Computational Fluid Dynamics

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Lecture No. # 10

Fundamentals of Discretization: Finite Element Method

In our previous lecture we introduced the concept of discretization. Let us try to recapitulate it a bit, so discretization principles.

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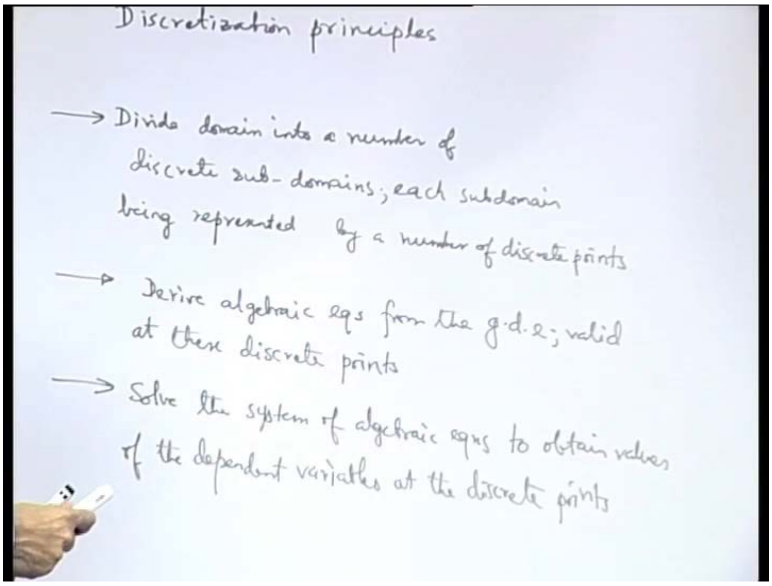

We should remember that this is a generic principle or a collection of generic principles, and it does not specify what is the method. So, it is a common philosophy of all discretization processes for simulation of continuum based equations. So, the first one was divide the domain into a number of discrete sub-domains; and each sub-domain being characterized by a… Or instead of characterized it is better to say represented by a number of discrete points. And once that is done then you derive algebraic equations from the governing differential equation valid at these discrete points. And then, solve the system of algebraic equations to obtain values of the dependent variables at the discrete points.

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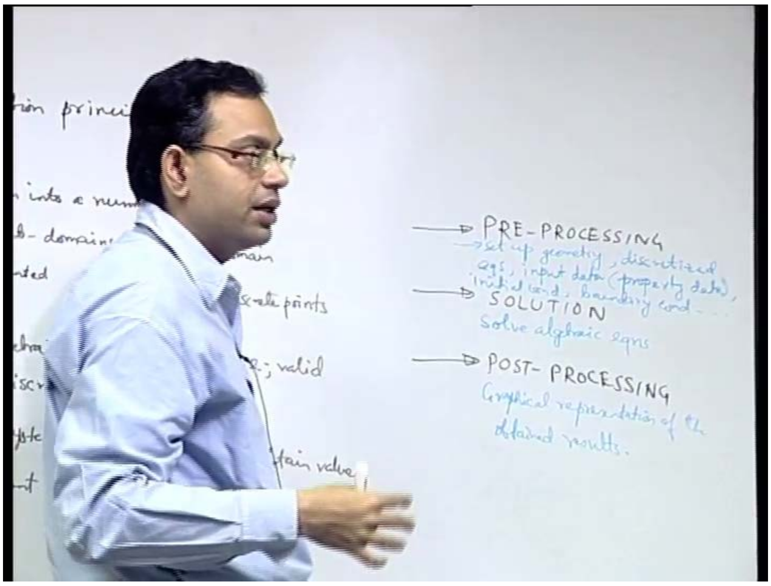

Now, if we if one follow this particular procedure then the overall analysis can be thought of as a collection of the following; one is pre-processing, another is solution and the third one is post-processing. These are the broad steps in the overall analysis. So, in pre-processing, what you are doing? You are setting of the problem. So, how do you setup the problem? You setup the geometry and discretized equations, input data that means property data as an example, initial condition, boundary condition, etcetera. So, these are the inputs which are necessary. And once this thing is properly setup then you basically required to solve algebraic equations. Once this algebraic equations are solved you get the value of the variable, say temperature or velocity at each and every point, and then post-processing is a graphical representation of the obtained results.

No matter whatever numerical method we are adopting, the method will go through these steps, so long as we are solving the problem in a continuum perspective. We will take up some examples of numerical methods which illustrate these principles, and the first example that we take up is the finite element method. So, let us consider the finite element method; we considered it first, because it evolves from the variational formulation that we have considered in some of our previous lectures.

