. But just to simplify the thing let us say that we are considering a two-dimensional situation in the x y plane. So, the temperature variation in the x y plane is important if it was not. So, it could be x z plane, but definitely along x direction the heat transfer is important. Now next is that if the in a x y plane it is important then there could be conduction along x, conduction along y out of these two, which one would you consider to be more important. Now if you see you have conduction along y takes place over a very short distance.

So, let us try to make an assessment let us say T c is the temperature of the center line of the fin and T S is the temperature of the surface of the fin. So, we can just make an approximate analysis T c minus T S by delta that is the that into k is the heat flux due to conduction that goes from the center line of the fin to surface of the fin. We are considering y direction heat transfer that is of the order of h into T S minus T infinity.

So, T c minus T S by T S minus T infinity is of the order of h delta by k which is called as biode number based on the length delta. So, to simplify our analysis we consider this to be small this is an assumption.

So, if this is small then what we can see; that means, that there is negligible temperature difference between the fin center line and the fin surface as compared to the difference between the surface and the ambient if that is. So, then there is no necessity of assessing the temperature difference between the center line and the surface and then there is; that means, there is no necessity of analyzing the heat conduction in the y direction. So, that renders the problem effectively a one-dimensional problem where the heat conduction you can analyze along x direction only.

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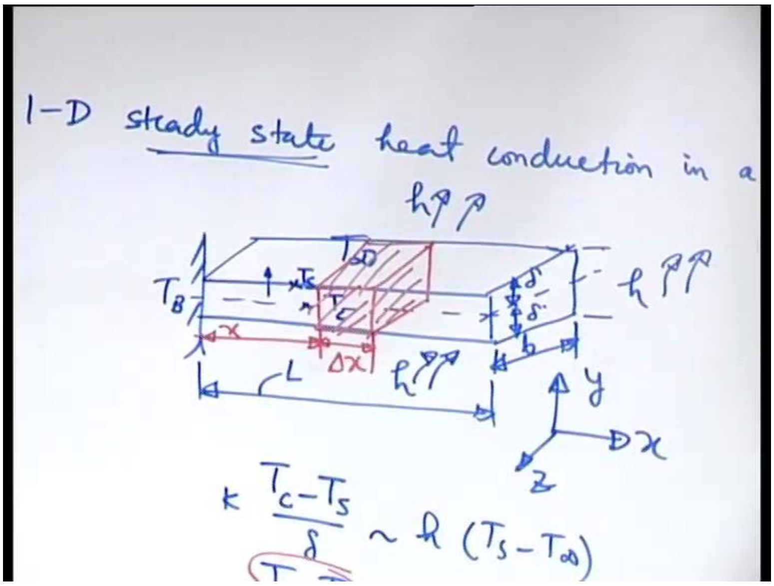

So, to do that let us say that we consider at a distance x of section of width delta x and write an energy balance. So, we are isolating the element this one and drawing it separately here.

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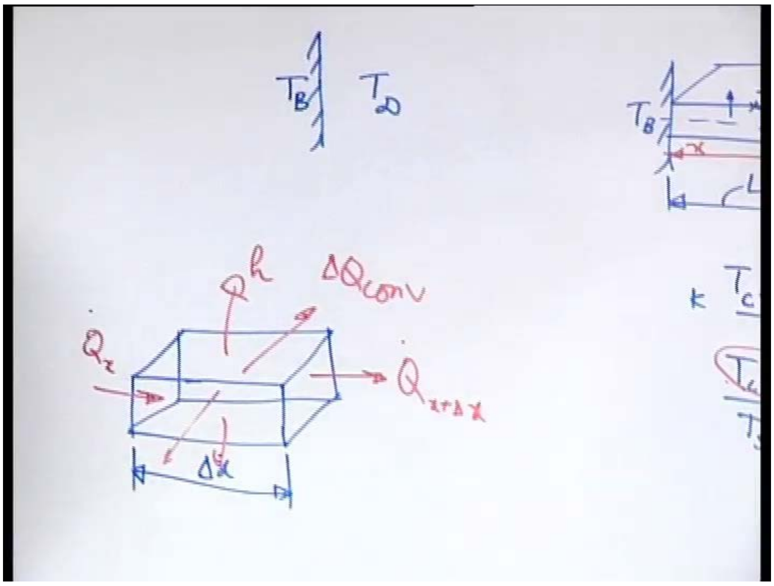

So, there is some heat that enters here that is q x the heat enters through the opposite face at the rate of q dot At x plus delta x whatever enters does the same leave here no because through the top, bottom, front and back heat gets lost to the surroundings by virtue of convective heat transfer with the surroundings. So that is given by the convective heat transfer coefficient h; and let’s call that as delta q convection.