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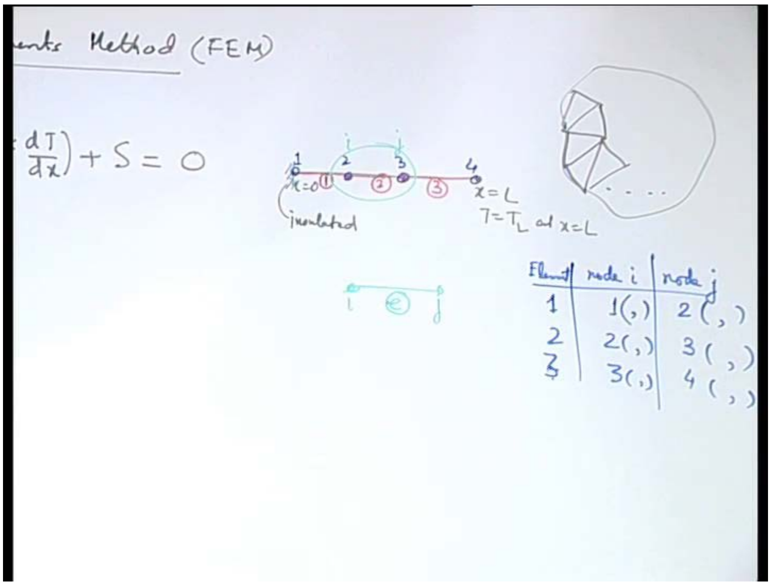

So, finite element method finite element method or FEM. Of course, we can go deeper and deeper in to the method, but we have to keep in mind that this is not a course totally dedicated towards finite volume finite element method. So, our objective will not be to go into the details of this method, but may be take up a generic simple example to illustrate the use of this method in perspective of the discretization policy - that we have just formulated. So, let us take an example and we will learn this method or the basics of this method through this example.

So example, say you are solving a one-dimensional steady state heat conduction equation. See we have considered this as a prototype equation for illustrating many of the principles, and let us say that we do it for illustrating the finite element method itself. Remember that the finite element method can be perceived in many ways, but in one of the ways in which it can be done is that one type of finite element method can be thought of as a discrete version of the Galerkin's method. So, the Galerkin's method if you recall that we had a trial function same as the weighting function, and we employed that trial function and weighting function over the entire domain.

Now, in the discretized version of that in the finite element method, we will employ that over each of the discrete sub domains in which the domain is decomposed, and each sub domain is called as an element in a finite element method. So, the first step if you remember, divide the domain into a number of discrete sub domains. So, if it is a two-dimensional or three-dimensional problem you could have sub domains of different shapes, may be in this way we can go on; this is a one-dimensional problem. So, here the sub domains are nothing but straight lines. So, you have this as a total domain. Let us say that this, the left hand side is x equal to 0, right hand side is x equal to L. And let us say that the boundary conditions are as follows; that it is insulated at x equal to 0 and temperature is specified at x equal to L.

See we have not yet formally discussed about the different types of boundary conditions. It is very, very important to discuss about that and how to implement that in a generic situation, and we will come in to that issue subsequently. But right now at least these types of boundary conditions we have we can demarcate as either the essential or the natural boundary conditions, which are very much defeating with the variational formulation. Now, we are interested to solve the temperature distribution within this domain. Let us say this is a rod, physically the rod is such that its cross sectional area is much smaller than its corresponding axial length, and major temperature variation takes place axially. There is some uniform heat source that is present which is given by this S.

Now, we divide the domain in to a number of sub domains. So, let us say that we divide it in to three sub domains. So, sub domain number 1, sub domain number 2, sub domain number 3; each sub domain we call as element. So, we have element number 1, element number 2, element number 3; each element in turn is represented by some discrete points - sets of points. So, these discrete set sets of points in the terminology of finite element, these are called as nodes. So, we give the node numbers 1, 2, 3, 4 like that. So, we can create a chart of element node connectivity. Remember that it is it not necessary that each element will have two nodes; each element can have whatever number of nodes, and we will see very soon that more the number of nodes contain by each element, you have higher and higher order approximating polynomial that can be that can be used for interpolating the dependent variable within the element. But obliviously more and more number of nodes will mean that you have higher computational expense, because you have to solve for the variable at those nodes. So, there is a sort of compromise that one looks for.

Now, you can prepare a chart where for each element, you can identify the nodes. So, for element number 1 the first node is 1, the second node is 2; for a element number 2 first node is 2, second node is 3; for element number 4 3 first node is 3 and second node is 4;

and each node is having its own coordinate. So, you can use whatever coordinate system you want to specify the coordinate of each node. It is a one-dimensional problem. So, although I have given a comma that is not necessary, you just require one coordinate - one x-coordinate, but if it were a two-dimensional problem, it could have been x comma y that is why I have put a comma or may be a three-dimensional problem a third coordinate. But here this comma is not necessary you just have only one coordinate specifying the the location of the point. So, this is a chart which is important, which is like an element node chart, which relates the connectivity of each element with the corresponding nodes.

Now, what we will do is, we will consider any isolated element. Let us say that we consider an element this one where the first node is i and the second node is j. So, we have one element which is starting with node i, ending with node j, and we are interested to write, at algebraic equation corresponding to this governing differential equation for that generic element; that is our objective. Let us say that the name of the element is e. So, what we will do? We will start with the basic principle of the Galerkin’s method. So, the first step will be to multiply the governing differential equation with a weighting function, and then integrating it, not now over the whole domain, but over each sub domain that is over each element, and then some that effect up over all the elements. So, effectively you are using the Galerkin’s method over the entire domain, but it is a discrete summation rather than a continuous integration over the entire domain. So, first you consider the Galerkin’s method implemented over each sub domain then you sum it up over all the sub domain. So that you get the net effect over each over the total domain, but in a discrete sense, not no more in a continuous sense.

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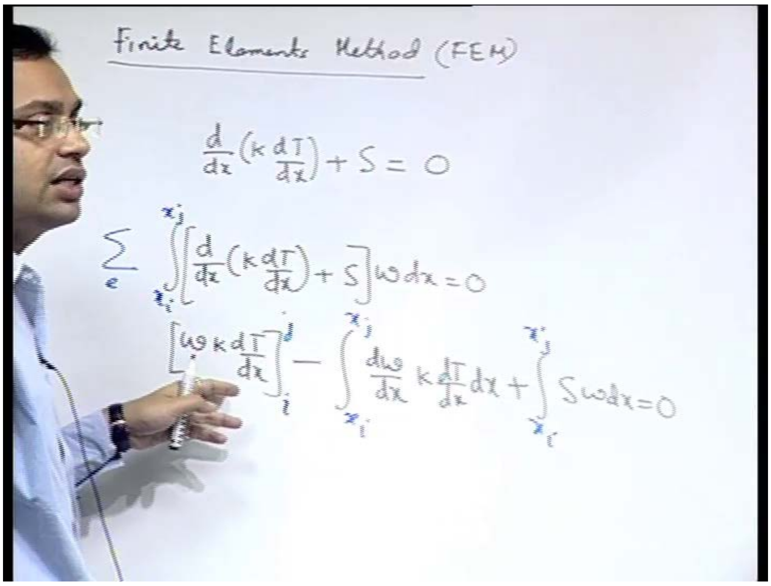

So, you write d dx of k dT dx plus S, we multiplied by a weighting function dx equal to 0; that is the first step. So, we are using discrete Galerkin's finite element method. There are many different ways by which finite element equations may be derived. So, do not develop a misunderstanding that this is the only way in which it can be done. But this is the discrete Galerkin's method by which we are showing this as an illustration. From x equal to 0 to x equal to L will be the limit of integration. Then what we will do is, we will integrate it by parts. So, w will be the first function and the remaining will be the second function. So, w into k dT dx minus integral of the derivative of the first dw dx into integral of the second k dT dx dx. That is a straightforward step, we have executed similar step for the variational formulation, but we have just doing it over each element that is the only difference. So, what will be the difference?

Now, the change from the variational formulation is that it will be no more from 0 to L, but from x i to x j and then this is summed up over all element that is the conceptual difference. So, what we will do is, we will consider first the formulation for each element and then we will sum it up or assemble it up later on. So, it will be from i to j, x i to x j, x i to x j. So, the first difference you follow from the general variational formulation. Then what we will do? We have to now assume a trial function and a weighting function. See we have reduced the order of the continuity requirement of the trial function by doing the integration by parts. So, the highest order derivative that you require is dT dx. So, linear function of T as a function of x we will do, and a linear function will require two coefficients for its specification.

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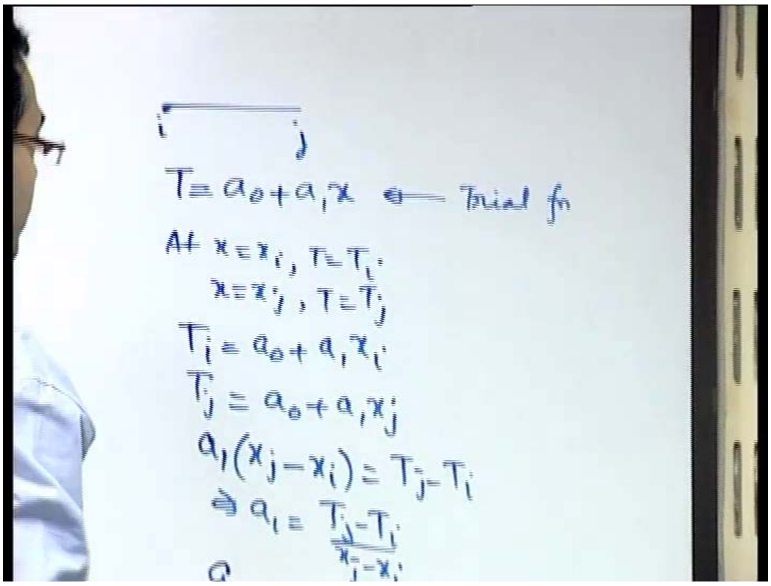

So, if you have a two nodded element which we sketched in our figure with each element has node i and j, then that would be sufficient. But if that was not the case, then you would have required a higher order derivative to specify. Next what you do? We will try to write the corresponding trial and weight function in this particular expression. So, the trial function, so you have two node points where you where you can specify the variable. So, you can have a trail function T is equal to a 0 plus a 1 x. This is a trial function. Remember the trial function you are making valid within each element, not a global trail function, but a trail function that is valid over individual elements. So, it is a piecewise linear interpolation that you are considering (( )) to say, it is not a global single linear interpolation.