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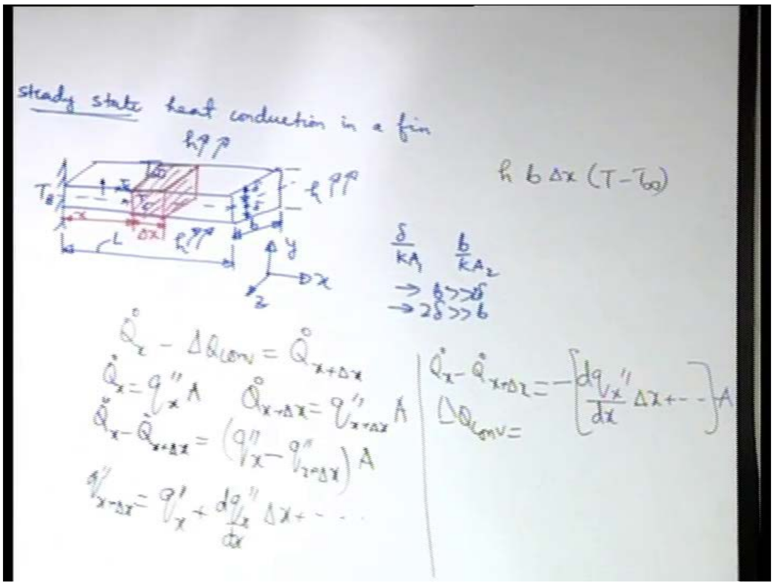

So, we can say let us write it separately q dot in minus delta Q convection is equal to q dot out or in is here x and out is x plus delta x. This is true at steady state. Let us say that the cross sectional area of the fin is a. So, Q dot x is minus or before writing it in terms of the temperature gradient, let us write it in terms of heat flux - that is heat flux at x times A. Q dot at x plus delta x is equal to heat flux at x plus delta x times A. And q dot at x minus Q dot at x plus delta x is equal to heat flux at x minus heat flux at x plus delta x times A.

We can write heat flux at x plus delta x as a function of heat flux at x. So, we can write this as heat flux at x plus this 1 using the Taylor series expansion which means that q dot x minus q dot x plus delta x is equal to in this way times the area and what is delta q convection. So, there are four surfaces through the top surface. So, what is that dimension of the top surface, B into delta x.

So, through the top surface the heat transfer will be B into delta x that into a h into T minus T infinity when we say T we imply that the entire cross section is at the same temperature T why because we have considered a small bio number. So, there is no difference between centre line and the surface temperature. So, at a particular x all y's have same temperature that we call as t.

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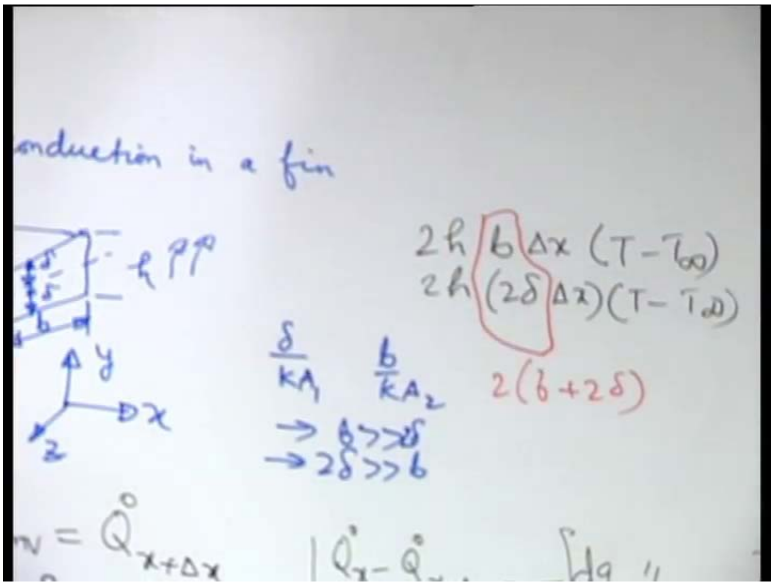

So, this into T minus T infinity there are two such surfaces for top and bottom and front and back front is 2 h into 2 delta x into T minus T infinity. So, if you add these two then 2 h delta x into T minus T infinity will be common and you have b plus 2 delta b plus 2 delta times 2 is what it is the perimeter of the cross section of the fin. So, delta q convection is nothing but the perimeter of cross section of the fin which we call as p into h into T minus T infinity, where p is equal to 2 into b plus 4 delta b plus 2 delta into delta x.