Now, you have the following information that at x equal to x i, you have T equal to T i. Which is an unknown? Which you intend to find out? An at x equal to x j, you have T equal to T j. So, it is physically must more much more interesting and intuitive, if you express a 0 and a 1 in terms of T i and T j. Because those are the unknowns that you need to solve, and those directly relate to the physical variable that you are interested in. So, we can write T i is equal to a 0 plus a 1 x i, and T j is equal to a 0 plus a 1 x j . So, now you can find out a 0 and a1. So for example, a 1 if you subtract this two a 1 into x j minus x i is equal to T j minus T i which means a 1 is T j minus T i by x j minus x i. And you can find out a 0 from anyone of the equations, say the first equation, it is T i minus a 1 x i, so T i minus T j minus T i by x j minus x i into x i. So, a 0 will be T i x j, minus T i x i and plus T i x i will cancel, so minus T j x i by x j minus x i.

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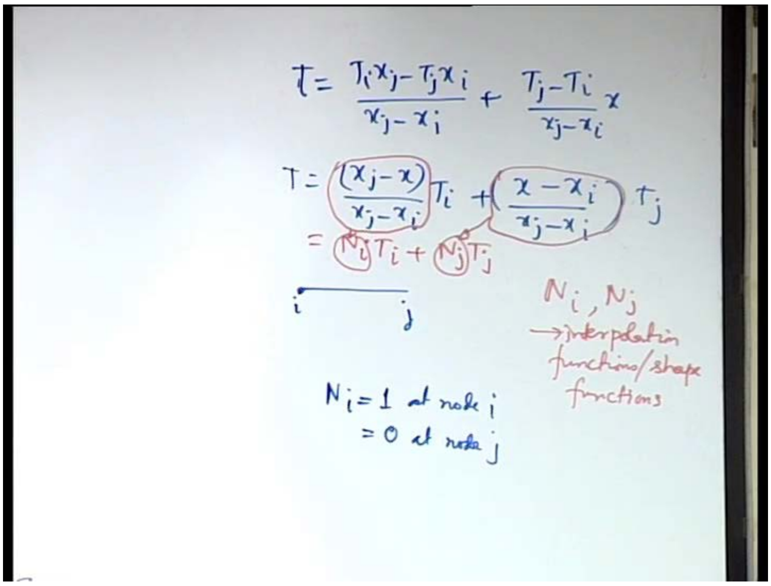

So, T will be equal to a 0, in place of a 0 let us write T i x j minus T j x i by x j minus x i plus a 1 x i plus T j minus T i by x j minus x i into x i in into x right not x i. So, you can collect the terms with T i and T j. So, T you can write x j minus x by x j minus x i T i plus x minus x i by x j minus x i T j. This is the form that eventually we need to keep in mind. So, what we do is? So, what we have achieved by this step? We have been successful in expressing T as a function of some function of x and function of values of T at the node points i and j. So, this we can write is equal to N i T i plus N j T j, where N i is this one and N j is this one. These are called as interpolation functions or shape functions. This is your trial function or shape function.

Interestingly you can see one very important property of this shape function. What is the property? What is the value of N i? See these functions are defined at discrete points i and j; N i equal to what at node i. At node i, x equal to x i, so N i equal to 1 at node i, and what is it at node j? It is 0 at node j, because it will become x j minus x j, 0 at node j. So, in general N i equal to 1 at at the node i and 0 at other nodes. That is the property of a shape function.

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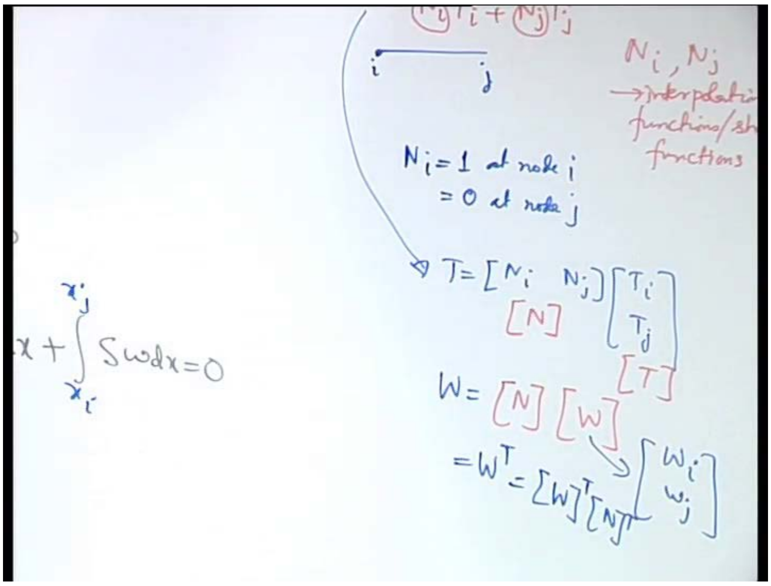

Now, what we will do is, so this is a trial function. So, you can write this particular expression T equal to N i T i plus N j T j in a matrix form; how, you can write this as N i N j T i T j. The objective of this is eventually your unknown that you need to solve may be written in a vector T i and T j, which you need to solve for each node point. So, this we give a name matrix N and this we give a name matrix T. Now, question is what is the weight function? In the Galerkin's method, the trial function and the weight function have the same form. So that you can write as N into W, where what is this W? W is W i and W j; conceptually they are like delta T i and delta T j. Since W is a scalar, this is same as W transpose; it is just a scalar, transpose of the scalar is scalar itself. So that is same as W transpose N transpose, because a into b transpose is b transpose into a transpose.

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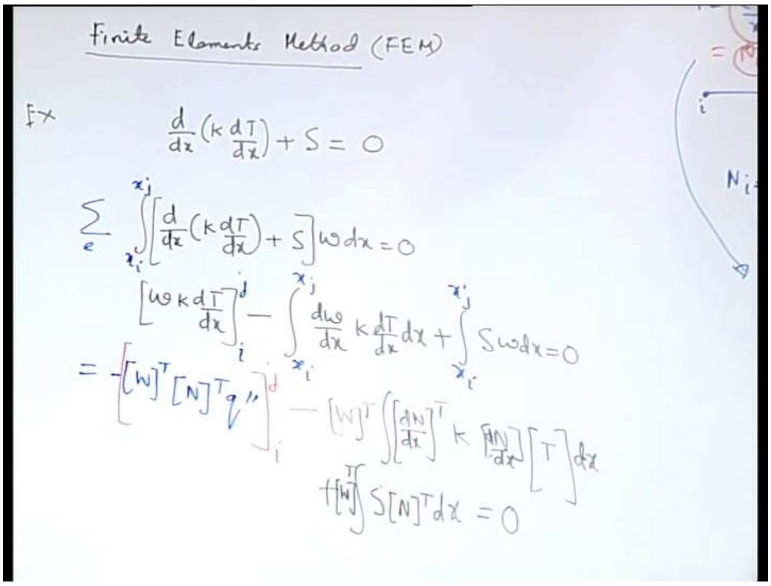

So, with that in mind let us try to write the corresponding form of this equation. So, in place of w, we will write W transpose into N transpose in place of k dT dx; see what is k dT dx? Minus k dT dx is the heat flux. So, k dT dx specification of that is basically a natural boundary condition specification. So, we will honor that concept in mind, and then instead of k dT dx we will write this as minus of the heat flux, where q double prime is the heat flux which is equal to minus k dT dx by Fourier’s law of heat conduction. This entire thing should be evaluated between i and j. Then minus dw dx, see if you that differentiating this with respect to x, remember x is contain within N only. So, you can write this as W transpose then integral of dN dx transpose into k into dT dx is N transpose dN dx into T dx. So, we have substituted dT dx as dN dx into T; remember this T does not contain x, these are the just the nodal values. Plus integral of now W transpose then integral of S, in place of W it is W transpose N transpose.

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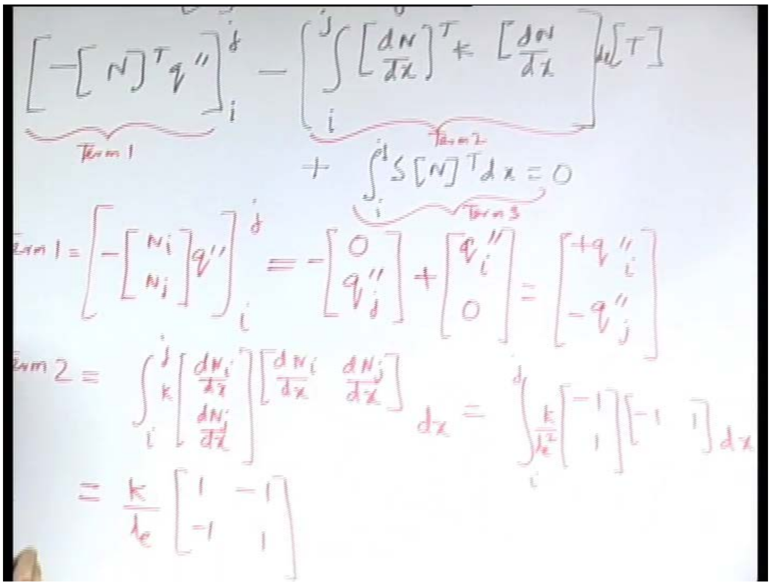

Now, since W transpose is arbitrary, it is just a variation. You are left with minus of N transpose q between i to j minus dN dN dx transpose k dN dx, whole thing multiplied with T, plus integral of S N transpose dx equal to 0. We will simplify this one by one. So, first consider this expression; minus N transpose means what? So, we are first considering this term, say term 1; minus N transpose means N i N j corresponding q then this entire thing between i and j. So, first you evaluate this quantity at j then subtract the same quantity at i. This is the upper limit minus the lower limit. So, what is the quantity at j? N i equal to what at j? 0, and N j is equal to 1 at j. So, it will be minus 0. So, one times the heat flux at j. This is the upper limit. Then the lower limit N i equal to 1 at 1 at i. So, q double prime i and this is 0. So, it is minus q double prime i sorry plus q double prime i and minus q double prime j.

Then let us consider the term two excluding the minus sign, so term 2. Term 2 integral i to j dN dx transpose. What is dN dx? dN i dx dN j dx. So, what is… So, let us write that dN i dx dN j dx, k is there, then dN i dx dN j dx.