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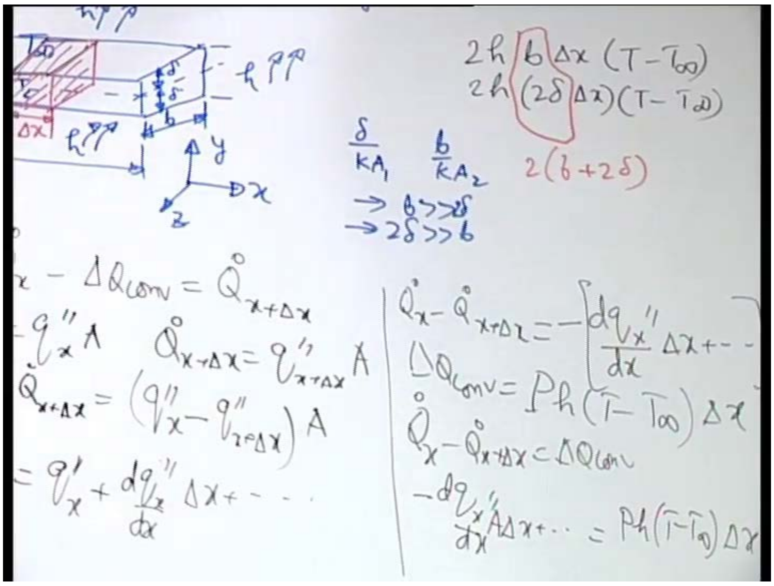

Since our balance is Q dot x minus q dot x plus delta x is equal to delta Q convection. So, we can write minus d q x dx delta x plus higher order term is equal to p h into T minus T infinity delta x, take the limit as delta x tends to 0 there is a area right here.

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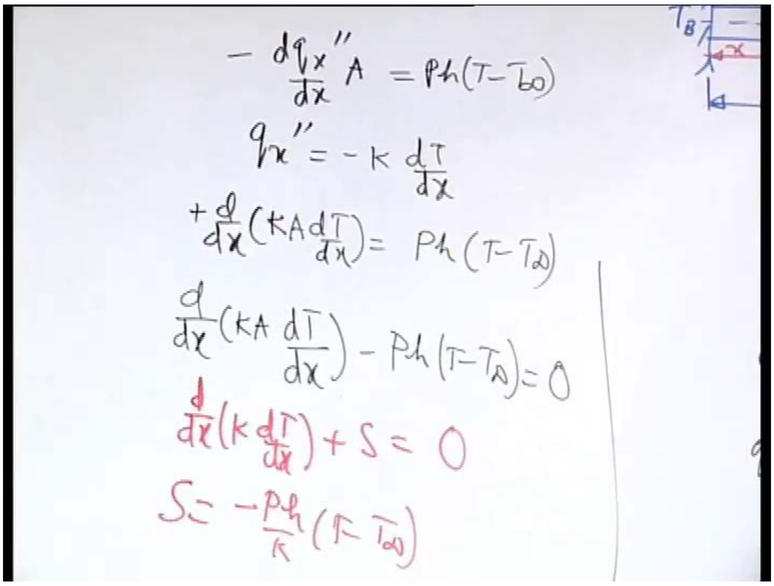

So, take the limit as delta x tends to 0. So, you have minus d q x dx A is equal to p h T minus T infinity and q is nothing but minus k d T dx here we are taking a fin of uniform cross sectional area. So, that is A constant, but if A is a variable it would be contain within d dx of A into q double prime so, but here since A is a constant we have minus d dx of k A k d T dx into a, but A we can put inside or outside whatever we want in this case because A is constant that this minus with this minus will become plus is equal to p h T minus T infinity.

So, in this particular problem k and A both are constants. So, dx of k A d T dx we can also write as k A into d 2 T dx 2 whatever you want, but let us write the more general form. So, that is the governing equation. So, how do you discretize this using the finite volume method. So, remember that this is of the form dx of k d T dx plus S equal to 0. So, what is S minus p h by k into T minus T infinity. Minus p h by A So, it is already a implicitly linear source term.