Term 2 exclude the T also, this is the unknown vector that you have to find out. So, we are excluding that from the evaluation of the term. So, what is dN i dx? So, dN i dx is minus 1 by x j minus x i and dN j dx is 1 by x j minus x i; x j minus x i is nothing but the length of each element. The difference between the x coordinates of the two ends. So, this is minus 1 by l e, this is 1 by l e. So, let us consider that for the integration.

So, this will become k by l e, 1, minus 1, minus 1, 1; 1 l e will come additionally integral of dx from i to j will be l e, so that l e will cancel with l e square to make 1 by l e. So, this is the term 2.

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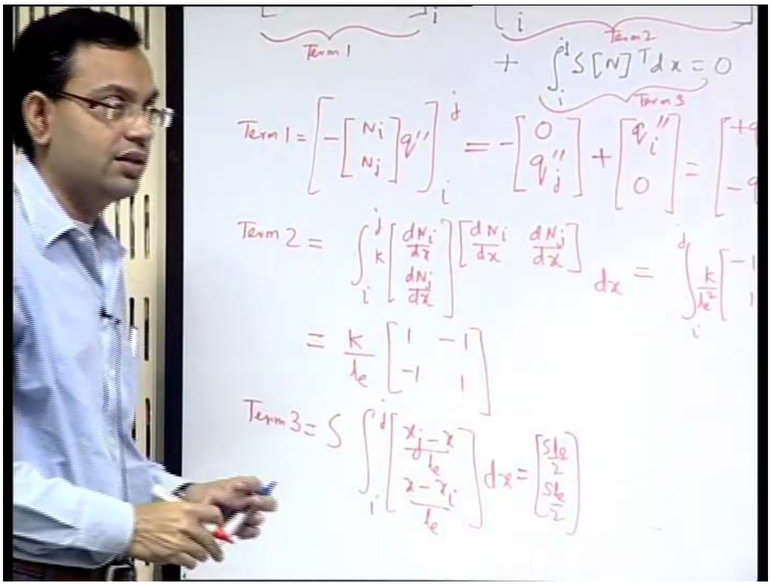

And term 3, let us consider this as the term 3; S is a constant then integral of i to j N transpose dx; so x j minus x by l e and x minus x i by l e, this is d x. So, you can do the integration like for the first case you can substitute x j minus x equal to a new variable. So, then this will be that new variable, let us say that new variable is y. So, it will be of the form integral of y dy from 0 to l e. So, it will be l e square by 2; that l e square by 2 divided by this l e will be l e by 2. So, this will be S l e by 2 and S l e by 2.

See this is the common sense appear. You have the total heat source as S in to l e, and that is dumped in the two nodes. Because if you see that eventually the entire effect of the element is manifested by the behavior of the nodes. The nodes have the sole responsibility of taking care of the burden of representing the behavior of the element. So, whatever is the total that is now apportioned between the nodes. So, because there is no bias or asymmetry between the nodes, it is just 50-50 share. So, it is a common sense thing even without doing the integration you can see, but of course by doing the integration you can exactly verify that. Now, let us assemble this equation.

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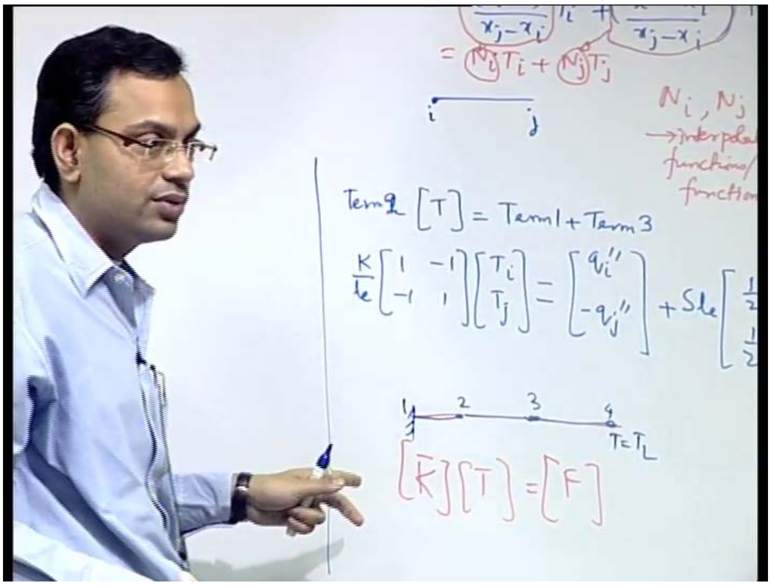

So, what is our equation? Our equation is term 1 or rather let us write term 2, take it in one side, term 2 into T is equal to term 1 plus term 3. So, k by l e 1, minus 1, minus 1, 1, T i T j is equal to… This is what you write for each element. Now, similar thing you can… So, this is what you write for each element means just your i and j will be different. So, for the first element i will be 1, j will be 2. So, let us reconstruct that sketch of the elements. You had three elements, so 1, 2, 3, 4; this was insulated and this is T equal to T L. So, for the first element i will be 1, j will be 2; for the second element i will be 2, j will be 3; for the third element i will be 3, j will be 4. So, you can write similar equations - similar three equations for each of the three elements. Question is now, how to assemble it globally? So that you get the behavior of the summation of all the elements or all the elements connected together.

Now, there are certain important things that you can observe from here. One is that for each element you are getting a 2 by 2 matrix. So, this sometimes is the left hand side is of such a form that you can write it as K into T where is capital K is different from this thermal conductivity, so let us say K bar, K bar into T equal to some F. This is the hallmark of a linear system; just like a linear spring mass system, you have K x equal to F, where K is the stiffness of the spring. Here similarly this is playing an artificial role of a stiffness. As if you are abstracting the physics from the mathematics. So, because of the mathematical analogy, this appears to be as if this is like a stiffness of the system. This is the as if the displacement.

So, here the stiffness the role of the stiffness is being played by k by l e which is like the thermal conductance, and these are like the displacements and these are like forces. So, the source term and the heat flux is like a force which is acting on the system. So, it is almost like a spring mass system. And why this analogy with the spring mass system one always gives, because the finite element method was originally developed for structural mechanics by engineers. It was not originally developed for solving heat transfer and fluid flow problems. So, still this type of matrix is known as a stiffness matrix. But we can we need to keep in mind that it is just a nomenclature, it has nothing to do with the stiffness in a heat transfer problem, the heat transfer problem in the physics; of course, has nothing to do here with the stiffness.

Now, if we need to assemble so and if you get a 2 by 2 equation for each element, then when it is summed up over all the elements what will be the size of the total matrix that we will get. So, if it is for if for 2 nodes it is 2 by 2, then for 4 nodes it must be 4 by 4. So, what we can see here is that we will get a square matrix as a coefficient matrix; so 4 by 4.

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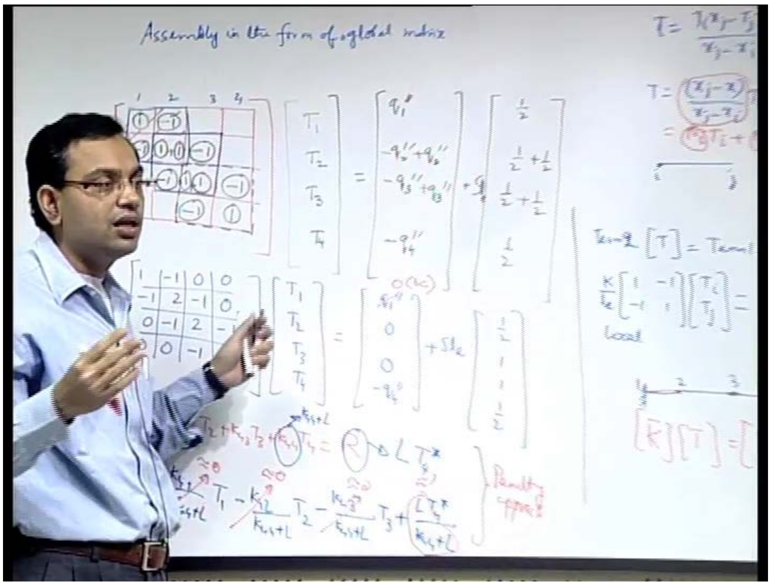

So, we just indicate the number of rows and columns by 1 2 3 4 and 1 2 3 4, just for our own convenience. So, what we are trying now? So, this is the local formulation; we are now trying to globalize it and assemble. So, assembly in the form of a global matrix that is what we are attempting. What are our unknowns? Unknowns are T 1 T 2 T 3 T 4, and right hand side you have a heat flux term and you have a heat source term. So, we need to… So, we have prepared the structure of the matrix form. Now, we have to put in or plug in the numbers. First for the element number 1, what are the node points which are involved with the element number 1? 1 and 2. So, it is…

This part of the coefficient matrix that will be activated. So, what will be the corresponding entries? Let us put the entries here. So, what will be the entry corresponding to 1? That is 1, then entry corresponding to 2 - minus 1. So, as if you are just copying that from the local matrix; minus 1 and this as 1. Right hand side, this is q 1, then this is minus q 2; then this is S l e, this is half and this is half. Then for the next element - 2 and 3 right; so, what what will be the activated part? 2 and 3 both from the row and the column side, so this will be the corresponding active part of the global matrix. So, you have 2 and 3 and here you also you have 2 and 3. Again it will be 1, so you have… Remember you are just linearly adding, because it is a summation - algebraic summation, you are summing it up over all elements; so 1, minus 1, minus 1 and 1.

Similarly, so remember for the first case it was like this, for the second case it is this one and… Let us complete the right hand side. So, for the right hand you have this as so here you have plus q 2 double prime, you have minus q 3 double prime, here you have plus half and here you have half. Then finally, you have this dotted portion 3 and 4, so 1, minus 1, minus 1, 1. Here plus q 3 and minus q 4; plus half and half. What is an important assumption that we have made? We have made an important assumption that k is a constant that is the thermal conductivity is a constant, it does not vary from one element to the other. But if it is a composite material, then it is possible that different elements have different thermal conductivities that is what we are not consider. And the other simplification is that we have considered each element to be of same length. That is not necessary, but just for algebraic simplification we have put in put it; otherwise different elements will have different length.