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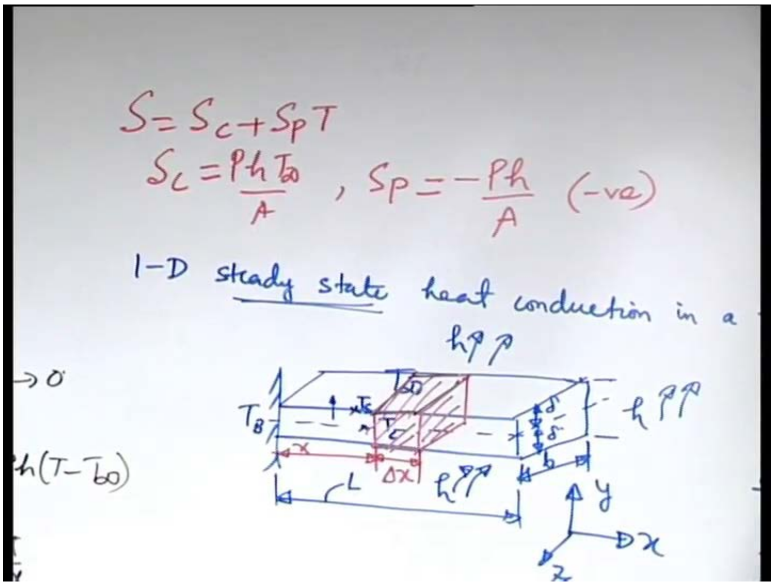

So, S can be written as a combination of S equal to S C plus S p into T what is S C, p h T infinity by a what is S p, minus p h by a this is negative. So, this is such a well designed problem to be implemented that you have to do basically nothing extra, because the source term is already linearised and it is linearised in such a way that no basic rule will be violated.

So, then you can cast this particular problem into that discretization mode of a one-Dimensional steady state problem with a source term that varies linearly with temperature we have discussed that particular problem and you can implement this particular problem using that strategy what boundary condition you will put. Let us consider an example. let us say that you consider that At x equal to 0 T is equal to T B and At x equal to L the fin is insulated.

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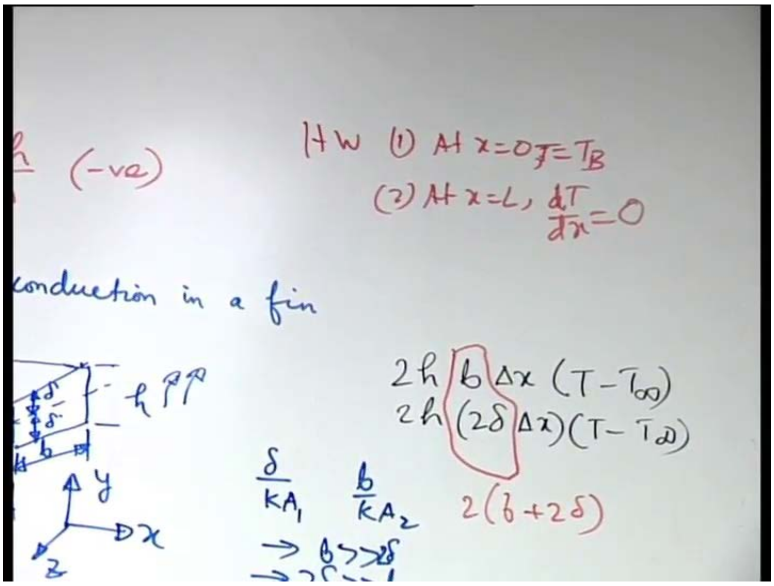

So, this I leave you on you as a homework that at x equal to 0 you have T equal to the base temperature at x is equal to L. You have d T dx equal to 0. So, using this boundary conditions you assume suitable numerical data as per your choice and obtain the temperature distribution along the fin. So, that is the first thing. So, you can discretize it into a number of control volumes say you non dimensionalise the entire equation.

So that your value of non-dimensional x will be which is x by L will be between 0 to 1. You can divide it into say five number of control volumes, and then you will get requisite number of grid points at each grid point you solve for the temperature by using the finite volume method. Also obtain the temperature distribution using finite difference method and finite element method and compared those with the analytical solution the analytical solution corresponding to this problem is also there.

So, you compare with the corresponding analytical solution in the same graph and comment on whether you are getting the same solution as the approximate solution or not the same as the analytical solution, and different approximate solutions whether they are same or not. Let us stop here today, and we will continue in the next class. Thank you