So, if you assemble this let us write the final form. So, 1, minus 1, minus 1, 2, minus 1, minus 1, 2, minus 1, minus 1, l, the remaining ones are zeros. Now, you can complete the problem by imposing the boundary conditions. What are the boundary condition that we are given? At x equal to 0 the heat flux is 0, it is insulated. So, this is 0, this is one boundary condition, and at 4, T 4 is specified. So, since T 4 is specified solution of that is not necessary. But let us say you are using a computer program which does not know that T 4 is specified, but it still intense to solve, despite the fact that it is specified. So, you can understand the problem, the problem is like this. You will be solving T 1 no doubt about it, you will be solving T 2, T 3 no doubt about it, but you will you do not want to solve T 4, because T 4 is already specified. But say every time you do not want to change your computer program to tune it with the fact that which is specified and which is not specified. So, even though T 4 is specified, you may want to solve for T 4 and find out the value exactly same as what is specified, then that will also solve the purpose.

Then you can use the same structure for solving the equation - system of equations, no matter whether T 1 is specified or T 4 is specified. So, how we can do that? So, the final equation for T 4 is like this k 11 T 1 sorry k 41 T 1 plus k 42 T 2 plus k 43 T 3 plus k 44 T 4 is equal to something some right hand side R. And let us say the value of the T 4 is specified as T 4 star. So, we will play a small trick with this one. What we will do? We will replace k 41 sorry k 44, because T 4 is specified sorry k 41 remains as it is. We will replace k 44 by that plus a large number, where L is a large number. And we will replace R with k 44 in to T 4 star sorry large number in to T 4 star not k 4 the same large number. We have to see how large it is. Whenever we say large or small, it is of course from a computational perspective (( )) it is a computational trick. So, what it will do?

So, how can you solve for T 4 from here? Say you divide all the terms by k 44 plus L. So, it is k 41 minus k 41 by k 44 plus L T 1 minus k 42 by k 44 plus L T 2 minus k 43 by k 44 plus L T 3 plus L T 4 star by k 44 plus L. So, what you want? You want the larger numbers to be so large that it is much, much larger in comparison to all the individual k 41, k 42, k 43, like that. So, when you do that this term will be very close to 0. So, largeness or smallness depends on the values of the individual numbers. So, you do not have a universal rule for that. Depending on the individual entries of k, you must keep it much, much larger in comparison to that. So, then this will also be closed to 0, this will be closed to 0 and this ratio will be approximately 1; k 44 is negligible in comparison to L. So that means if you solve it you will get T 4 equal to T 4 star numerically. This approach is known as penalty approach. That is wherever a value is given still if you want to solve it from the system of algebraic equations, then what is the approach that you could follow.

Of course, if you are doing it manually, you can get rid of the problem by not solving that equation itself. But if you are not doing it manually and if you are doing it through a computer program then may be some other point temperature is also specified. So, you need to jumble up with the system of equations and manipulate manually, which you can avoid by making its structured in a way that you can apply it no matter which point at which point temperature is specified and at which point it is not.

So, to sum it up what what we have achieved by this time? We could convert the governing differential equation into a system of algebraic equation. The system of algebraic equation is in the form of K T equal to F where the T includes the temperature at the unknown points. So, the next problem boils down to the solution of the system of algebraic equation. So, once you solve the system of algebraic equations you get the values of temperatures at the discrete points. You no more get a continuous variation, but what you can always do is, you if you take the points sufficiently close to each other, you will get a very close variation. So that it will be approximately like a continuous variation; it will be discrete points joint together through some interpolation. But if there are sufficient number of points it will look like almost a continuous variation. So, one of the important things that remains of course is how to solve this system of algebraic equations. We will consider it or take it up in a through a separate chapter, not through this particular chapter.

So, in the next class what we will do is, so we have seen that how to come up with the system of algebraic equations from a vibrational formulation. Next class we will see that how to come up with the system of algebraic equation through discretization directly from the deformulation that is a differential equation formulation. And we will take up an example as a finite difference method which tries to do that. Then what will do is, we will try to search for some method which sort of is the compromise or something in between in terms of its behavior, in terms of the finite element method and the finite difference method, and that method which we will extensively follow throughout this course is the finite volume method. And the finite volume method we will see that in the limiting cases, it can be shown as a particular variant of the Galerkin’s weighted residual method.

But it may also be it it may also be perceived as a control volume based finite difference method. So, there are different perspectives in which it can be looked at. But our important objective will not be initially to to detail the finite element, finite difference and the finite volume method. But to understand that basic essential elements, see their important features, and once we learn this basic methods we will go deeper and deeper into the finite volume method for solving the fluid dynamics problems computationally. That aspect we will take up through a subsequent lectures, but in our next lecture we will be starting with the finite difference method. See philosophically, how it how it differs from the finite element method. And then we will identify some plus and minus points of the finite element and the finite difference method, which will motivate us towards approaching for the looking for alternative method in terms of the finite volume method. Thank you.